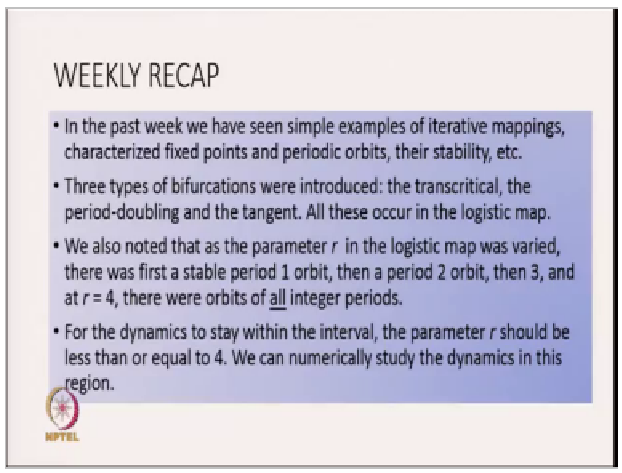
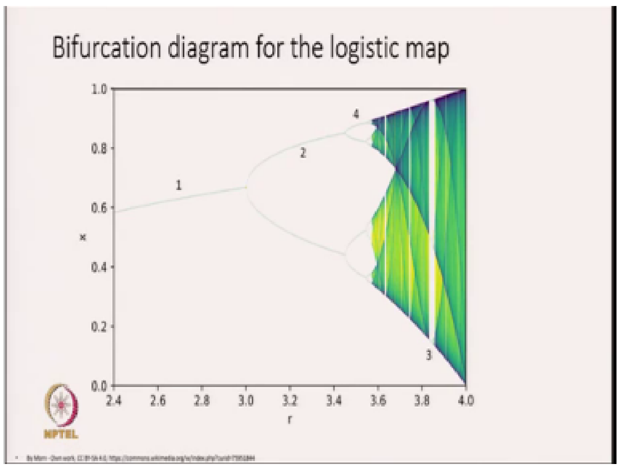
Introductory Nonlinear Dynamics
Prof. Ramakrishna Ramaswamy
Department of Chemistry
Indian Institute of Technology, Delhi
Lecture - 07
Bifurcation diagrams. Period 3 implies Chaos. Characterizing Chaos

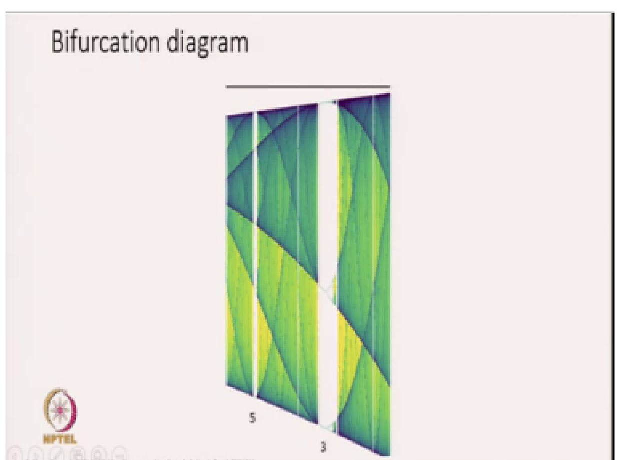
Hello. In this set of lectures, we will start Characterizing Chaos or Characterizing Chaotic dynamics using some of the stuff that we have built up over the last few weeks. (Refer Slide Time: 00:37)



