Introductory Nonlinear Dynamics

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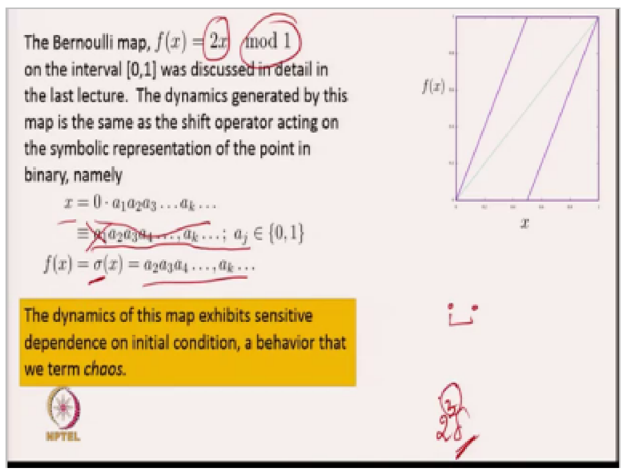
Lecture 06

Logistic map

Simple Examples of bifurcations.

Hello, we continue our discussion today on maps and flows and we look at some Simple Examples of bifurcations. (Refer Slide Time: 00:37)



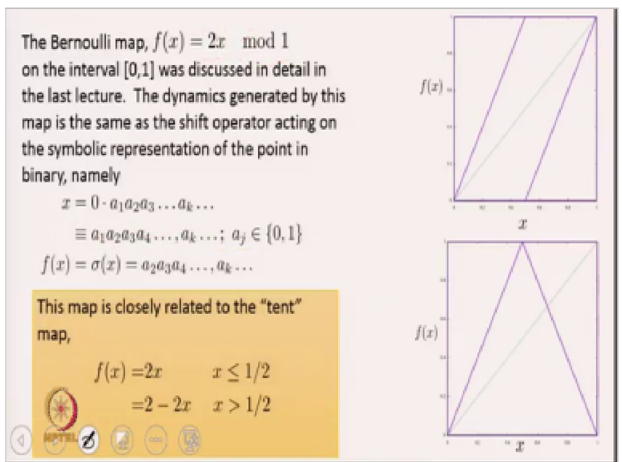
Recall that in the last lecture, we had considered the Bernoulli map

$$f(x) = 2xmod1$$

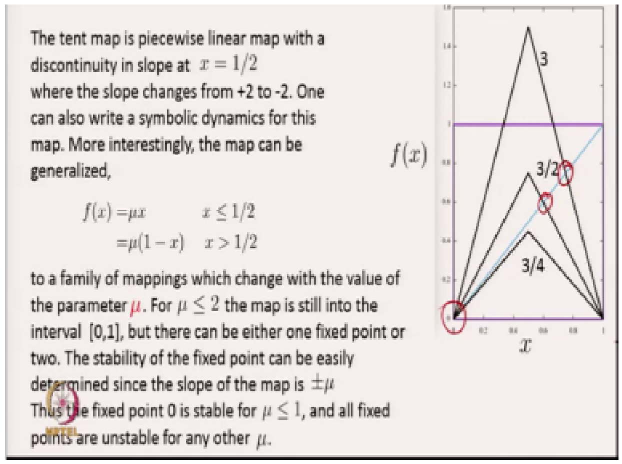
on the interval [0,1], this was discussed in detail and we looked at the dynamics on this map. When we noticed that the dynamics is the same as if a shift operator was acting on the symbolic representation of the point in binary, namely if the point

$$x = 0.a_1a_2a_3...a_k...$$

whereas as are either 0 or 1. Then this strings the symbolic string a 1 all the way onwards is a representation of the point. And $f(x)$ namely this multiplying by 2 and taking the modulo 1 operation was the same as acting a shift operator on x. The purpose of the shift or the action of the shift operator is to simply forget the first symbol the leading symbol there and consider this to be the new point in the phase space. We also saw that the dynamics of this map shows what is called sensitive dependence on initial conditions, we also call this behavior chaos. But basically if there are 2 points that start out at some distance delta apart, then after 1 step; 1 operation of the shift the distance between these two points is 2 times delta, after one more operation it is 2 square times delta, after one more it is 2 cubed so and so on. So, no matter how small delta might be after a sufficient number of steps the distance between these two points is of order 1. So, this kind of sensitivity to initial conditions is the behavior that has in the popular sorry, if the description is that of chaos. (Refer Slide Time: 02:58)



