Introductory Nonlinear Dynamics
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Lecture 05
Maps and Flows
Simple Examples of Dynamical Systems.

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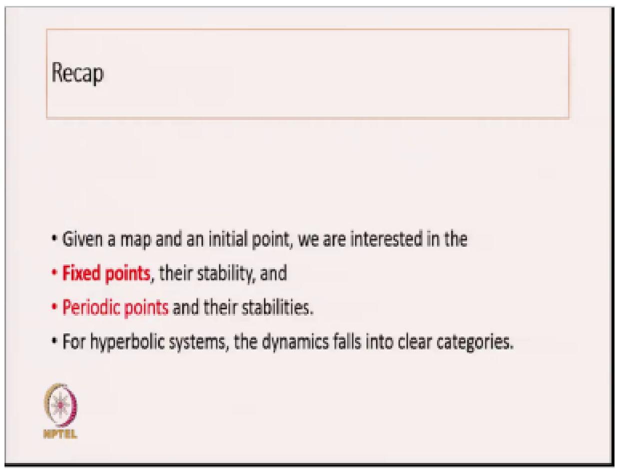
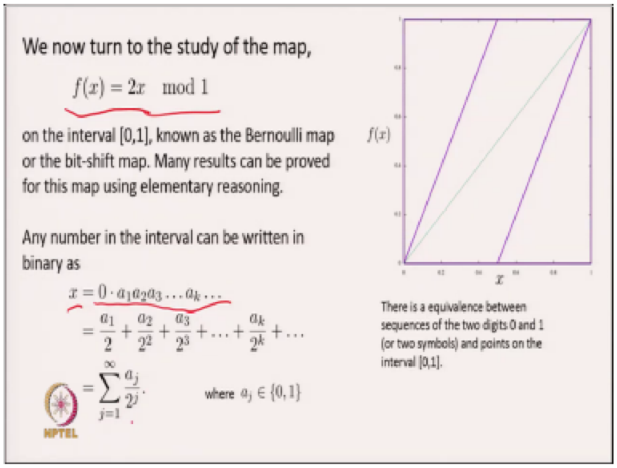


Hello. Continuing our study this week of Maps and Flows, we would look at some Simple Dynamical Systems. Recap what we did in the last lectures: given a map and an initial point, we are interested in examining fixed points, their stability; we would like to know what are the periodic points and find out what their stabilities are. And, if the system is hyperbolic, namely if the derivative of the map is less than one in magnitude at a fixed point or at a periodic point then the dynamics falls into clear categories of either being attractors or repellers. (Refer Slide Time: 01:13)



Let us look at a particular map to get some of these ideas more concrete. The simplest map that one can look at is where the function f of x that is where the function f of x. Let us look at one of the simplest maps which is known as the Bernoulli map or the bit-shift map and reasons for this name would become obvious. Given a number x multiplied by 2 and do the mod operations. So, this is a map $f(x) = 2xmod1$. actually on the interval [0,1] or you can think of it as a map on the circle of identifying the point 0 and 1 and this map has plays a very important role in getting ideas on of non-linear dynamics in place ok. So, f goes to f(x), f takes x from x to 2x mod 1 and many of the many results in this area can be proved for this map using very elementary reasoning ok. When you graph the map it looks something like this from 0 to half it doubles over here, and then once you cross half it starts again. So, this map is a discontinuous map with the discontinuity at the point x is equal to one half because, 0 and 1 can be identified. There is some another way of looking at this map is to see it is action on a point x and a point x I can write now I would like to write it in not in decimal, but in binary and I write it as a

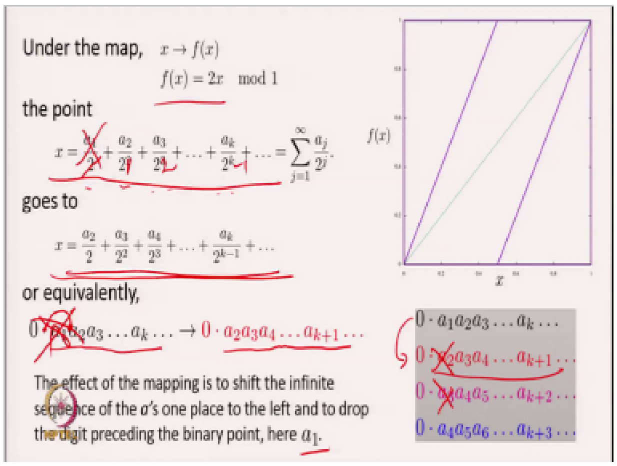
$$x = 0.a_1a_2a_3...a_k...$$

$$= \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + ... + \frac{a_k}{2^k} + ...$$

