Introductory Nonlinear Dynamics
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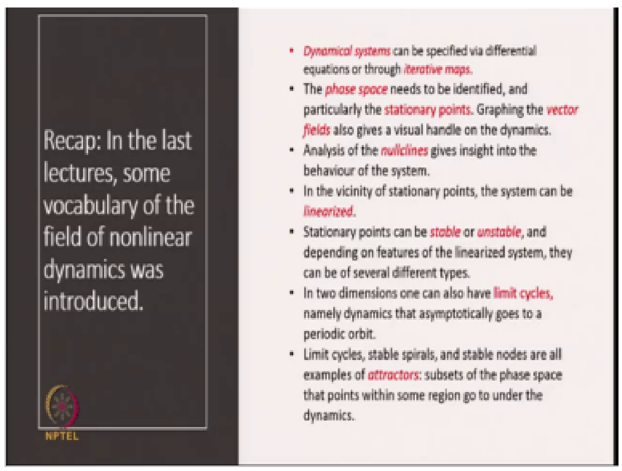
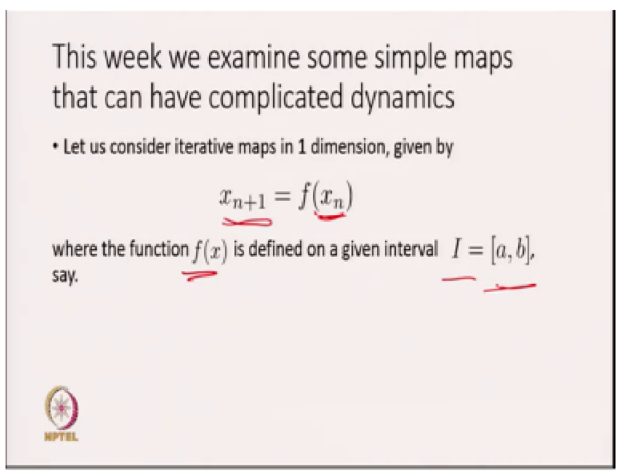
Lecture 04
Maps and Flows
Simple Examples of Dynamical Systems

This week we look at Maps and Flows in Simple Examples of Dynamical Systems. Recall that in the last lectures, some of the vocabulary of the field of Nonlinear Dynamics was introduced. (Refer Slide Time: 00:32)

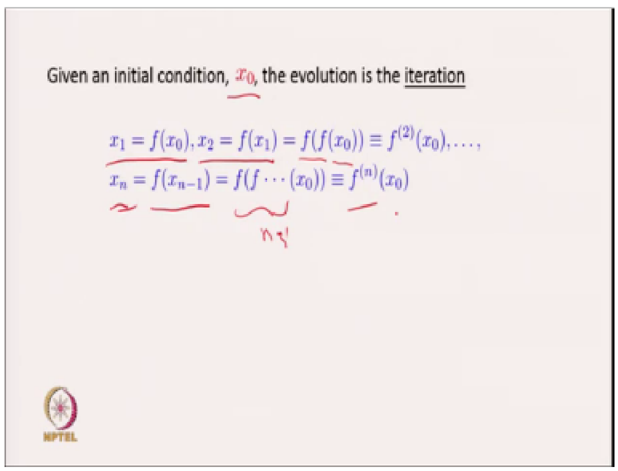


In particular we define what is the dynamical system and showed that this can be specified either via coupled differential equations, or through iterative maps. Given any dynamical system, it is important to identify the phase space and characterize how the motion goes around in the phase space and in order to do this. First, we would like to know where are the stationary points, namely those points in phase space where nothing happens. A second way of looking at the dynamics qualitatively is to graph the vector fields, namely to see what is the velocity at all points in the phase space and this gives us a visual handle on how a point might move through the phase space. Looking at lines where either the velocity in one direction or the other is vanishing, this is what is called a nullcline. This also gives us great insight into the behavior of the system and this will become apparent as we go on, but the important idea was to use was to introduce the idea of nullclines. And, the intersection of the nullclines gives you the stationary points. In the vicinity of which the system can be linearized, namely instead of the full non-linear system, we can replace it by a linear approximation. Analysis of these of the linearized equations of motion helped us to decide whether the stationary points are stable or unstable. Because, they depend on essentially on the Eigenvalues of the matrix of that the corresponds to the linearized system at the fixed point. So, once you have linearized the system and looked at the stationary points, you can figure out that they are of various kinds. In one-dimension they are either stable or unstable, in two-dimensions there are 6 different possibilities, three of which are stable and 3 of which are unstable. In more dimensions there will be many many more possibilities, essentially going up along with the size of the of the phase space. In two-dimensions though we also looked at some very interesting behavior, namely that of limit cycles, the dynamics does not start out or wherever it starts out will asymptotically go to a periodic orbit and, this follows from a very important theorem by Poincare and Bendixon. So, looking at one in two dimensions, we saw that there were certain kinds of asymptotic dynamics. Limit cycles, stable spirals, and stable nodes, were all examples of what are called attractors. Namely, subsets of the phase space that points within some region will go to under the dynamics. And, this point this feature of attraction is such that once the dynamics comes on to an attractor, it stays there which is why the named attractor. And, that part of phase space from where all initial points will go to an attractor is called the basin of the attractor. (Refer Slide Time: 04:20)