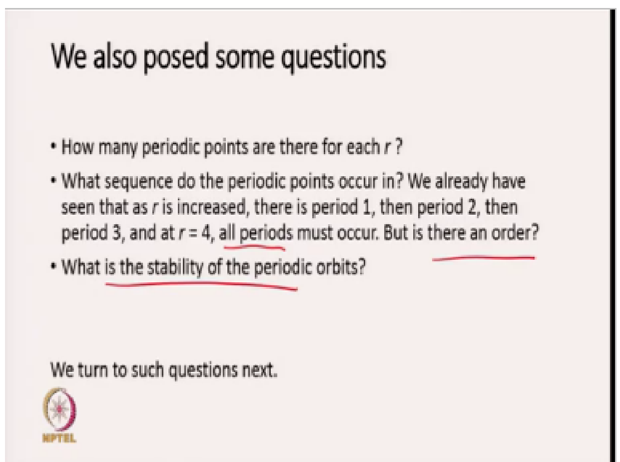
So, to recap in the past week we have seen simple examples of mappings, we have characterized fixed points, periodic orbits, looked at their stability and so on. We saw that as you change the parameter on the system the behavior could abruptly change, these points of abrupt change of behavior are termed bifurcations and they come in various different types. We looked at some of the simplest forms of bifurcations, the transcritical bifurcation, the period doubling bifurcation and the tangent bifurcation. All these types of bifurcations occur in the logistic map which we are going to study in some detail. There are other bifurcations some of which also occur in the logistic map, but in they do not many of these other bifurcations occur in different kinds of systems. We noted also that the parameter r in the logistic map the non-linearity parameter. As you varied it first there was a stable periodic orbit of period 1, then there was this bifurcation the period doubling bifurcation and then you had in an orbit of period 2. And, at some later point in time we noticed that there could be a an orbit of period 3. And then when the parameter r takes the value 4, there were all bits of all possible integer periods. How does this happen? And well one of the things is that in the logistic map, for the dynamics to stay within the interval 0 to 1 the parameter should be less than or equal to 4. It is a simple algebraic exercise to show that if r is bigger than 4, then the midpoint which is the maximum, which is half, exceeds it maps to a point outside the interval. So, we always look at the interval 0 to 4 as far as r is concerned. Now, we can look at the dynamics of this system numerically as well. (Refer Slide Time: 03:07)



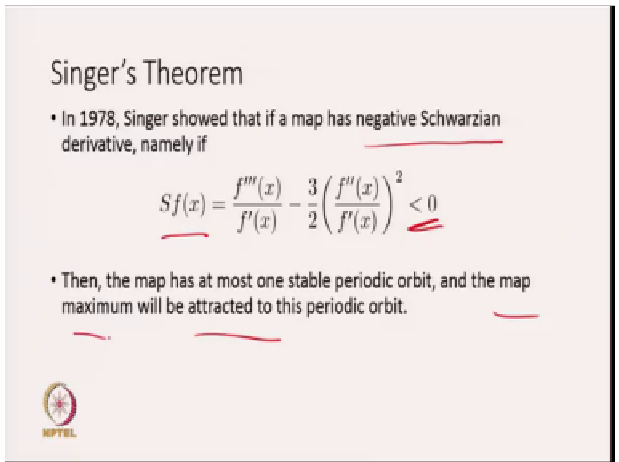
But, the questions that are raised in this slide are of course, they are interesting and important to look at. (Refer Slide Time: 03:17)



Here is a bifurcation diagram for the logistic map namely as you vary the parameter r. What are the different orbits that are seen? So, we notice that up till r is equal to 3, we see a period 1 orbit. At 3 there is this bifurcation and then you see a period 2 orbit and then you can see over here we can just numerically count that is a period 4 orbit. And that looks pretty much like a period 8 orbit and so on and so forth. And the diagram is sort of it begins to look interesting and complicated as you move along over here. And around this point you notice that there is a period 3 orbit and that is we have already calculated and seen when that happens. So, here is where there is a period 3 orbit, but you know to the left of period 3 you notice that there is period 5. So, there was period 1, period 2, period 4, period 8, then there is period 5 over here and if one looks and counts carefully there is a period 7, I think over here and there is period 3 at this particular point. (Refer Slide Time: 04:43)



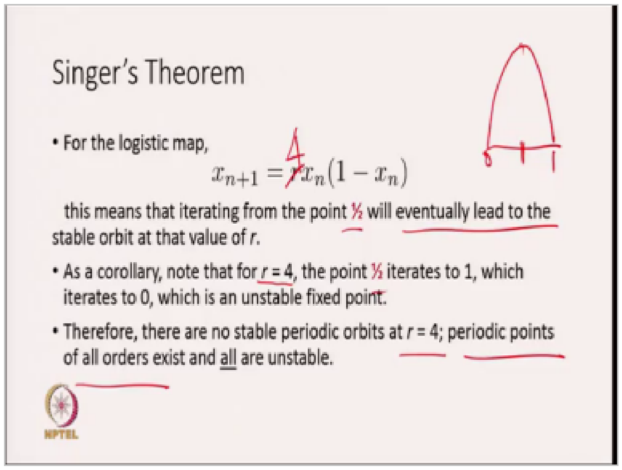
Now, how there was some questions that we had posed in the earlier lectures 1 started as starting off with how many periodic points are there at each r. Now, you look over here and you see that at each value of r, it appears that there is a single periodic orbit. But, you know, what is the reality? A more interesting question related interesting question is what is the sequence of these periodic points? We have already seen as we increased r that there is period 1, then there is period 2 and then beyond that in the figure we saw there was period 4, period 8, then we saw that there was period 5, then period 3 and at r equals 4 all periods must occur. But is there any order? And again as far as the visibility of these periodic orbits is concerned, what is the stability of there of these orbits? Because, this will tell you whether you can find them easily or not so, we turn to these questions now. (Refer Slide Time: 05:53)



