Introductory Nonlinear Dynamics
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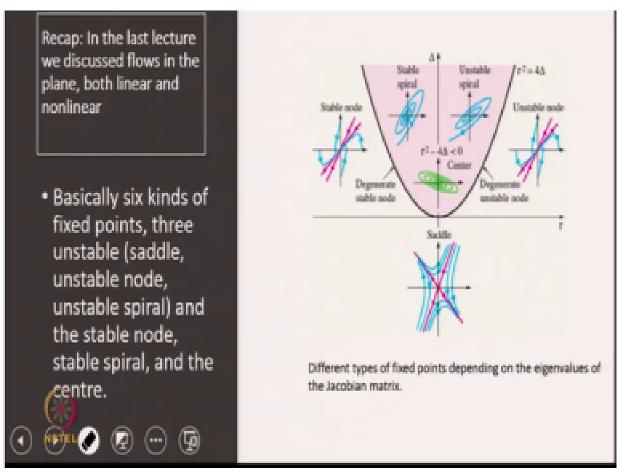
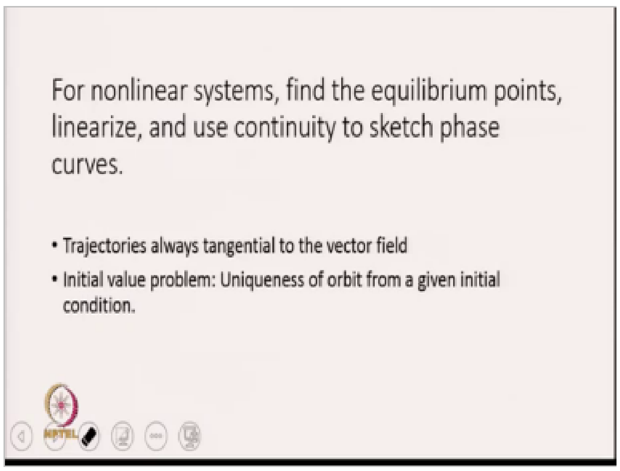
Lecture - 3
Introductory, Stability, Phase space and Invariant sets

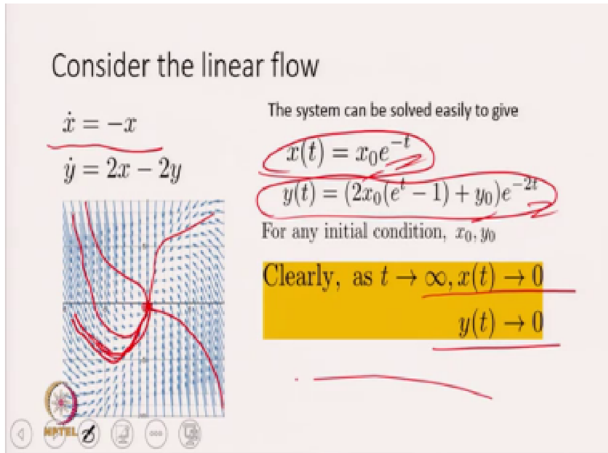
Hello in todays lecture we will look at flows in two dimensions, look at fixed points, what kind of fixed points one can have and other kinds of limiting behavior that one can have namely limit cycle attractors. And finally, we will discuss the Stability of such objects. To start with let us just recap what we did in the last lecture in the last lecture we looked at flows in the on the plane and we looked at both linear and non-linear flows and by analyzing in some detail the linear flow in two dimensions, we found that there are basically six kinds of fixed points; three of which are unstable namely the saddle, the unstable node and the unstable spiral and then we have the stable node, and the stable spiral and the center which is a kind of marginal behavior. (Refer Slide Time: 00:34)