So, with this background let us start looking at some simple maps that can have complicated dynamics. There is a very lovely paper by Robert may in

1976 with the title from where that line is drawn. And, in order to do that it is simpler to look at iterative maps in one dimension. And, they are given essentially by the following kind of equation, namely that excuse me, namely that given any point x_n; the recipe for going from x_n to x_{n+1} is to basically operate by some function. This function f of x is defined in some domain in one-dimension this could be maybe it is defined on an interval, and let us consider this simplest case, where it is defined on the interval from a to b and it is a closed interval in just in this example, different examples can be drawn. Now, what are we going to be interested in? (Refer Slide Time: 05:39)

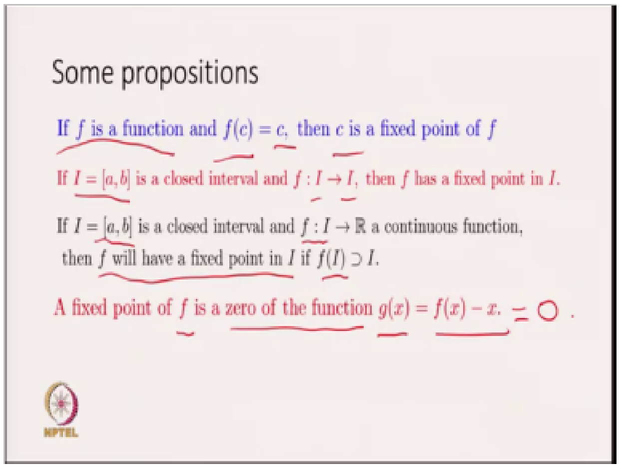


By and large we are interested in the evolution starting from some initial point; x_0 we would like to know what happens to x_0 under the map well as we have seen given x_0 over here,

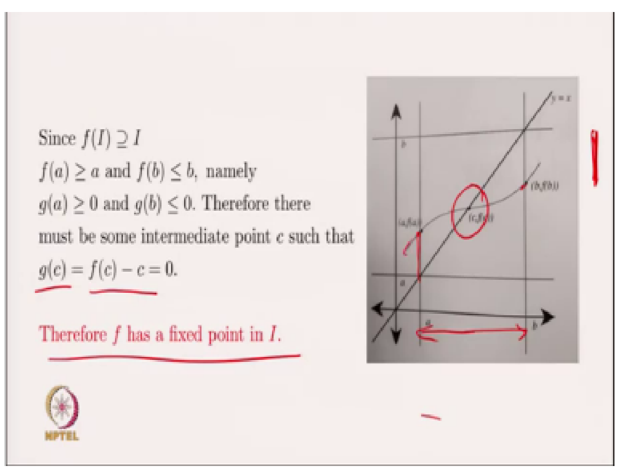
$$x_1 = f(x_0), x_2 = f(x_1) = f(f(x_0)) = f^2(x_0), ...$$

$$x_n = f(x_{n-1}) = f(f...(x_0)) = f^(n)(x_0)$$

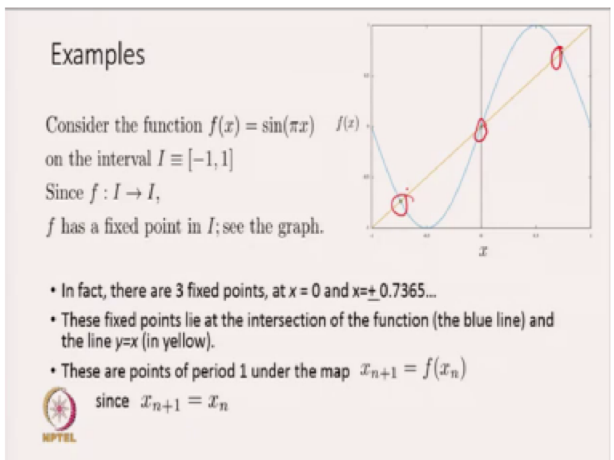
. So, x_n is given as $f^(n)(x_0)$. So, once you are given a particular initial condition one can just plug it in into the set of equations and go on and find out where is x n, after n steps. Now, there would be some let us just introduce some propositions, which will help us to now fix some ideas. A fixed point is such that the iterative the fixed point is itself, namely if f is a function and $f(c) = c$, then c is a fixed point of f. (Refer Slide Time: 07:05)