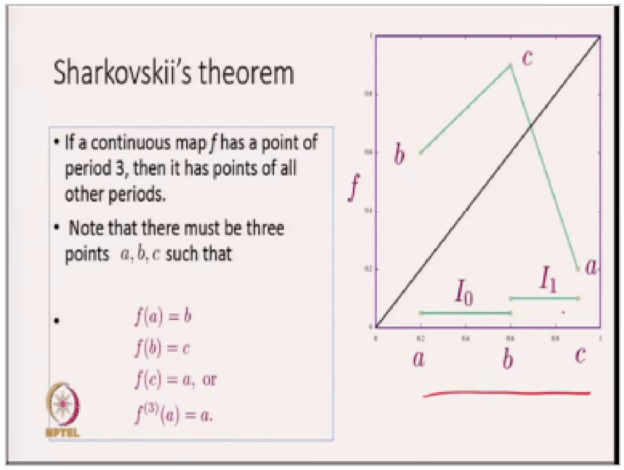
In 1978, you know singer showed that, if a map has a negative Schwarzian derivative, that is this quantity $Sf(x)$ the Schwarzian derivative of the function f, it is given as.

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}(\frac{f''(x)}{f'(x)})^2 < 0$$

If, this quantity is negative in the region that we are considering, then the map has at most one stable periodic orbit. And, furthermore the maximum of the map will be attracted to this periodic orbit. So, what you need is a map with a negative Schwarzian derivative, then there can be at most one stable periodic orbit. And, in order to find this periodic orbit all you really have to do is to start with the map maximum and keep iterating it. So, eventually you will reach this periodic orbit. Now, for the logistic map. $x_{n+1} = rx_n(1-x_n)$ what does this mean? First of all as you will check in your in a homework assignment, you can verify that the Schwarzian derivative for the logistic map is negative on you know in the region that we are looking at which is from 0 to 1. (Refer Slide Time: 07:31)



And for r less than 4 and this means that if you start from the point half, because the map maximum the map goes it is on 0 to 1 and the maximum is at the midpoint which is the half. So, if you start at half, then you will eventually go to the stable orbit at that value of r. Then, there is an interesting corollary, that when r is equal to 4 the point half iterates to 1, as you can see from here 4 times 1 half into 1 half is 1 1 on the other hand will then iterate to 0, because if xn is one then xn plus 1 is 0. And 0 we know is an unstable fixed point above r is equal to 1. So, this contradicts the statement that if there is a stable orbit, then half will eventually lead to the stable orbit. Therefore, we must infer that there are no stable periodic orbits at r equals 4. There are; however, periodic points of all orders, but all of them are unstable. So, the logistic map with r is equal to 4, which is a map that was first studied by Ulam and Neumann, this has a huge number of periodic points, but all of them are unstable. And, this formed the original basis for using this particular map as a random number generator, because there was no periodic points at all which are stable. The ordering in which these points appear has been is determined by what is called Sharkovskiis theorem . (Refer Slide Time: 09:32)