The Bernoulli map is very closely related to the tent map, the Bernoulli map recall is just $f(x) = 2x$ mod 1, and the tent map takes this portion of this branch of the Bernoulli map and flips it around to make a tent shape. And the equations for this tent map that $f(x) = 2x$ on the interval [0,1/2] and it is $2-2x$ from this point onwards. This is slight difference between these 2 maps particularly, because now the fix point moves from being at the corner of the square over here at the point 1 to a point which is somewhere intermediate that point is something that we can figure out easily. But I am introducing the tent map for a different reason which is a basically to introduce maps that have a parameter in it. (Refer Slide Time: 04:01)



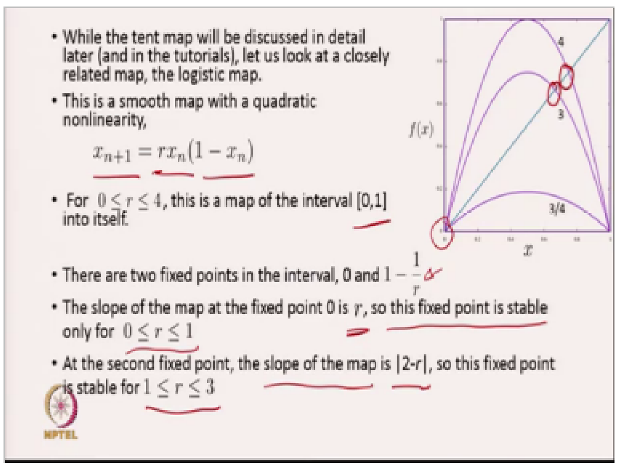
Now, the tent map is a piecewise linear map and there is a discontinuity in slope at the midpoint $x = 1/2$, the slope changes over here from +2 to -2 in this case. So, here the slope was 2, here the slope is minus 2, but the modulus of the slope is the same. Now if instead of the map has written in the previous slide I just now change it to

$$f(x) = \mu x, x \leq 1/2$$

and

$$= \mu(1 - x), x > 1/2$$

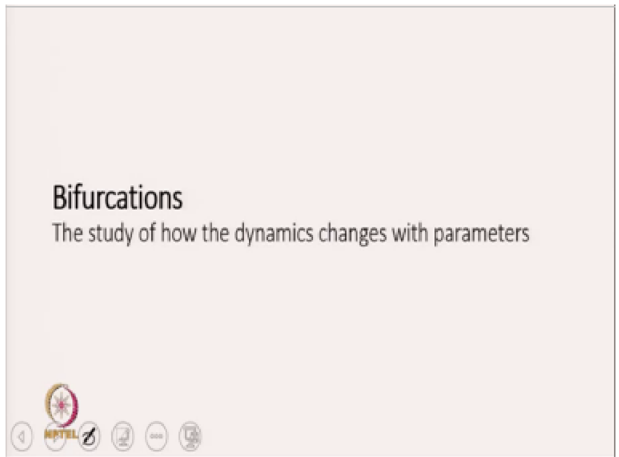
. Then I get a 1 parameter family of maps. For the map with μ equals 2 you can write a symbolic dynamics and show that it has exactly the same sensitivity to initial conditions that the Bernoulli shift map had. But for this map over here if I change μ, you can see that the map changes because, the maximum of the map is exactly at μ is equal to; at x is equal to half and there it takes the value μ by 2. So, when μ is 3, I find that the map goes up to the point 1.5 3 by 2 that is and the tent map escapes from the square. When $\mu$is equal to 3 by 2, I find that the map is over here, you know but the maximum is at 0.75. And at this particular point if μ is equal to 3 by 4 then we find that the map is actually the tent itself lies below the diagonal line. So, clearly you can see that if the tent lies below the diagonal line 0 is the only fixed point. If the tents now goes above the diagonal line, we have this fixed point as well as 0, so there are two fixed points for μ bigger than 1. Now, you can easily see what is the slope of this map? The slope of this map is always mu on this branch and minus μ on this branch so, I you know the modulus of the slope as I said earlier is μ And we have already seen that at the fixed point, if the slope of the map is less than 1, then we have a stable fixed point. So, long as μ is less than 1 then 0 is the only stable point because, we can see that the slope of this map at the fixed point is exactly μ. Once mu crosses 1, the slope over here exceeds 1 and the slope over here also exceeds 1 in modulus and therefore both the fixed points are unstable for any μ bigger than 1. Notice that of course, for all values of μ there are always two fixed points in the interval, the fixed point at 0 and the fixed point at wherever the intersection happens to lie. Today we will leave the tent map at this point and come back to it later on, and perhaps in the tutorials. (Refer Slide Time: 07:51)