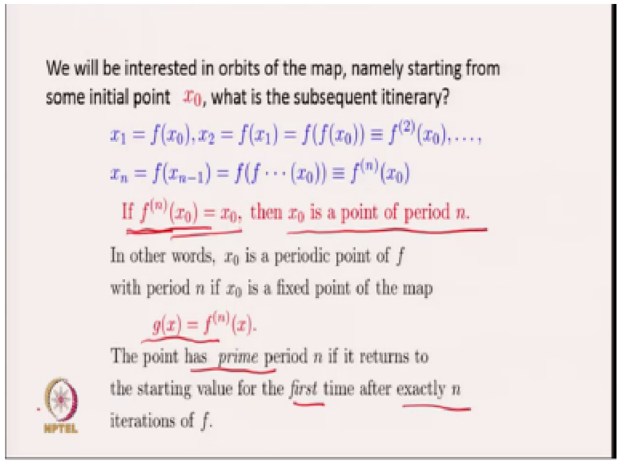
So, it is a pretty obvious thing, but I just like to point it out, because we are going to use these ideas as we go along. When can we have a fixed point, if, your function is defined on an interval, it is taking points from the interval and mapping them somewhere. So, if the interval I is this closed interval a to b and if f maps I to I, then f must have a fixed point in I. It could have many fixed points in I, but it must have at least 1 fixed point in I. Likewise, if f if I is the same closed interval and f is not mapping to the end to itself, but is mapping to R, then and R is; if f is a continuous function, then f will have a fixed point in I, if f(I) is a superset of I. So, if f(I) contains I, then there is going to be a fixed point, a fixed point of f is a zero of the function $g(x) = f(x) - x$. So, every time you would like to find a fixed point of f we write this equation for g(x) and set it is equal to 0, and any root of that equation will give you a fixed point of f. (Refer Slide Time: 09:06)



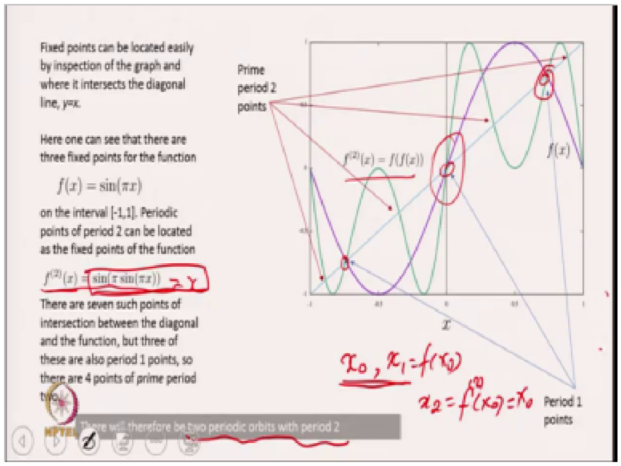
Now, since f(I) is superset of I, let us understand why there must be a fixed point. Now, f is mapping a, b it is mapping a to b into some region which is contained inside. So, here is f(a) and here is f(b); now, $f(a) >= a$ and $f(b) =< b$. And, therefore, $g(a) >= 0$ and $g(b) < 0$. Therefore, somewhere in between g must be equal to 0, namely there must be some point g(c) such that $g(c) = f(c) - c = 0$ and, therefore, f has a fixed point in I namely the point c. Notice, that we have also done something you know shown this in one other way, we have drawn the function f which is here, and we have also drawn the diagonal line y is equal to x. And, the intersection of this diagonal line with this function tells us, where is the fixed point. We are going to use this technique frequently. (Refer Slide Time: 10:42)