These fixed points depend on the eigenvalues of the Jacobian matrix and for a two dimensional linear flow where you have the Jacobian matrix having trace tau and determinant delta, the behavior in this tau delta plane can be essentially broken out into these regions. If tau squared is greater than 4 delta, one has unstable nodes and depending on the magnitude of the eigenvalues you can have unstable nodes. If tau squared is less than 4 delta, one has the stable behavior. If delta is negative and one has saddles and if delta is positive one can either have a center or stable and unstable spirals. So, these are the kinds of behaviors that one can find for systems on the plane. Now, although the analysis was done based on the linear equations we found that for non non-linear systems, you could repeat this analysis quite easily by first finding the equilibrium points, linearizing around the equilibrium points and then identifying the behavior at these equilibria and use continuity to sketch phase curves. (Refer Slide Time: 02:39)



By this I mean and be illustrated this the last time that just as in one dimension between any two stable fixed points there must be an unstable point, between any two unstable fixed points there must be a stable point. In the same way not all of these kinds of the behaviors are compatible with one another and one can easily find out how to make a continuity work for you in sketching phase curves. We will take up some of these issues in the homework assignments and so on which we will be discussing subsequently. The important part to remember is that frequently and in the more complicated non-linear systems, it is almost always necessary to use computational tools and trajectories and use computational tools in a variety of ways. One can have the computational tool to sketch the vector fields and any trajectory of the system has to be tangential to the vectors at that point. The second thing that helps us especially when doing numerical calculations is that through any point there can be one and one trajectory. So, this is I mean you remember that this is an initial value problem and therefore, the orbits are unique. (Refer Slide Time: 04:38)

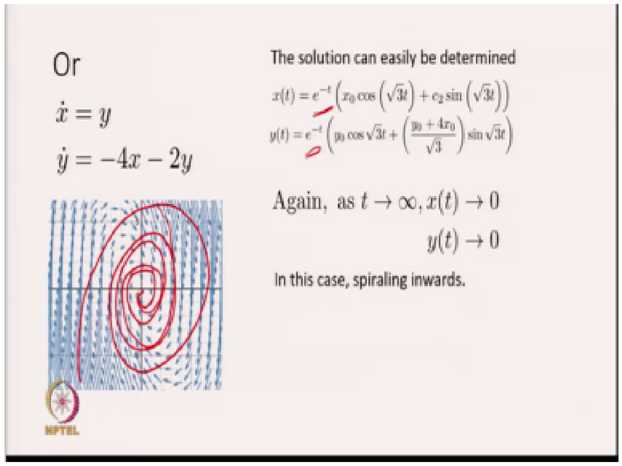


Let us look at some specific examples, partly for illustration, but also partly to introduce a concept of attraction. If we look at the linear flow and it is a very simple one

$$\dot{x} = -x$$

$$\dot{y} = 2x - 2y$$

, using the tool that I pointed out in the last lecture there must one can draw the vector fields and the vector fields are actually pretty the straightforward and you can see that everything is pointing inwards. A little analysis will show you that this is a stable node and the equations are so simple that one can actually write down the solution. So, the first one is simple enough and one has just a second while I get my pen on right; so, the first one which is $\dot{x} = -x$ is easily solved to give us this exponentially decaying solution for x. Putting that into y and solving that gives us another rather simple solution for y. Now, seeing that you have got an exponential factor over here of e^{-2t} and e^{-t} over here, as $t \to$ inf clearly x(t) is going to go to 0 and y(t) is also going to go to 0 regardless of the value of x naught or y naught. Therefore, starting anywhere, you will come eventually to the point (0,0). If you start over here you will go to (0,0) and so on and so forth ok. So, the point 0 seems to attract all initial conditions coming from wherever you are wherever you find yourself initially eventually you will always land up over here. Now, the question of tangency to the vector field at all points is important. So, this orbit that I have drawn over here is actually tangent or at least in my approximation it is always tangent. And these other orbits are not tangent so, you will you will see you can easily prove for yourself that a valid trajectory must follow these flow lines and come to the origin which is the attractor. (Refer Slide Time: 07:21)