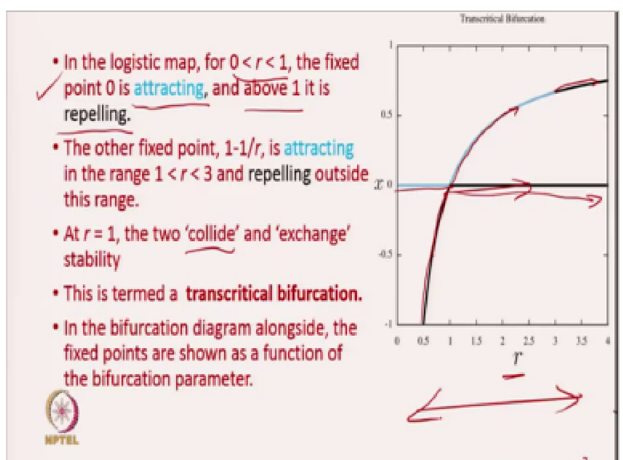
But let me now look at a related family of maps called the logistic map or the quadratic family. This map is given by

$$x_{n+1} = rx_n(1 - x_n).$$

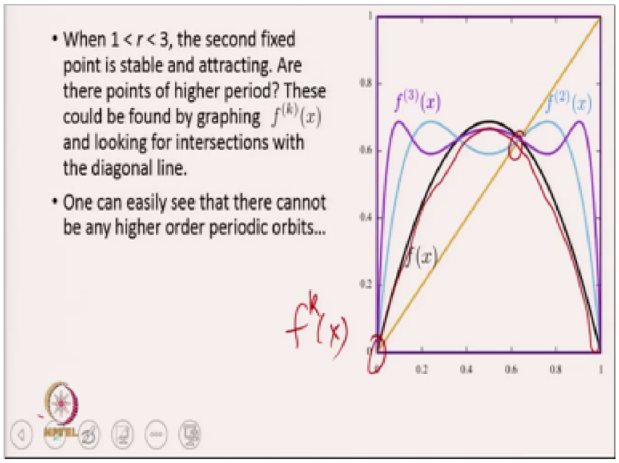
As you can see this map is a quadratic function it vanishes at the point x is equal to 0 and at the point x is equal to 1 and so long as r is between 0 and 4 this maps the interval 0 1 into itself. Like the tent map there are two fixed points for this map 1 is at the point 0 and the other you can easily work it out is at 1 minus 1 by r, and so long as r is bigger than 1, the second fixed point also lies in the interval. The slope of the map at the fixed point 0 is r, so this fixed point is stable only for r lying between 0 and 1. At the second fixed point, the slope of the map is 2 minus r modulus you can easily work that out. So, this fixed point is stable between r between 1 and 3. So, the slope at this point is basically 2 minus r the modulus of the slope at that point is 2 minus r, and so this fixed point is going to be stable between r lying between 1 and 3. (Refer Slide Time: 09:55)