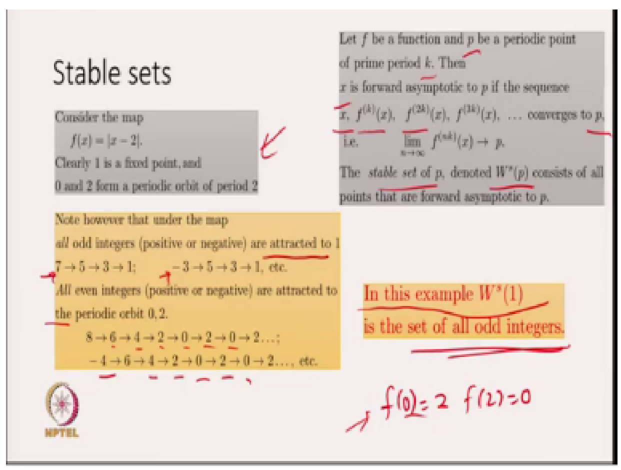
So, here is an example consider the function $f(x) = sin(\pi x)$ and the interval $I = [-1, 1]$, f is mapping I to I and therefore, we know that we have a fixed point. And, you can visually see that there we do not have just 1 fix point, but we have 3. And, these three fixed points are given here at 0 and at these 2 values which are plus or minus 0.7365 something, something alright. Now, these fixed points lie at the intersection of this function namely the blue line, the blue curve, and this diagonal line y is equal to x which is drawn here in yellow. And, as you can see these are points of period 1, because at this point xn plus 1 will be exactly equal to x n, you can see it trivially for x n is equal to 0, then f of x n is also equal to 0. (Refer Slide Time: 11:52)



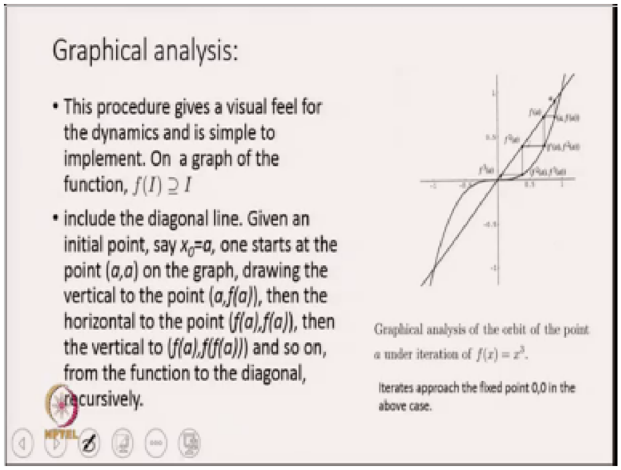
In general though given some arbitrary function, we are going to be looking at orbits of the map. Namely starting from some initial point x naught, what is the subsequent itinerary x 1, x 2, x 3, x 4 etcetera etcetera alright. Now, if your initial condition comes back to the starting point after n iterations, then x naught is said to be a point of period n; namely, if f to the n of x naught is equal to x naught, x naught is a periodic point of period n. In other words x naught is a fixed point of the function g of x is equal to f to the n of x. So, a periodic orbit of f is a fixed point of the map g of x, which is the nth composed map. The point is said to have a prime period n, if it returns to the starting value for the first time after exactly n iterations of f. This is to cover yourself from you know, because any point which is a fixed point, will also be back at the same place after 2 steps, after 3 steps, and so on. So, we want to make a distinction between points that return after every step or points that return only after n steps. And, clearly we are more interested in those points that will return, after exactly n steps rather than any other condition. (Refer Slide Time: 13:49)



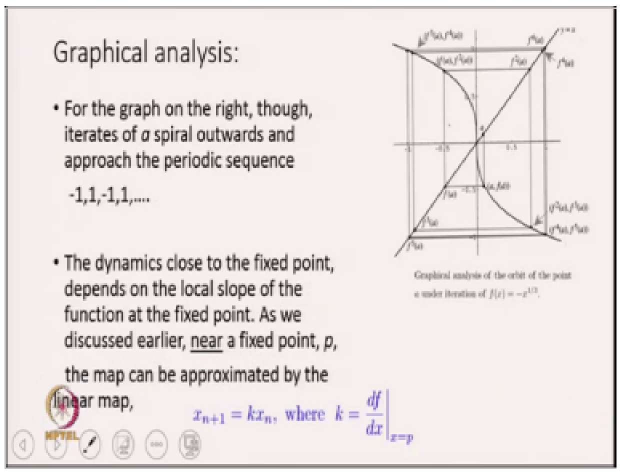
So, now let us look at the same map that we had observed, namely $f(x) = sin(\pi x)$, and ask what are the fixed points we already know that there are 3 fixed points of period 1, ok; so, a fixed point is a point of period 1, because it comes back to itself after 1 step. To find higher order periodic orbits, we need to look at as we have seen over here, they are fixed points of the nth composed map. So, if I am interested in looking at points which have got period 2, I have to look at the map $f^2(x) = f(f(x)$and in this particular example, it is $sin(sin(\pi x))$. Now, this is a function which looks like this green curve that you see over here. The purple curve is your original 1, which is sin pi x and the green curve is f 2, which is f of f of x. Now, as you can see on this particular graph just a second let me get the laser pointer out. As you can see over here, this is a fixed point of period 1, it is also a fixed point of period 2. This is a point of period 1 and also; obviously, a period 2 as is this last point over here both points of period 1 as period 2. What is new over here are these four points 1 2 3 and 4, which are intersections of this curve $f^2(x)$ namely $f(f(x))$ with the diagonal line. So, these are fixed points of the twice composed map, on and they are the intersections of this map with the diagonal line f(x), I am sorry with the diagonal line $y = x$. Now, there are 7 such points of intersection of this particular curve excuse me, there are 7 such points of intersection between the green curve that is the twice composed map with the diagonal, 3 of which are period 1. So, we discount these, but there are 4 new points which are of period 2. And, these points have got prime period 2. Now, any period 2 orbit must have 2 points in it is composition; namely if I have got x naught and then I have got x 1 which is f of x naught and x 2 which is f 2 of x naught is equal to x naught. So, I need two elements on a period 2 orbit and, therefore, these 4 points which have got prime period to form 2 distinct periodic orbits which have got period 2. So, a combination of graphical analysis and a little simple arithmetic over here tells us, that this map not only has got period 1 points and it is also got period 2 points, and we can see now how to get points of any period that we desire. Either by solving an algebraic equation, solving this equation that is sin pi x sin pi x is equal to x, this particular equation is somewhat difficult to solve. So, you know 1 cannot do it without numerical techniques, but visually and graphically when can easily see that one has the following fixed points. And, one can also identify as to where they are, this one lies below minus 0.73, the next one is between minus 0.73 and 0 and so on and so forth. Now, given these various fixed points, one can see that there either stable or unstable depending on the slope of the function at these fixed points. And, we can do the analysis and we will have to do that algebraically. But one can also identify an important feature of these of these fixed points, and this is to ask what is the stable set of a given fix point. (Refer Slide Time: 19:01)