So, much for the question of so, much for the question of the stability of a periodic point and how initial conditions are attracted to it right. Having seen singers theorem, which tells us about the negative Schwarzian and derivative condition and how the map maximum is attracted to the stable periodic point. We now turn to the question of what is the order in which these periodic orbits are seamed. And this ordering owes it is named or to a very beautiful theorem by Sharkovskiis which was proved in the early 1960s. Sharkovskiis theorem is an extremely simple theorem to state. And the statement is as follows, if a continuous map f has a point of period 3, then it has points of all other periods. For a map to have a point of period 3 there must be 3 points, let us call them a, b, and c such that a maps to b. So,

$$f(a) = b$$

$$f(b) = c$$

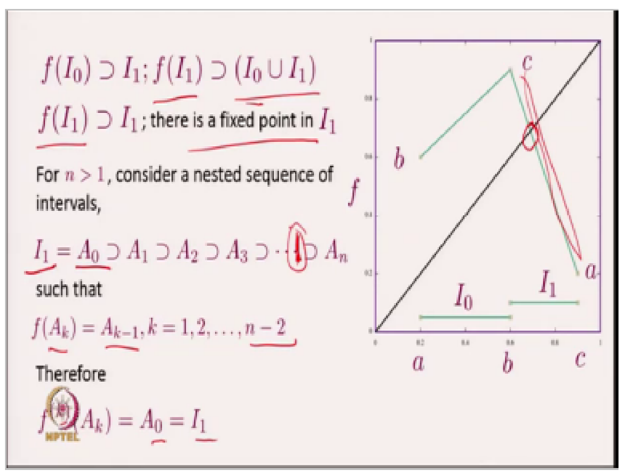
$$f(c) = a$$

or

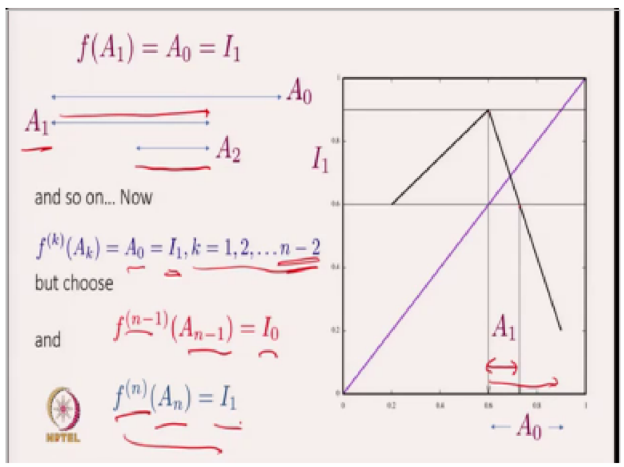
$$f^3(a) = a$$

. Of course, the ordering could be different in the sense that a could map to c and not to b and, but this is just a minor rearrangement of everything over here. So, we will take this particular ordering a, b, and c, a less than b less than c right. So, the map says that if the theorem says that if there is a map which has a point of period 3, then it necessarily has points of all other periods and the flavor of this proof is somewhat simple to understand.

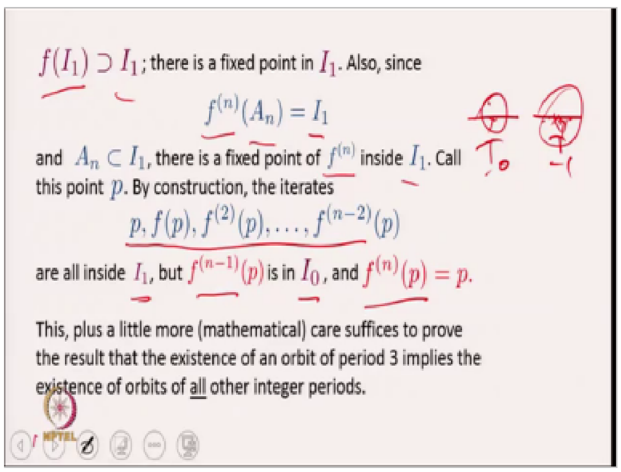
So, we are going to just cover this in the next couple of slides. For that let me note that these 3 points a, b and c they allow us to define 2 intervals. Let me call this interval from a to b, I call that I 0 and the interval from b to c, I call it I 1. The map itself is just continuous it does not have to be differentiable or anything like that. So, it could be and it in particular it could be quite complicated between these points a, b and c, but I am just drawing it as a piecewise linear map for convenience all right. So, the basic idea of this proof rests on the fact, that if I take this interval a to b, if I take the interval a to b, the action of the map is to take it to take a to b and b to c. So, it converts this particular interval I, 0 into something that could contain that does contain I 1 ok. So, it contains. So, f of I 0 is the superset of I 1. What happens to I 1 on the other hand you know under the action of the map, b goes to c and c goes to a. So, the action of the map on I 1 is to stretch out that interval I 1. So, that it contains both I 0 and I 1. (Refer Slide Time: 13:45)