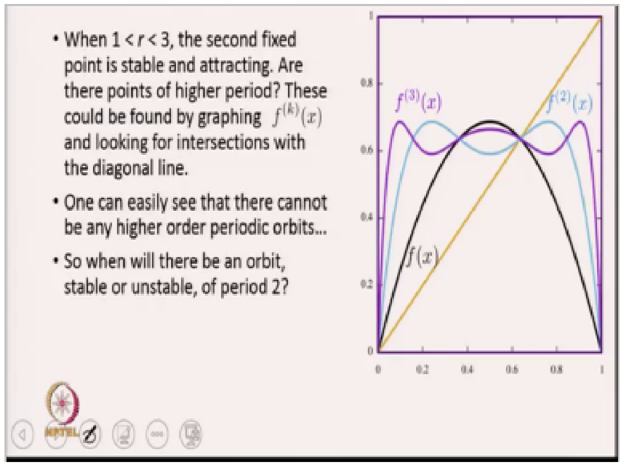
How does the behavior change with parameters? This generally when the when the dynamics changes abruptly as you change a parameter smoothly, this is termed a bifurcation and in this logistic family the quadratic family of maps. We have a number of different bifurcations and we will be introduced to 3 of them during this lecture, ok. (Refer Slide Time: 10:27)



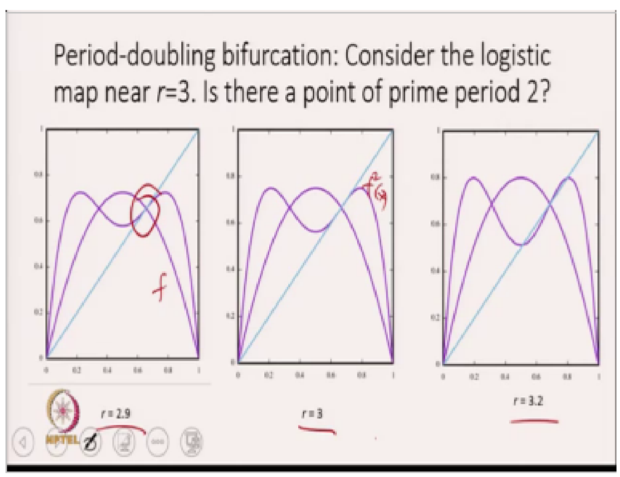
Consider the fixed point at 0; the logistic map so, long as r lies between 0 and 1 the fixed point 0 is attracting, because the slope is r and the slope is less than 1. Above the value r is equal to 1, this fixed point is repelling, the second fixed point 1 minus 1 by r on the other hand is attracting in the range r lying between 1 and 3 and it is repelling outside this range. At r equals 1, both these points at r is equal to 1 the fixed point is 0; in the fixed point is 0 is always 0 of course, the fixed point at 1 minus 1 by r it takes the value 0 when r is equal to 1. So, if you draw the locus of the first fixed point you just get this horizontal line that we have over here that is the x axis, and the second fixed point 1 minus 1 by r that has it is this second curve over here. I have drawn in blue, the portion of this fixed point the region where this is stable notice that on the x axis over here I have the parameter r. So, the 0 fixed points is stable from r going from 0 to 1 and it is unstable after this entire range. The second fixed point is unstable for r going from 0 to 1 and it is again unstable for r bigger than 3, but in this region it is stable. Now as we change the parameter r from 0 up to 1 and beyond 1, it looks as if these two fixed points they come and collide with each other, they merge at this point and subsequent to this point what was an unstable fixed point now becomes stable, what was a stable fixed point now becomes unstable. So, they exchange the stability such a bifurcation is termed a transcritical bifurcation, for r less than 1, 0 would be an attracting fixed point for r above 1 let us say it the value 2 the fixed point will be 1 minus 1 by r. This pictures that we have on the right hand side is what is called a bifurcation diagram, where you plot the fixed points as a function of the bifurcation parameter, you plot the stable fixed points. And then you indicate the stable or the unstable regions of these fixed points either by colors or by dots or sometimes just by their being there and not being there. So, this is one of the simplest bifurcations one can have the transcritical bifurcation. (Refer Slide Time: 13:57)