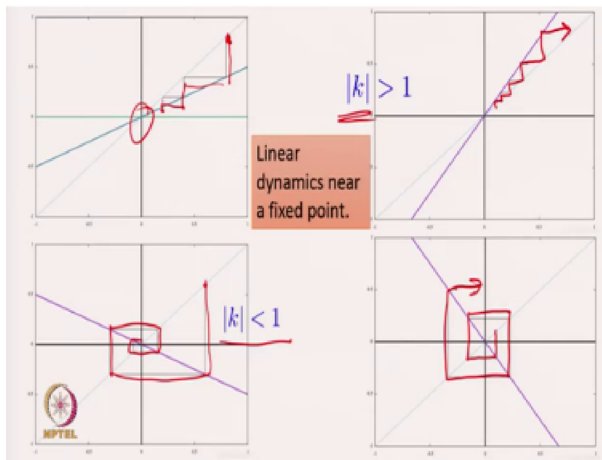
In order to understand what the importance of the stable sets are let us consider a slightly different map a simpler 1, and I will take the case of $f(x) =| x - 2 |$. Now, clearly the point x is equal to 1 is a fixed point associated with any periodic orbit or with a fixed point is the notion of a stable set. Namely, what are all the points that are eventually attracted to this particular periodic point? In order to introduce this particular notion, let us look at a simpler example; the example of the map $f(x) =| x - 2 |$. Now, clearly for this function 1 is a fixed point, because f of 1 is 1 minus 2 is equal to minus 1 modulus of that is 1 so, 1 is a fixed point. You can also see that the 0.0 and 2 these integers, they will form a periodic point a periodic orbit of period 2, because f of 0 is equal to 2 and f of 2 is equal to 0, therefore, f 2 of 0 is equal to 0 and it is a periodic orbit. Now, just looking at the integers on the line we note that under this map all the odd integers either positive or negative are attracted to 1. We start with 7, from 7, I go to 5, from 5 we go to 3, and from 3 you go to 1, and 1 we know is a fixed point. On the other hand if I start with minus 3 I go to 5, then I go to 3 and then I go to minus y, then I go to 1 which is also the fixed point. So, all and you can see that all the odd integers are going to do this whether they are positive or negative. On the other hand all the even integers positive or negative are attracted to the periodic orbit 0 and 2. If, I start with 8, I go to 6, I go to 4, I go to 2, I go to 0 to 0 and so on and so forth. On the other hand if I start with minus 4, I go to 6 minus 4 minus 2 is minus 6 modulus is 6, which then goes to 4, then goes to 2 to 0 2 to 2 0 and so on and so forth all right. The stable set is the set of all the points that will either go to the fixed point or to the periodic orbit. So, to be mathematically little more formal if f is a function and p is a periodic point of prime period k so, one has got period 1 and 0 and 2 have got prime period 2, then the point x is said to be forward asymptotic to p. If, this sequence x, f k of x f to k of x etcetera, converges to p namely the limit of f to the n k of x goes to p as n goes to infinity. The stable set of p is denoted W s of p and it consists of all the points that are forward asymptotic to p. So, if I start somewhere and I keep iterating and I eventually land up at a particular fixed point, then that point belongs to the stable set of the fixed point. So, in this particular example W s of 1 is the set of all odd integers. The idea of the stable set of a fixed point and later on as we will see the idea of an unstable set also of a fixed point. These are important in the theory of non-linear dynamical systems and we can see it we introduce very simply over here. (Refer Slide Time: 23:45)