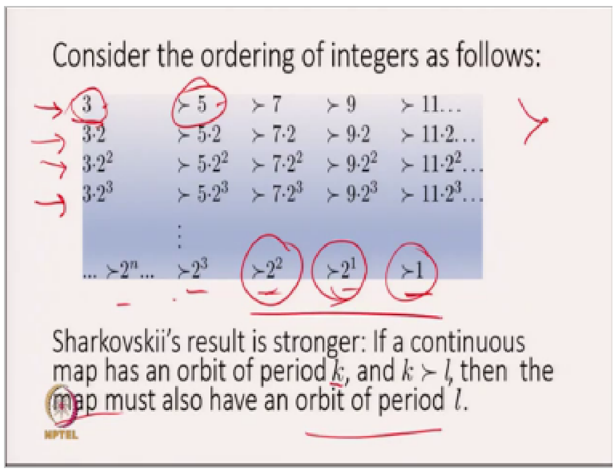
So, to put it formally the f of I 0 is a superset of I 1, f of I 1 is a superset of the union of I 0 and I 1. Now, since f of I 1 contains I 1, there is a fixed point inside I 1. So, f of I 1, it contains I 1, somewhere and you see clearly that there is a fixed point of the map inside this interval I 1. The idea behind the proof to show that you must therefore, have a points of all other prime periods is to do the following. We consider a nested sequence of subintervals; let me call these A 0, A 1, A 2, A 3 etcetera etcetera. So, A 0 contains A 1, which contains A 2. So, these are shrinking intervals all contained inside A 0, which I take to be the interval I 1. So, here is the interval A 0 within that there is an interval A 1 within that there is an interval A 2 and so on and so forth. Now, up to the subscript n minus 2, what these subintervals, what property these subintervals possess is the following? When I act the map on A k I must get A k minus 1. And, this is easy to see that I must do other I could do that. Because, since f of I 1 contains I is not union I 1, then the action of f is to stretch whatever is there inside this particular sub interval and stretch it out to contain both these. And, therefore, it is possible for me to construct a sub interval, such that f of that sub interval is the previous sub interval in which it is contained. Now, if f of A k is A k minus 1, f 2 of A k would be A k minus 2 and so on. So, that f to the k of A k must be A 0 which is I 1 ok. I hope this is clear now. Each time the action of the map is to stretch, because it is pulling outwards as you can see over here it pulls it out. Therefore, the action at each stage is to expand a sub interval to contain the whole of this interval. (Refer Slide Time: 16:52)



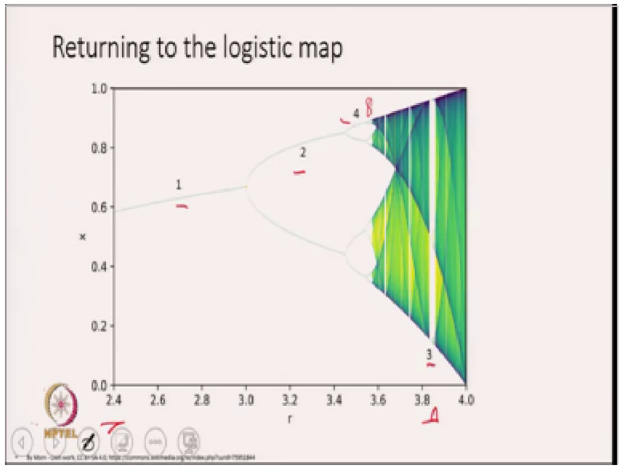
So, here again just to show you what is A naught A naught is this interval I 1 which we had called in the previous graph, but that is A naught which goes up till there. A 1 on the other hand is just this portion over here, because when A 1 is mapped it maps into this entire portion over here, which is I 1 or A naught. So, the sub interval A 1 is this portion of A naught, you can notice also that the orientation is flipped a little this edge of A 1 goes to c this one edge of A 1 goes to b all right. So, the next interval A 2 is actually on this side. And when I add the map on this will again flip the intervals, flip the edges over because of the way in which this map acts on these intervals. So, f to the k of A k eventually gives me A naught which is I 1, for k going from 1 2 all the way to n minus 2, but notice that when I act f on the interval A naught, I get both I 1 and I naught, namely I get A super set which contains both these intervals I 0 and I 1. So, now, for the in this construction I need to construct n subintervals. So, the A nth minus 1 A n minus 1th sub interval over here, which is contained inside A n minus 2. This I choose in a way that f to the n minus 1 of A n minus 1 is not I 1, but it is I naught. So, this is the way in which all these worked was that when I expand the sub interval I get I 1, but for this particular step I just go back and I ask it to be inside I naught and then the last sub interval which is again contained inside that is such that f n of A n is I 1. Now, this is actually the gist of the entire argument all hinges on this. (Refer Slide Time: 19:34)