Now, if you looked at another system and here I have chosen it differently

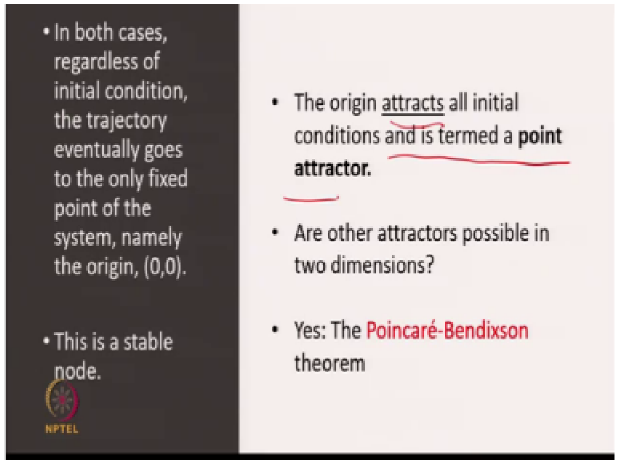
$$\dot{x} = y$$

$$\dot{y} = -4x - 2y$$

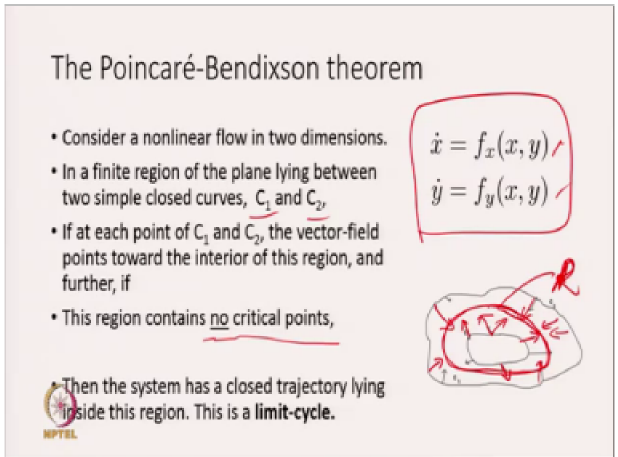
. Analysis will show you that this is a stable spiral and again the solutions are not too difficult to determine. Here they are here are the solutions for you;

$$x(t) = x_0 e^{-t}$$

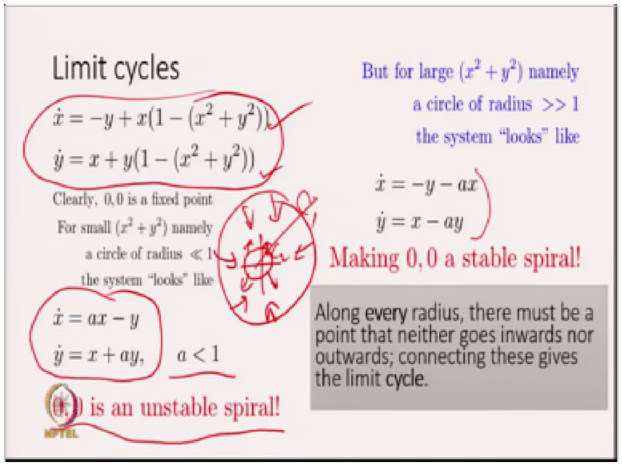
multiplying some oscillatory some trigonometric functions and wide y has got e to the minus t again multiplying another bunch of trigonometric functions. And without much more than just inspection you can see that as $t \to \infty$ x(t) and y(t) go to 0. But in this case they are; obviously, now going to be spiraling inwards all right. So, again we find that the origin 0,0 is the limit for all orbits in the plane. No matter where we start from you are going to eventually go down to 0,0. (Refer Slide Time: 08:38)



Now, in both cases therefore, regardless of initial condition the trajectory eventually goes to the only fixed point of the system, which is the origin. And in both cases this was and this is why they were chosen as examples here these are stable nodes; both the origin in both the examples is a stable node. When one case is just a simple stable node in the other case a spiral. The origin therefore, can be said to attract all initial conditions and it is termed a point attractor because it is a point. Now, given that you have got a plane and if it is a linear system there is only one fixed point then as we have seen this analysis over here there is only one type of attractor namely a point attractor. And the question which is of interest and has been of interest for over a 100 years is that in the plane if your equations of motion are non-linear are any other kinds of attractors possible and it there was a very important result in this area and its known as the Poincare-Bendixson theorem and which basically says that in addition to a fixed point attractor you can also have another kind of an attractor which is a cycle. (Refer Slide Time: 10:01)