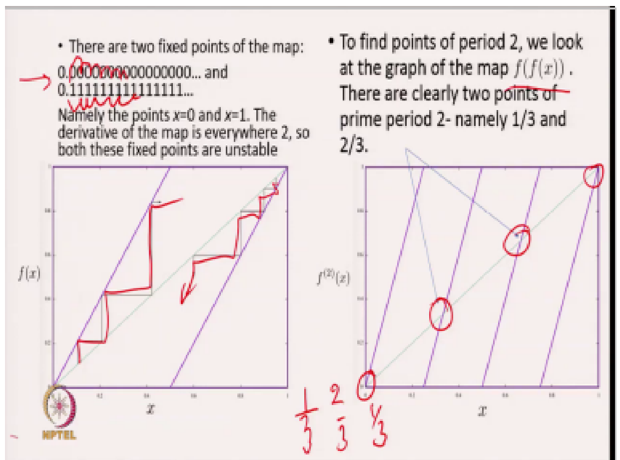
$$= \Sigma_{j=1}^{\infty} \frac{a_j}{2^j}$$

. And, any number that lies between 0 and 1 can be written as so, with the a js because we are using binary representation the a js can be either

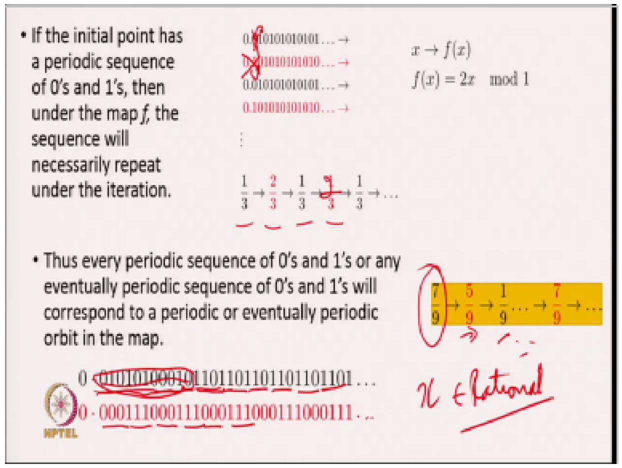
0 or 1. So, any sequence of the two digits 0 and 1 have a correspondence unique correspondence between this sequence of two digits and any point on the interval. (Refer Slide Time: 04:03)



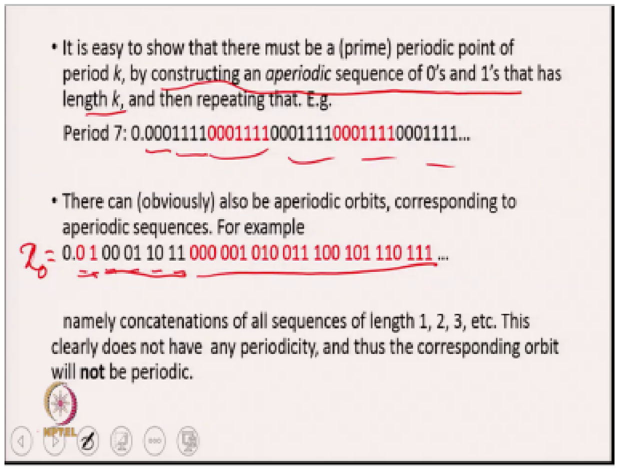
The action of the map $x \to f(x)$ where $f(x) = 2xmod1$ is to take the point x which was as we have just seen a 1 divided by 2 plus a 2 divided by 2 squared etcetera etcetera multiplied by 2 and remove any part which is bigger than 1. So, clearly what happens is that if I multiply x by 2, I remove that I remove a power over here the 3 becomes 2 and this becomes k-1 and so on as I have written over here. And, if a 1 is 0 we do not have to consider it and if a 1 is bigger than 1 if a 1 is 1 then the number is bigger than 1 and in the modulo 1 operation this will get dropped. So, the point x will go to this point which is a 2 by 2 a 3 by 2 squared etcetera etcetera. Equivalently we can say that x which started out as a 1 a 2 a 3 all the way through to infinity will now get transformed to a 2 a 3 a 4 etcetera; just removing the first digit. I could have taken this binary point over here and just shifted it here and removed the first symbol and that would have done the same effect. So, the effect of the mapping is to shift this infinite sequence of as one place to the left should that be to the right; sorry I shift this infinite sequence of as one place to the left and I dropped the digit preceding this binary point and in this first example it was a 1. Likewise what so, what I have just discussed is this first transformation at the next instant I take this entire sequence shift it one place to the left and drop a 2 again over here I shift it and drop a 3 and so on and so forth. This is why this particular map is called the bit-shift operator because the symbols a 1 a 2 a 3 etcetera either 0 or 1 they can only take two values and the effect of the map is to basically shift by one place either to the to the left and to drop the drop the leading symbol. (Refer Slide Time: 07:04)