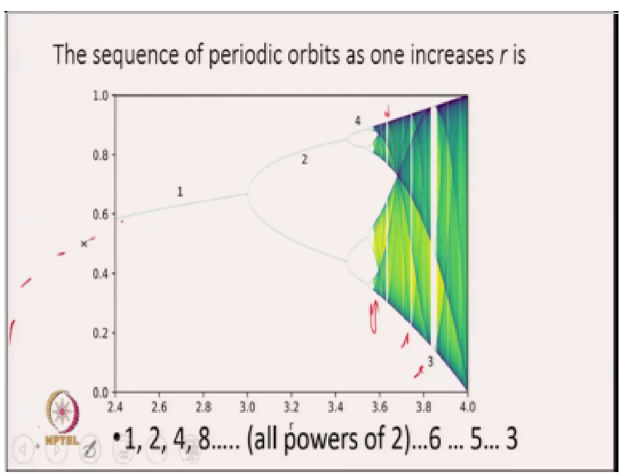
Because, we now notice that since f of I 1 contains I 1, there is A fixed point inside I 1 that was the simplest one that we found, but since f n of A n is I 1 and this I n is contained inside I 1 there is a fixed point of f of n inside I 1. So, let us call this point p, there is a fixed point of f to the n inside I 1 and this is the point p. Now, by construction because of this very clever way in which these subintervals have been chosen the first n minus 2, the n minus 2 iterates, they are all inside I 1, but the n minus 1th iterate is in I 0 and then the nth iterate is back inside I 1 it is p. So, here we had these 2 subintervals I 0 and I 1, the n minus 1 iterates are all inside I 1 the nth iterate is over here and the last one is over here, at the n th minus 1 iterate is over here the last iterate is over here and this completes the periodic point of order p. Now, because all the iterates are on this side except for one of them there can be no smaller periodic orbit that you know that is doubling into this or multiplying any particular factor into this. And therefore, one can with a little more mathematical care, you can show that this implies the existence of an a prime periodic orbit of any other integer period by this construction. You want an orbit of period 5 well construct 5 sub intervals A 1, A 2, A 3, A 4, A 5 and that will be enough to show that there must be a fixed point inside A sub 5. But, you know remarkable as this statement is Sharkovskiis squeeze theorem is actually much more spectacular the result is stronger. (Refer Slide Time: 21:57)



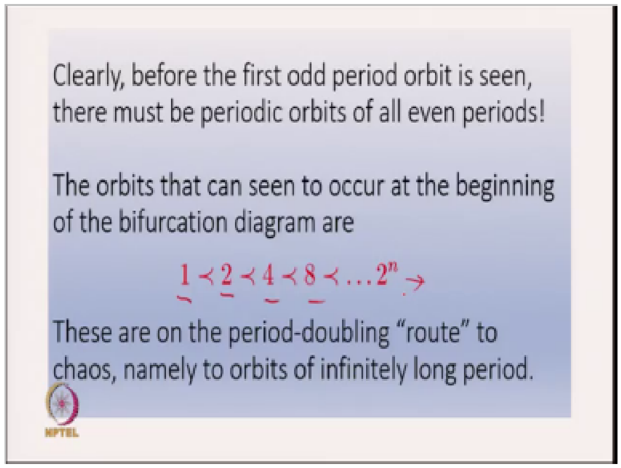
And he gives an ordering of the integers, which is as follows you first list all the odd integers in increasing sequence 3 5 7 9 11 13 and so on and so forth. After, that odd integers are exhausted, we list out 2 times each of the odd integers, exhaust that, then 4 times all the odd integers exhausts that 8 times and then you notice that these are all the powers of 2. So, you now you start listing all the different powers of 2 times the odd integers in sequence. And when all that is exhausted you end the sequence with the pure powers of 2 ending with 2 to the n etcetera etcetera 2 to the 3 2 to the 2 2 to the 1 1. Now, this is an ordering where you can easily see that all the integers appear once and once only. So, here is 1 here is 2 here is 3 here is 4 5 and so on and so forth, you can see all the integers are here. Sharkovskiis result says the following that if a continuous map has an orbit of period k and k precedes l, in this ordering this symbol is for proceeding, in this particular order, then the map must also have an orbit of period l. So, if you have an orbit of period 3 you must have all other periods. If you have an orbit of period 4, you will have to have an orbit of period 2 and an orbit of period 1, but no others ok. So, this tells you in a way how all the different orbits are occurred or in what ordering they may be seen, it is possible to construct maps that have a period 5 orbit, but no period 3 orbit, just as this other example over here indicates. (Refer Slide Time: 24:29)