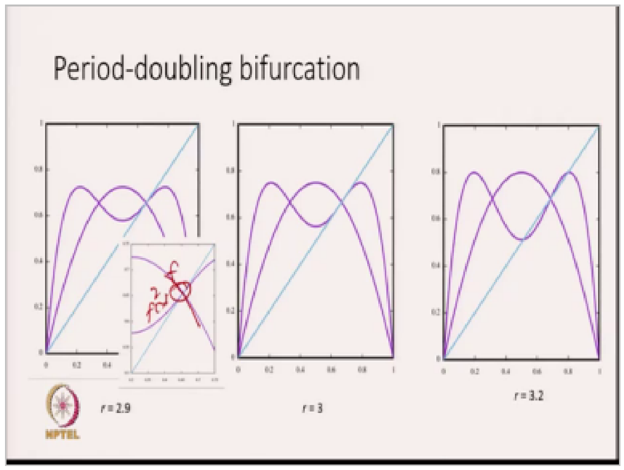
Now, as you have increased r from 0 to 1, the second fixed point became stable between r is equal to 1 and r is equal to 3, the second fixed point is stable and attractive. Now what we would like to know is if this fixed point becomes unstable beyond r is equal to 3. Are they points of higher period at all, how do we find them? As we discussed in one of the earlier lectures a simple way of finding periodic points of higher order is to graph the function f to the k, the k th the composition of x and look for intersections of this with the diagonal line. If you do that in this particular case for r lying between 1 and 3, we notice that the black curve over here is f(x), the blue curve over here is $f^2(x)$, the purple curve is $f^3(x)$ and I could draw many many more of them. But you can see that given the fact that f of x is has this parabolic shape; f 2 of x has two humps, but the depth of this hump is related to how high the peak is. So, since the peak is not very high compared to the fixed point, this valley over here is also not very low and you can see that f 3 loops you know pretty again smooth out that way f 4 will have more oscillations but always 1 intersection. You can convince yourself by graphing and also by looking at it a little with some algebra that there can be no higher order periodic orbits. Because f to the k of x is not going to intersect this diagonal line except at 0 and at the fixed point of period 1. So, all periodic orbits are just going to be period 1 orbits repeated many times. (Refer Slide Time: 16:33)



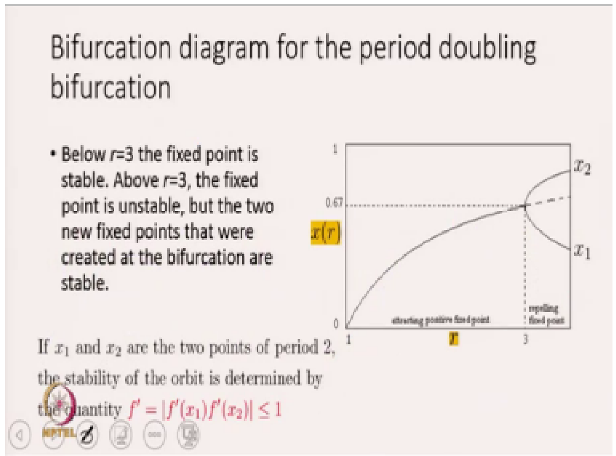
So, one can ask this question when will there be an orbit either stable or unstable that we can have which has got let us say period 2, period 1 is the simplest one next one we look for is period 2. So, when are we going to have a period 2 orbit, to do that we have to keep increasing r or we have to ask questions related to this family of maps; so, let us look at r near 3. (Refer Slide Time: 17:02)