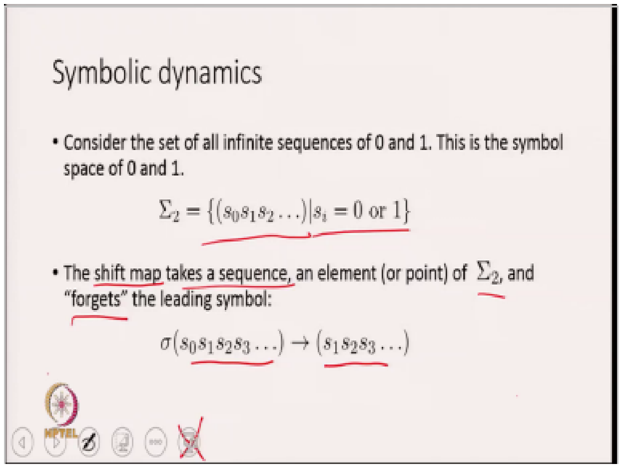
There are two fixed points of this map and what are the fixed points they are either 0 or 1 because two times 0 is 0 and 2 times 1 is 2 which is 1 which is also which is a fixed point and you can see that very simply because if I have this symbolic sequence. Or, if I have this sequence binary representation 000 each time I shift one point if I just keep shifting by one point I will just get the same thing repeated. So, it is a fixed point. And, similarly for 1 if I just keep shifting I do not change the sequence at all. The derivative of this map everywhere is 2 as you can see this is just the derivative is just 2 and so, since this is bigger than 1 in modulus these fixed points are both unstable. Starting at a point close by I can see that I just move outwards if I am starting over here I just moved down and outwards. So, both these are unstable fixed points. To find points of period 2 we look at the graph of the map f of f of x and f of f of x is the is the map I have shown over here in purple 0 to 0.250 to a quartered gets mapped into 0 to 1, quartered to half again gets mapped 0 to 1 and so on and so forth ok. So, you can see that this map is the twice composed map you have the fixed points at 0 and at 1 and now you have two new fixed points which are created over here these are prime period 2 points and it is very easy to figure out that there must be the points one third because one third doubled becomes two thirds and two thirds double becomes 4 by 3 which modulo 1 becomes 1 by 3. So, these two are just the two fixed points of period 2. (Refer Slide Time: 09:21)