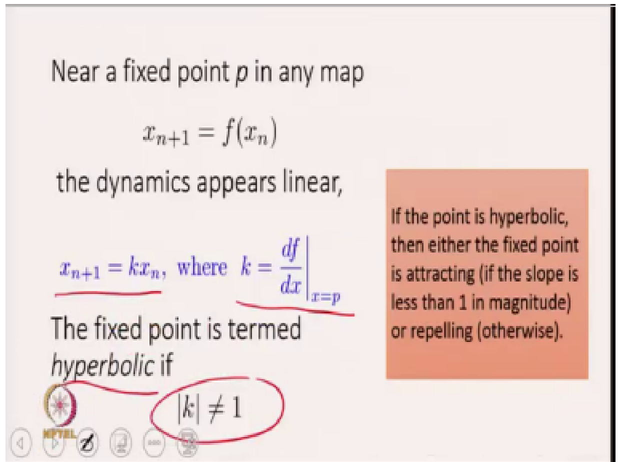
I have already introduced you to the idea, that graphical analysis is a powerful way of analyzing some features of the dynamics. We saw that looking at the intersection of the function f and the diagonal line, made it very easy for us to identify the fixed points. Now, you can do a little more than that. Supposing, I would like to know how does a, what is the orbit of a given in you know initial point. One very simple way to do this is through such you know through graphical analysis in the following way. So, you draw the function f of x and you draw the diagonal line. So, you draw the function f of x and you graph the diagonal line y is equal to x. The intersection of the function and the diagonal line will tell you where the fixed points are. How does the point; as the how does the initial condition a iterate, well you start with the point a on the diagonal and you draw a; you draw a vertical and go up to the point where you intersect the curve. So, you draw the vertical line from a to this is the point a and you draw it to a f of a, and that will tell you what is the next iterate a and then f of a. Where does f of a iterate to find that, we need to iterate from the diagonal point f of a, f of a and see where that goes and that goes to the point f of a, f 2 a. And, clearly now you can continue and find out where f to a and we will iterate by drawing the horizontal line to the diagonal, and then again dropping to the curve and then going to the diagonal etcetera alright. So, this is a very simple rule you start at the diagonal point go to the curve, go to the diagonal, go to the curve, go to the diagonal, go to the curve and so on. Starting close to a fixed point, but not at the fixed point because the fixed point iterates onto itself, but starting close to a fixed point and continuing the iteration can immediately tell you whether that fixed point is stable or unstable. So, this graphical analysis of the orbit of the point a under this iteration, which is x f of x is x cubed, tells us that this 4 point fixed point x is equal to 1 is unstable, because any small displacement from there will just move away move downwards towards this second fixed point which is 0, which is stable. So, the iterates over here will approach the fixed point 0 in this above case . So, you start anywhere you will come down to the origin same it goes if you started somewhere over here. (Refer Slide Time: 27:05)



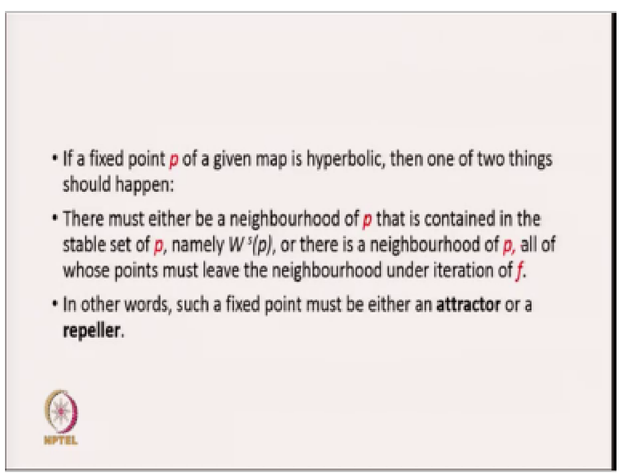
Now, this graphical analysis is actually quite powerful, because it gives you an instant feel for how things are. And, in another example which we have over here I have taken this function f of x is equal to minus x to the one-third both these examples are taken from the book by Holmgren. This function as you can see is this particular curve over here in dark and if I start close to the origin at the point a, I start from the diagonal I go to the curve, I go to the diagonal I go to the curve, I go to the diagonal I go to the curve and you can see that I am spiraling outwards. But then I spiral out to another side to a cycle, which is shown here and sort of is getting darkened by the emphasis. So, f 6 is almost a fixed point over here, but it is now a fixed point that will iterate from the diagonal to the curve, to the diagonal, to the curve, back to itself. So, it goes to a period 2 point. So, and this period to point is the point minus 1, 1, minus 1, 1 this is your periodic orbit of period 2. Close to the fixed point over here, the local dynamics depends on the slope of the function near the fixed point. As we saw in the earlier lectures, the map near a fixed point can be approximated by the linear map xn plus 1 is k times x of n, where k is the derivative of this function evaluated near this fixed point p. It is easy to see now, why points move in or out away from a fixed points depending on the slope. And, you can do that by examining just the simple example over here of linear dynamics near the fixed point. (Refer Slide Time: 29:19)