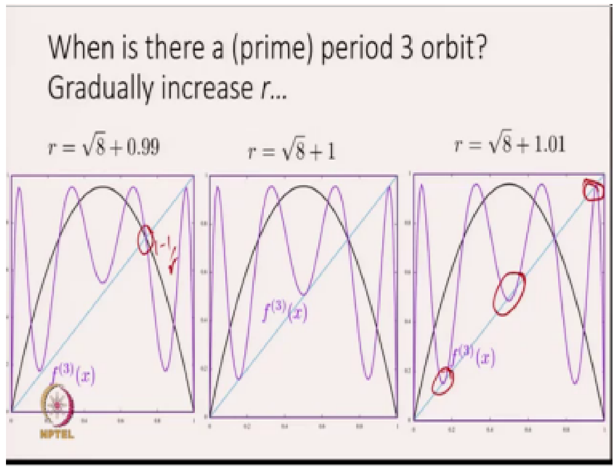
So, here is a graph of the function $f(x)$ and the function $f^2(x)$ at r is equal to 2.9 in this case 3 and 3.2, and the behavior is slightly different even just glancing at this we can see that there are 2 intersections of this map with a diagonal which is not a period 1 orbit. So, there definitely is a period 2 orbit at this particular point, but what has happened in between. This is the so called period doubling bifurcation which occurs at r is equal to 3 and to understand that let us look at a blow up of this region of this figure, alright. (Refer Slide Time: 18:00)



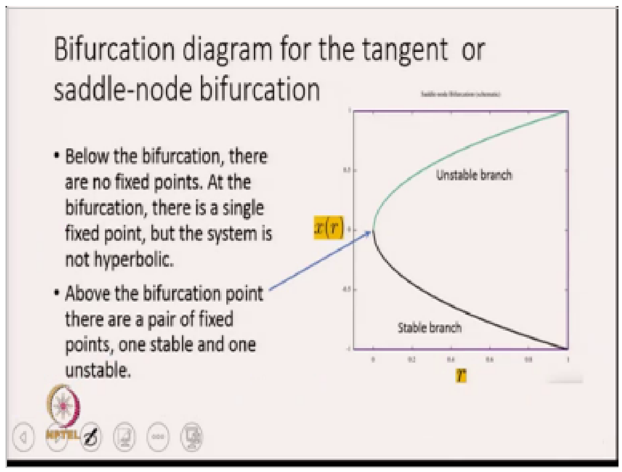
If you look at the blow up of this region, near the figure you see that near the at the intersection this is the function f of x this is the function f 2 of x and f 2 of x just smoothly crosses the same point at, it just crosses the diagonal line transversely over there. The slope over here you can see is clearly the slope at this particular point for f 2 of x is less than 1, alright. At r is equal to 3, at r is equal to 3 what we find is that the f 2 of x that is this curve over here is exactly tangential to the fixed point over here. Its tangential to the diagonal at the fixed point again there is only a single intersection between f of x and f 2 of x, but at this point the slope of f 2 of x is equal to 1. At r is equal to 3 point 2, actually 3 point just immediately after 3, but it is easier to see if you are a little further away from the bifurcation point, f of x has the old intersection f 2 of x has this intersection of course, but it also has 2 new intersections. And these two intersections are purely period 2 points, they are intersections of f 2 of x with the diagonal they are not period 1 points, so this is a prime period to orbit. We can also see just by the way in which it has 1 second let me just bring it back, just by the way in which these curves are drawn that the slope over here is less than 1 and the slope over here is less than 1. (Refer Slide Time: 20:10)



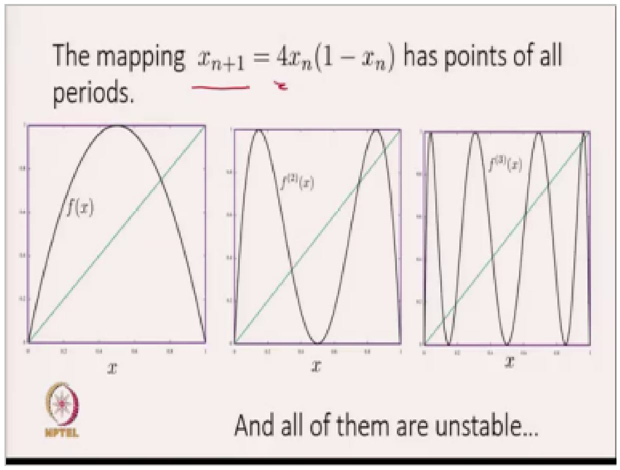
So, when the period 2 orbit is born, the stability is determined by the product of the slopes of this function at the 2 fixed points, at the period 2 points and that slope product is less than 1 in modulus, so the period 2 orbit is born stable. So, the bifurcation diagram for this bifurcation looks something like this. We had a period 1 point which was attracting for r between 1 and 3. Above r is equal to 3 that period 1 orbit went unstable, so we indicate this in dashed lines over here. Just at r is equal to 3 and above we found that there were two points of period 2 that were created let us call them x 1 and x 2, they are the two points that we found over here this is x 1 this is x 2 all right. And these two points are such that the product of the slopes at these two points tells you the stability of the period 2 orbit and just add the bifurcation and a little above it the product of these 2 is less than 1. We will come back to the stability of periodic orbits later on. But the important point to note here is that at a period doubling bifurcation, a period 1 orbit becomes unstable and a period 2 orbit is born. So, the period doubling bifurcation is characterized by the disappearance of an orbit of a particular period and the birth of an orbit of twice that period. (Refer Slide Time: 22:12)