Note that if the initial point has a periodic sequence of 0s and 1s, then under the map f the sequence will necessarily repeat under iteration. So, you take this first example here which is represented in the binary representation of the point one third it is 0.01010101 etcetera all the way to infinity. When I apply the map to it goes to the point two third which is the same as having shifted this point over here and removed the initial 0 so, that gives me the 0.101010 all the way out to infinity. Applying the map again I now shift from here shifted to the next point remove the leading one and I get again 0101 etcetera all the way to infinity and this process will repeat. This is exactly equivalent to writing in you know in decimal notation 1 by 3 goes to 2 by 3 goes to 1 by 3 goes to 2 by 3 and so on. So, every periodic sequence of 0s and 1s or any eventually periodic sequence of 0s and 1s will correspond to a periodic or an eventually periodic orbit in this map. So, here for example, have started out with some sequence which just terminates in the following sequence 1101111 sorry, 110110110110 etcetera etcetera. So, this is going to eventually be a periodic orbit even though initially it was not ok. So, initially it was something and then eventually it becomes periodic. Here I have started with another sequence which is 000111000111000111 and so on, all the way to infinity and this is another periodic orbit. Any rational number like 7 by 9 in this example over here 7 by 9 will go to 5 by 9 which will go to 10 by 9 modulo 1 is 1 by 9 and so on and so forth and it will eventually start repeating. So, any rational any point x which is rational will either be a periodic part of a periodic orbit or it will be something which will eventually become periodic. (Refer Slide Time: 12:34)



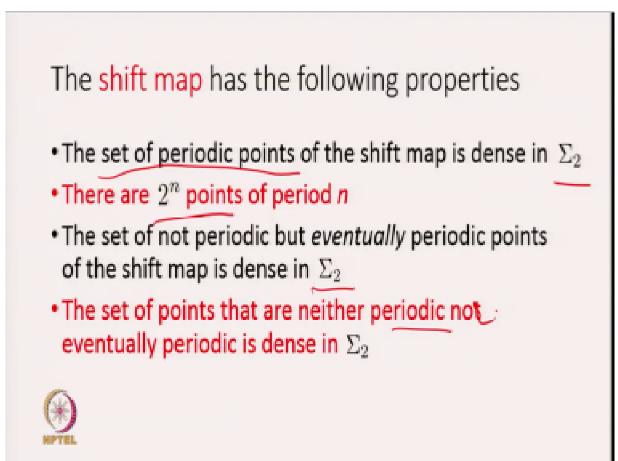
It is also easy to show that there must be prime periodic orbits of period k by simply constructing one. You start by taking an a periodic sequence of 0s and 1s which has got length k, and then simply repeat it ad infinitum. So, for example, to get a period 7 point I just have to start with three 0s four 1s this is itself a periodic and then just simply repeat it. If I do that now this is an orbit which has prime period 7 more than that. They can also be a periodic orbit strictly a periodic orbits which correspond to a periodic sequences. For example, if I start with the following number which is 0.0100011011 and what you can see over here is that I have taken concatenations of all sub-sequences of length one of length 2 of length 3 and so on and so forth. If I now just keep on extending this, this sequence will have no periodicity and therefore, if I start with some x naught which is this particular point this point will also not be periodic. So, for this rather simple map, this dyadic map or the Bernoulli map one can show that one must have periodic orbits, eventually periodic orbits, a periodic orbits and so on. This has been very nicely systematized through what is known as symbolic dynamics and there is as we have seen over here the binary representation of any point in the interval 0 to 1 uses just these two symbols 0 and 1 and the action of the map is merely to shift the this binary point from 1 location 2 to a single iteration is to move it one place to the right or to move the sequence one place to the left. This can be systematized in what is known as symbolic dynamics. (Refer Slide Time: 15:05)



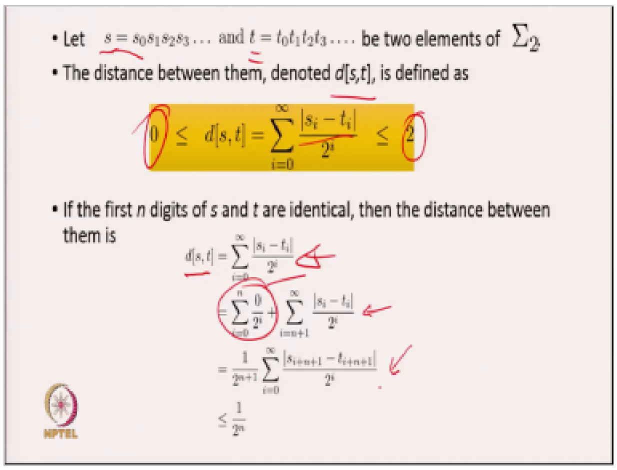
And, in order to do that let us consider this space sigma 2 of all infinite sequences of 0 and 1. This is also known as the symbol space of 0 and 1. So,

$$\Sigma_2 = (s_0 s_1 s_2 ...)|s_i = 0 or 1$$

. Along with this space Σ_2 let us consider the shift map. The shift map takes a sequence, an element or a point of Σ_2 and forgets the leading symbol. So, if I apply the shift map on to the sequence s 0 s 1 s 2 s 3 etcetera that just drops s naught and gives me s 1 s 2 s 3 etcetera ok. So, this is the very simple action of the shift map on the symbol sequence. (Refer Slide Time: 16:06)