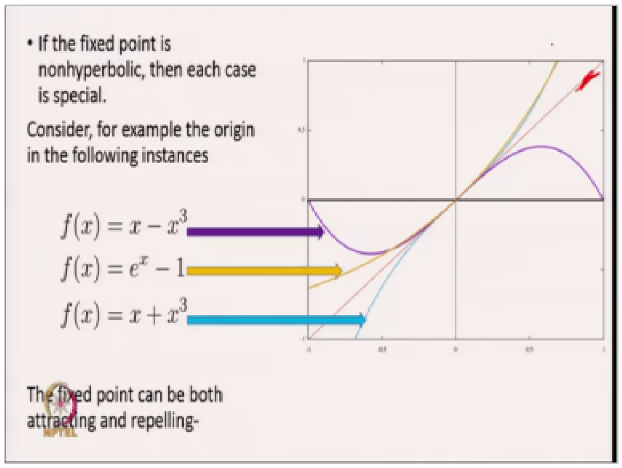
So, over here I am taking the case k less than 1, which we know in modulus which we know from our earlier discussion to correspond to a stable fixed point. If, the slope of the function is less than 1 and positive, then you can see that if I start at this point over here, I start on the diagonal, I go to the curve, I go to the diagonal, I go to the curve go to the diagonal outwards, and as you can see I am just very quickly going to the fixed point of x is equal to 0. If the slope is less than 1 in modulus, but it is a negative slope, then you can again see that if I start at the diagonal I go to the curve, go to the diagonal, go to the curve and I am now spiraling inwards to this fixed point. On the other hand if k is bigger than 1 in modulus, then if I start close to the fixed point, I very rapidly start moving outwards and we will just move far away from the fixed point. On the other hand if I am close to if the slope is negative, but bigger than 1 in modulus, then starting over here it is pretty clear that I have to spiral outwards. So, the slope of the map near the fixed point determines whether or not the dynamics is going to be stable. (Refer Slide Time: 31:09)



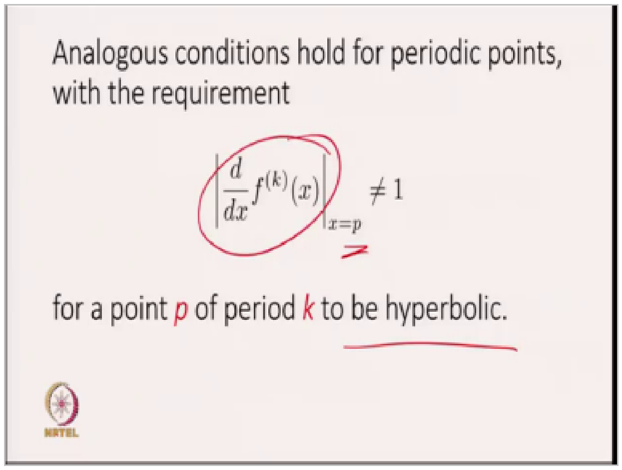
So, near a fixed point p in any map the dynamics appears linear, and this linearized dynamics is represented by $x_{n+1} = kx_n$, where k is just the value of the derivative at the fixed point. If, $k =| \ 1 \ |$ the fixed point is termed hyperbolic. And, this is an important idea to have, because if the fixed point is hyperbolic, it is either attracting or repelling, if the slope is less than 1 in magnitude, it is going to be attracting, as we saw over here, if the slope is less than 1, you either just go in dive in or spiral n. If the slope is bigger than 1 you zoom out or spiral outwards. And, both these behaviors or let us say that so long as k is not equal to 1, the system is said to be hyperbolic. (Refer Slide Time: 32:14)