We will briefly discuss the Poincare-Bendixson theorem just using the salient parts of the theory of the theorem without being very rigorous about the proof. In fact, the proof requires some very serious mathematical analysis but let me just try to give you a flavor of what this theorem attempts to prove. So, in its simplified version the Poincare-Bendixson theorem says that if you have a non-linear flow in two dimensions and here is an example of a non-linear flow in two dimensions very general; if in the fine in a finite region of the plane which is the phase space of this particular problem ok, we consider two curves C1 and C2 and the C1 and C2 are of the following kind, the vector field specified by the right hand side of both these equations, at each point of C1 the vector field points inwards and at each point of C2 the vector field points outwards ah, but into this annulus. So, all along the curve C1 we have the vector fields pointing inwards, all along the curve C2 we have the vector fields pointing outwards and furthermore if this annular region over here, if this region which is just call it R, if this region R does not contain any critical points then the Poincare-Bendixson theorem says the system has a closed trajectory lying inside this region and this is a limit cycle. It is a limit cycle, because the vector field points away from here and it points away from here and so, the curve that is lying in between is a limiting behavior in some sense and this is a very important result showing that you can have other kinds of attracting behavior. (Refer Slide Time: 12:28)

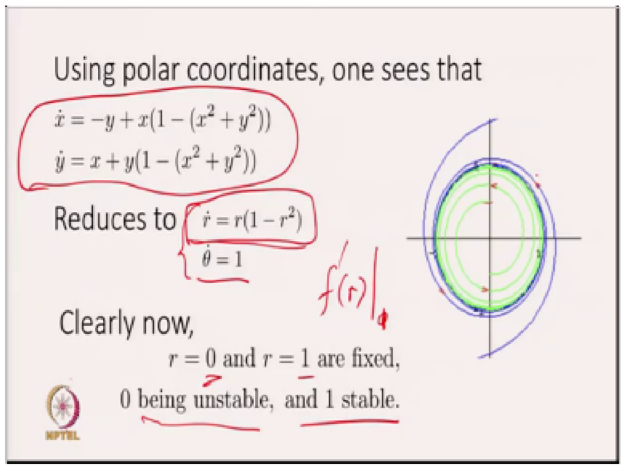


Now, the Poincare-Bendixson theorem has been used to show that that you know that various systems can have limit cycles and here is a very standard example. So, if you consider the following 2 dimensional flow and this is a non-linear system now, all right you can by inspection see that 0,0 is a fixed point. Now, for small x squared plus y squared namely x squared plus y squared describes a circle around the origin so, it is around 0 0. So, for as around the circle of very small radius around the origin the system more or less looks like $ax - y$ for this equation and $x + ay$ for this equation, where a is some number which is less than 1, because its 1 minus R squared. So, we choose it R square to be less than 1 and you finally, you find that these are the equations more or less what the system looks like namely 0,0 is an unstable spiral. More importantly on the circle which is got radius square root of 1 minus a what you find is that all the points, because it is an unstable spiral at the center all the vector fields are pointing outwards. On the other hand if you go to a very large circle and circle which has got radius much larger than 1, then x squared plus y squared is a number which is bigger than 1 and this term is negative. So, it looks something like