Now, the shift map has the following properties. The set of periodic points of the shift map is dense in sigma 2. There are 2 to the n points which have got period n. The set of not periodic, but eventually periodic points of the shift map is dense in Σ_2 and the set of points that are neither periodic not event or nor eventually periodic that itself is dense in sigma 2. A lot of what happens in Σ_2 is mirrored in what happens in a Bernoulli map, but importantly over here we can introduce the idea of a metric on Σ_2 by defining the distance between two points. (Refer Slide Time: 17:00)



This is easier on Σ_2 then to do it on the interval in some ways. So, let us just consider how do we characterize the distance between a point s and a point t which are 2 elements of Σ_2. So, $s = s_0 s_1 s_3 ...$ and $t = t_0 t_1 t_2 ...$. And, the distance between them which we will denote by $d[s,t]$ is given by this following expression.

$$0 \leq d[s,t] = \Sigma_{i=0}^{\infty} \frac{\mid s_i - t_i \mid}{2^i} \leq 2$$

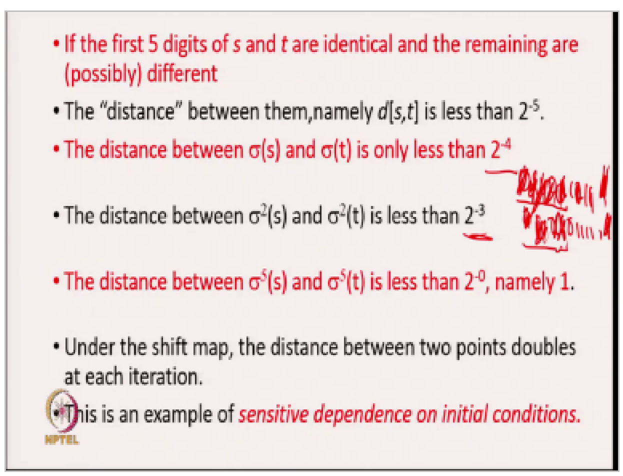
If this is the sum from 0 to infinity of s i minus t i divided by 2 to the power i modulus of s i s i minus t i. So, this is a Hamming distance and since the maximum value that s i minus t i can take is 1. This has to be less than the sum of 1 over 2 to the i summed over infinity and that geometric series has the value 2 and the least value that s i and t i can have s i minus t i can have is 0 that is if the point s and t are identical then it is then it would take the value of 0. So, and you can easily show and you will show in the homeworks and the tutorials that d is a distance ok, it satisfies the triangle inequality and, one can it has all the properties of a distance. What is useful about this idea of a distance is to note that since the distance is defined in this way.

Supposing s and t are identical for n places. So, if the first ns symbols of s and of t are identical, then this value over here s i minus t i will be 0. So, this sum this sum that I have which tells me what this distance is can be split into two; one part which is definitely 0 that is the terms from i equals 0 all the way up till n and the terms which are not equal which are not or possibly not equal to 0 and that is this particular term over here which is the sum from n plus 1 to infinity.