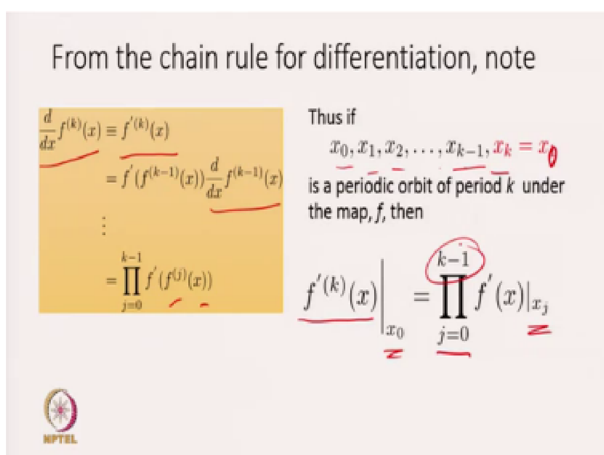
So, the; to just to reiterate, they must either be a neighborhood of this fixed point that is contained in the stable set of p namely W s of p or there is a neighborhood of p all of whose points must leave the neighborhood under iteration of f. In other words such a fixed point must either be attracting or repelling. (Refer Slide Time: 32:37)



In case the system is non-hyperbolic, where the slope is exactly equal to 1, then it turns out that you can have a different types of behavior noting generalizable, but something which you know which has to be looked at from each case. So, here are 3 examples where the slope at the fixed point the fixed point is the origin and, it is exactly equal to 1 in all these 3 cases. If, I take the first example which is this purple curve over here. The purple curve and you can see that this is the diagonal line that I have also drawn in brown I think ok. So, the point is attracting from both sides. If, I take this function $e^x - 1$, which is the yellow curve then from one side the slope is less than 1. So, it is attracting and from the other side the slope is bigger than 1 so, it is repelling. And, in the third example over here I am sorry it is sort of it is repelling on both the sides and then the third example over here, if the slope here is less than 1. So, it is attracting and over here it is bigger than 1 so, it is repelling. So, again you have this rather complicated behavior, which is neither attracting or repelling and this is typical of what happens in the hyperbolic case alright. (Refer Slide Time: 34:05)



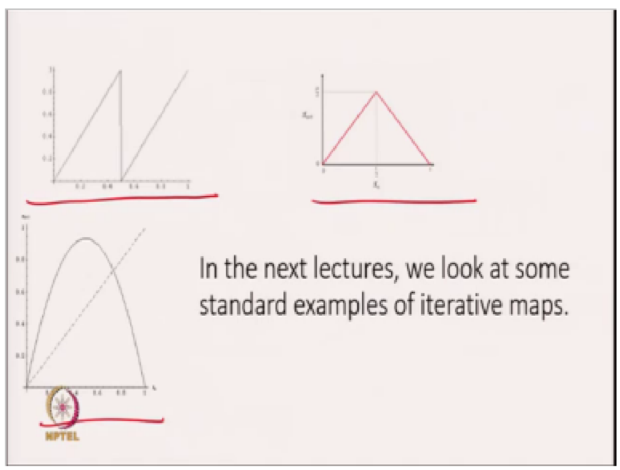
Now, analogous conditions also hold for periodic points with the requirement, that the derivative of | $\frac{d}{dx}f^{(k)}(x)$ | and this fixed point or in this periodic point, that should not be equal to 1, for a point p of period k to be called hyperbolic. For a fixed point p of period k to be hyperbolic, there is an analogous condition that will hold. Namely, that the derivative of the k th composed map, evaluated at the periodic point that derivative should be less than 1 or should be different from 1 in absolute magnitude. Now, how does one take the derivative of the k th composed map, that is done very simply using the chain rule for differentiation. (Refer Slide Time: 34:55)



Note for example, that

$$\frac{d}{dx}f^{(k)}(x)$$

from the chain rule and I am the notation over here is I am putting a prime on the f prime k of x, from the chain rule this is just equal to f prime of f k minus 1 of x times the derivative of f k minus 1 times of x. Now, this of course, is the same as this 1 with k going to k minus 1. So, recursively if we come down, we know that this is the product of j going from 0 to k minus 1 of f prime of f j of x. Thus if you have x 0 x 1 x 2 etcetera all the way up till x k minus 1 and x k is equal to x naught as a periodic point of period k. Then, $f'^{(k)}(x)$ evaluated at x_0 is just the product of $f'(x)$ each time evaluated at x_j, j going from 1 to k-1. So, evaluating the derivative of the k th composed function is multiplying the derivative of the function at each of the k points along the trajectory. We are going to use this frequently in other applications, but and we will see it is utility in when we start discussing different examples. (Refer Slide Time: 36:48)



In the next lecture we will look at the Bernoulli map and subsequently we look at the tent map or the logistic map, these are very standard examples of iterative maps, which have been studied extensively since the 1970s and we will turn to this in the next lecture of the series.