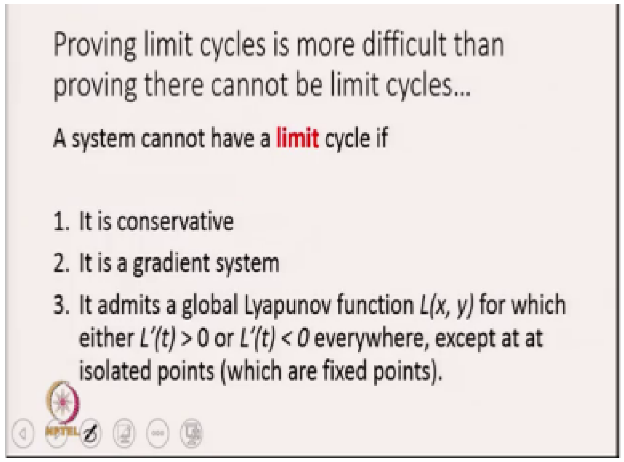
$$\dot{x} = -y - ax$$

$$\dot{y} = x - ay$$

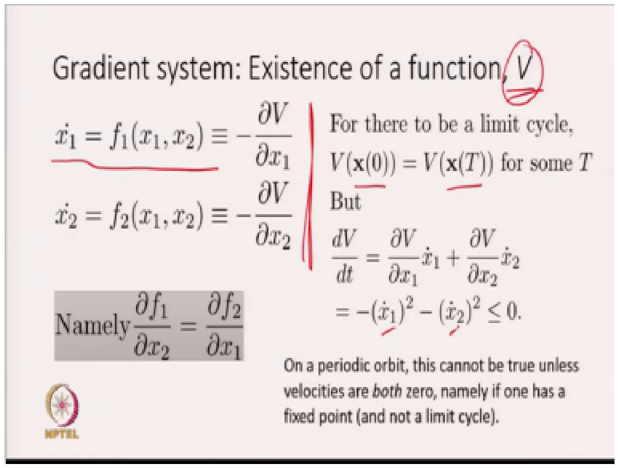
and again little analysis will show you that (0,0) in this scenario is stable spiral; which is to say that on this larger circle around the origin all the lines the vector fields are pointing inwards, because it is a stable behavior. And now this has got exactly the condition that the Poincare-Bendixson theorem requires. Namely you have two curves this is C1 and this is C2. Along C1 the vector field points n along C2 the vector field points out. Therefore, somewhere in between along every radius there must be some point that neither goes in or out and you know connect all these points you get yourself a limit cycle. (Refer Slide Time: 15:25)



You can simplify the analysis a little by just considering polar coordinates and if you use polar coordinates you can see that this equation this non-linear equation over here reduces to the following coupled equations well. It is not quite coupled, the motion in theta is very simple, because you can integrate this and this just says that theta is time and this these two equations reduced to r dot is equal to r into 1 minus r squared. Now, one can clearly see that this looks like a one dimensional equation for the radial coordinate. But when considering this one dimensional equation, I have got two fixed points on this in this one dimensional equation these are actually two fixed circles if you like. One of them has got radius 0 so, it is just a point and the other one has got radius 1 and both these are fixed circles if you like. Now again doing the analysis namely looking at the derivative f prime of r, if you look at f prime of r at r is equal to 0 you find that 0 that that the derivative is equal to 1, positive number and there 0 is an unstable fixed point or an unstable circle and again doing the same analysis f prime at r is equal to 0 turns out to be negative. And therefore 1 is a stable fixed point a fixed point namely a stable circle and orbits look like this namely if you start from inside the circle of radius 1 you will spiral outwards until you hit this circle. If you start from any point outside the circle of radius 1 you will spiral inwards until you hit this cycle. (Refer Slide Time: 17:25)



So, this helps us to understand that on the plane you can have somewhat more complicated behavior than just a fixed point and a fixed point which is either stable or unstable or a saddle or a center or whatever. Now, to prove that you have a limit cycle sometimes is more difficult than proving that they cannot be a limit cycle. It turns out that there are many situations in which without doing any analysis and without even trying to prove the Poincare-Bendixson theorem for your system, you can show that a system cannot have a limit cycle in the following situations. One that it is a conservative system, that is there some conserved quantity. Two that it is a gradient system and I will get to that in a moment and three that it admits a global Lyapunov function L(x,y) for which either $L'(t) > 0$ or $L'(t) < 0$ is negative everywhere except at isolated fixed points. So, what is what are these various conditions? Conservative I will come to subsequently, but first let me look at what is a gradient function. (Refer Slide Time: 18:50)



A gradient system is one where there is a function V, such that the equations of motion that namely