$$0d[s,t] = \Sigma_{i=0}^{\infty} \frac{\mid s_i - t_i \mid}{2^i}$$

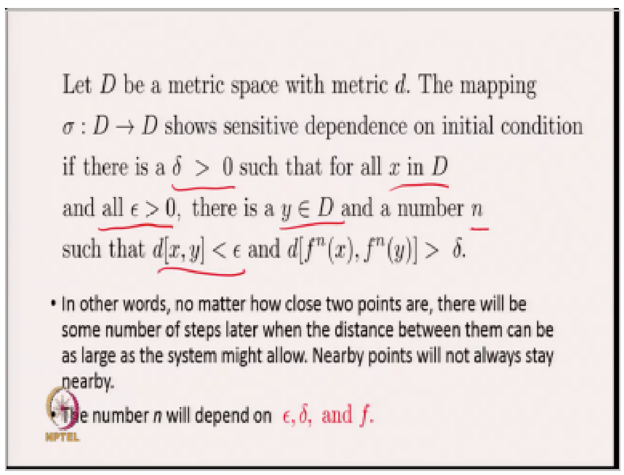
$$\leq \frac{1}{2^n}$$

Now, I can rewrite this taking out 2 to the power n minus 1 out of this in this following way and this shows that this entire distance must be less than 1 over 2 to the n. So, if things are matching for n places in the symbolic sequence then the distance between these two symbols is less than 1 over 2 to the n. This gives you an idea about distances in this space sigma 2 to make this more concrete. (Refer Slide Time: 21:11)

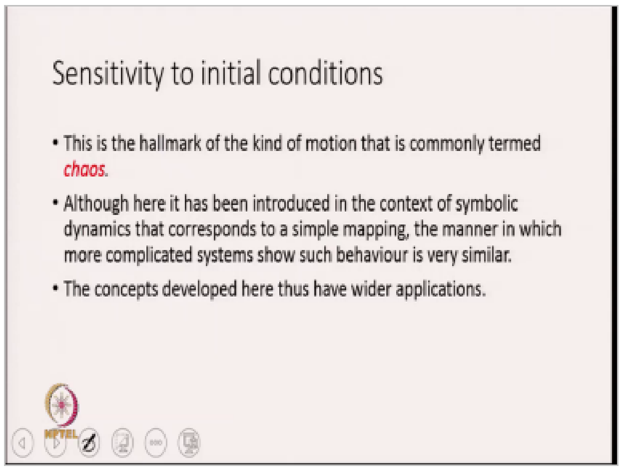


If the first 5 digits of s and t are identical and the remaining are possibly different, so, you know say s and t are s is 011011101111 etcetera something and this is s and t I write as 01101, oops and then it could be 011111 etcetera it could be any old thing. So, they are matching in the first five places they are identical in the first five places they are identical. Then, the distance between them has to be less than 2 to the minus 5 or 1 by 32. Now, if

I operate by sigma I have to drop the first digit and so, these two new sequences are only matching for four places and so, the distance is only less than 1 over 2 to the 4. If I apply sigma again then if I apply sigma again I drop one more place and now these two symbol sequences are only matching for three places. If that is if that is so, then the distance between them is only 2 to the minus 3 and if I apply the shift operator 5 times I have I wipeout the first 5 digits of both of them the first 5 symbols of both of them. And, now the two sequences do not have any think necessarily in common and so, the distance has got to be only of the order of 2 to the 0 which is 1. So, under the shift map the distance between two points doubles at each iteration and this is a very important idea in this field and it is known as sensitive dependence on initial conditions. (Refer Slide Time: 22:37)



Formally, if D is a metric space with metric d and that is what we have done over here with sigma 2 and the metric defined by small d. The mapping sigma which is in this particular case a shift shows sensitive dependence on initial condition, if there is some delta positive such that for all x in D and all epsilon greater than 0, there is some y and a number n such that the distance between x and y initially starts out by being less than epsilon and the distance between the n-th composed the map applied n times to x and the map applied n times to y that becomes bigger than delta. In other words, no matter how close two points are there will be some number of steps later when the distance between them can be as large as the system might allow. Nearby points will not always stay nearby. How long will one have to wait to see you know this the separation of these two initial points? This number n will depend on epsilon, it will depend on delta and it will depend on this function f; namely, how close do the initial points be and how far away do they get that depends on the map f. (Refer Slide Time: 24:24)



But, this idea of sensitivity to initial condition is extremely important in this field and it is the hallmark of the kind of motion that is commonly termed chaos or chaotic dynamics. Now, we have introduced this in the context of symbolic dynamics that corresponds to a very simple mapping, the manner in which more complicated systems show such behavior is actually very similar. Namely, at each step the distance between two nearby points increases by a factor which is larger than 1 and here in this example it was 2, but it just has to be bigger than 1 and so, two nearby points will eventually become further and further apart. So, the concepts developed here have wider applications. In the next lecture, we will take up some of these other applications in similar, but slightly different mappings.