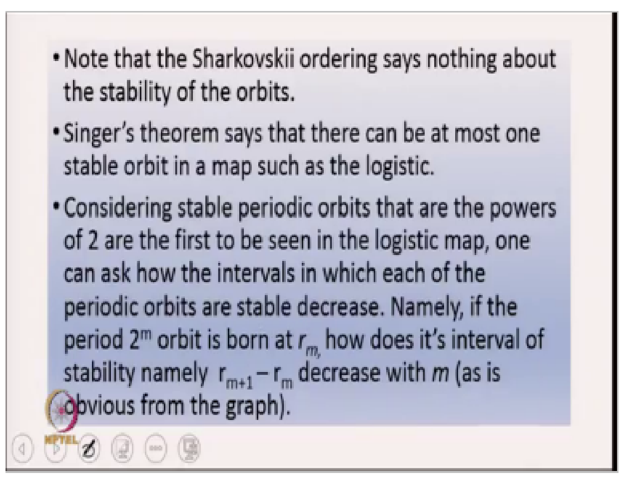
Now, returning to the logistic map and what does the Sharkovskiis theorem have to say to us about this we noticed that there is period 1, there is period 2, there is period 4 and clearly there is a period 8 over here and so on and so forth period 3 over here all the odd ones are over here. So, somewhere between this point over here and the point where period 3 occurs all other periods occur all other integer periods have occurred. (Refer Slide Time: 25:05)



This sequence of periodic orbits as one increases are on this side is 1, 2, 4, 8 and all the powers of 2 that seem to happen by this point over here then at this point you can notice that there is an orbit of period 6, there is an orbit of period 5, there is an orbit of period 3, and so on and so forth. (Refer Slide Time: 25:32)



So, clearly before the first odd period orbit is seen, there must be periodic orbits of all even periods, because of the Sharkovskiis ordering. At the beginning of the bifurcation diagram, we see that the orbits are 1, 2, 4, 8 and so on and so forth and these are the period doubling you know because the period 1 is double to 2 double to 4 double to 8 etcetera. And this is on the period doubling route to chaos and as n goes to infinity you get orbits of infinitely long period. (Refer Slide Time: 26:15)



Few points over here the Sharkovskiis ordering actually is not A state it is A statement about the ordering of the orbits, but not about their stability.

Singers theorem says that for a map like the logistic which has got negative Schwarzian derivative, there can be at most one stable orbit in a map. Now, we have seen that when you construct this bifurcation diagram by numerical means, when, what you see is really all the stable behavior. So, considering the stable periodic orbits that are powers of 2, the first to be seen in the logistic map are 1, 2, 4, 8 etcetera. As you can see from the diagram the interval on which period 1 is stable goes all the way from 1 to 3, period 2 is stable for not quite that much in terms of parameter space, it is actually stable only until about 3.4. Then period 4 seems to occupy even less space. So, a natural question to ask is how do these windows in which these periodic orbits exist, how do these windows keep changing as a function of the order? So, if period 2^m orbit is born at r_m, how does the interval of stability namely $r_{m+1} - r_m$, how does this decrease with m as is evident from the graph, we will turn to such questions in the next lecture.