Proceeding along the lines, we can ask when is there going to be a prime period 3 orbit. Now, we know that period 1 has gone unstable at r is equal to 3 and period 2 was born, but in order to find period 3 we have to travel quite a bit more. We travel to the region of r close to the square root of 8 plus 1 why this is and so on will come up in one of the home works. But let us look at the map near this value of r is equal to square root of 8 plus 1. To find an orbit of prime period 3, we have to look at the third composition of the map f of x which we have abbreviated as f to the 3 of x. So, here is the graph of f 3 of x for square root of 8 plus 0.99, square root of 8 plus 1 and square root of 8 plus 1.01. Now, you can see from this particular curve this figure over here, that the third composition of the map has only one intersection with the diagonal which is the same intersection as the map itself. So, f 3 has an intersection it happens to be the same as the intersection of f which is 1 minus 1 over r and that is all there is to it, there is no point which is purely of period 3. But if we look at the map in the vicinity we can see that this minimum over here is coming downwards towards the diagonal, but it is not quite touching the diagonal. If we now go to the square root of 8 plus 1, a little expansion of this region shows that this curve is now tangential to the diagonal the slope here is exactly equal to 1. And as we go above square root of 8 plus 1 the map now clearly intersects the diagonal twice. So, we suddenly have two fixed points being created. Now this is the third composition of the map, so what is happening over here if I may just go back alright. So, there are two points of intersection here, there are two points of intersection there and there are two points of intersection here. So, at one by as the parameter is varied there is suddenly the creation of six new points of period 3 and these form 2 different orbits of period 3. Now one of the interesting things about this particular bifurcation is that when these points are created as you can see one of them ok; so, at this point the slope of the map is more than 1. So, this point is going to be unstable at this point the slope of the map is less than 1. So, the map is so this particular fixed point is stable. (Refer Slide Time: 25:44)



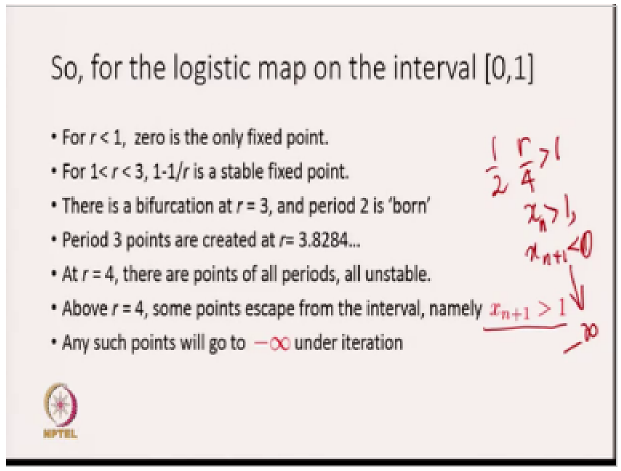
And this is a characteristic of the so called tangent or Saddle node bifurcation, that below the bifurcation there are no fixed points. As we saw over here, there was no fixed point over here a point of tangency and then two fixed points. So, below the bifurcation there are no fixed points, at the bifurcation there is a single fixed point which is non hyperbolic and above the bifurcation point there are a pair of fixed points one of which is stable and one of which is unstable. In the case of the period 3 map, you will have 1 period 3 orbit which is stable and 1 period 3 orbit which is unstable. And we will come to that to discussing the period 3 period 3 orbit when we look at the logistic map in some more detail. For today what I would like to emphasize is at the tangent bifurcation or the saddle node bifurcation, you always have fixed points created in pairs one stable one unstable. (Refer Slide Time: 27:06)



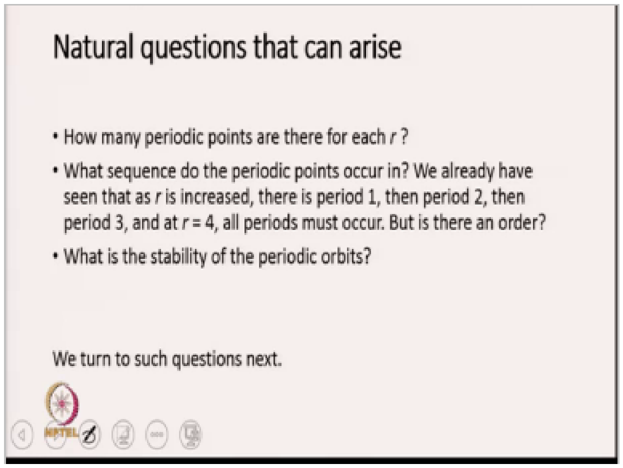
Notice that the map this quadratic map when the parameter takes the value 4 so, that is

$$x_{n+1} = 4x_n(1 - x_n)$$

. Actually has points of all periods and the simplest way to see this is graphically. If I draw $4x(1 - x)$ this is a inverted parabola which goes from 0 to 1 to 0, so it completely it is a it goes over the entire interval from 0 to 1. Now if the map itself goes from 0 to 1, f^2 is going to go from 0 to 1 twice. So, there will be clearly two new intersections in addition to this one which already was there for period 1. If I go to f^3 again because this map is going up and down as many times as I iterate it, I am going to find that there will be new points which are created just by the k th iteration of this map. Now the slope of this map is almost everywhere bigger than one in modulus. So, it turns out and you can prove it by other means that all these periodic orbits that this map possesses of all periods all turn out to be unstable. (Refer Slide Time: 28:30)



So, now for the logistic map the behavior on the interval 0 to 1 can be summarized as follows. For r less than 1 0 is the only fixed point, when r lies between 1 and 3, we have $1 - \frac{1}{r}$ as the stable fixed point and 0 is unstable. There is the bifurcation at r is equal to 3 and we find that period 1 disappears and period 2 is born. The period 3 points are created out of nothing at the point r is equal to 3.8284 etcetera, that is to say the square root of 8 plus 1, at r is equal to 4 there are points of all periods and all of them are unstable. Now if r is bigger than 4, then the midpoint at the point half will become some number r by 4 which is bigger than 1 and if x n is bigger than 1, then you can easily show that xn plus 1 is less than 1, sorry is less than 0 and this will pretty soon go to minus infinity under iteration. So, the logistic the behavior in this logistic map in the quadratic family above r is equal to 4, there will always be points that escape from the interval. And once x n is bigger than 1 or x n plus 1 is bigger than 1, then eventually these points will iterate to minus infinity. (Refer Slide Time: 30:29)



Natural questions that can arise when you are faced with these kinds of you know these are statements of this is what happens in this logistic map, someone could ask that I am sorry prompting you to ask the following questions. Given any particular value of r I have talked as if there is only one possible periodic orbit is that true, can there be more than one periodic point at a given value of r. We also saw that we you know for small r we had period 1, then we had period 2, then at some point we had period 3 and these were all you know born stable. But then we also note I also indicated that at r is equal to 4 all periods must occur, what is the order in which these appear is there some sequence 1 2 3 4 etcetera etcetera or is it something more interesting and more complicated and we will see that the answer to that is yes. There is a very natural way in which these periodic orbits occur. And equally important what is the stability of these periodic orbits, are they you know we have say that it r is equal to 4 all periods must occur. But all of them are unstable, are they stable periodic orbits of period k. In the next lectures of this course will turn to such questions.