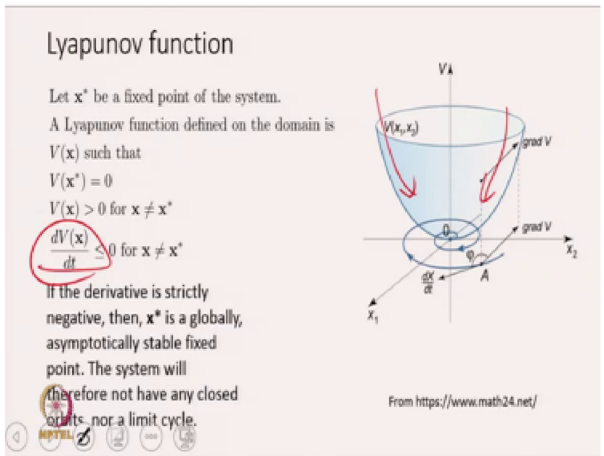
$$\dot{x_1} = f_1(x_1, x_2)$$

$$\dot{x_2} = f_2(x_1, x_2)$$

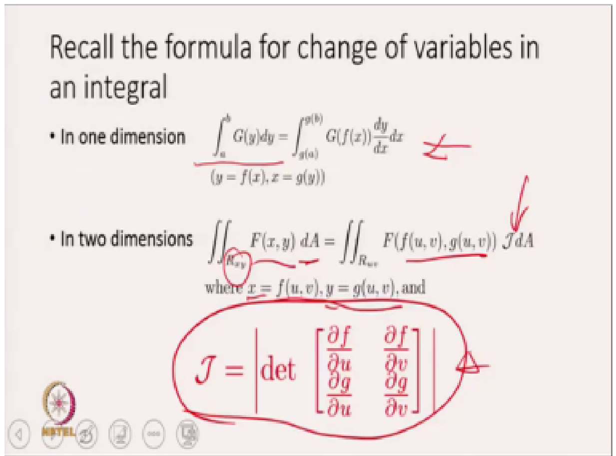
these can be written as the gradient of some other function. This is very similar to the way in which potentials are introduced, but this is not quite the same thing right. You will also notice that I go back and forth between the notation x,y for the coordinates on the plane and x1,x2 some of it is deliberate and some of it is for convenience of notation because sometimes I need to use a bold vector x and so on and so forth, but I hope that this should be straightforward and clear. Now, if there is a function V which whose derivative is the these functions f1 and f2 whose partial derivatives and these two different variables is x1 and x2, consider that we have a limit cycle. Now, this limit cycle is a is some kind of an orbit that goes back around its and meets itself. So, at time t equals 0 it is at the position x where we started out and after time t you come back to exactly the same point and so, this function V takes exactly the same values. But now let us look at the derivative of V as a function of time. Because of these relationships

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_2}\dot{x_2} + \frac{\partial V}{\partial x_2}\dot{x_2}$$

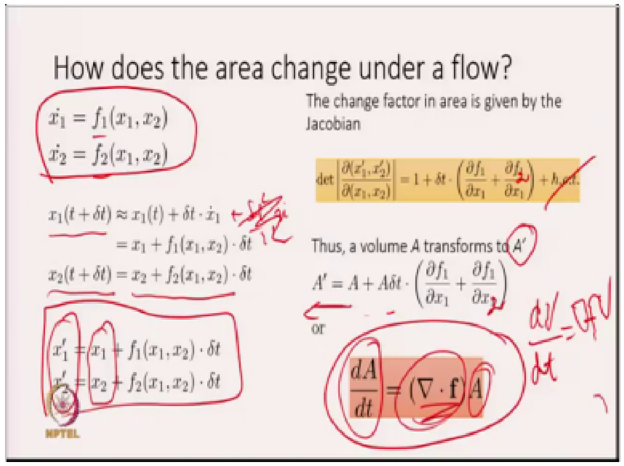
and this is negative definite. So, on a periodic orbit this cannot be true unless both the velocities that is this and this are equal to 0 and if both these velocities are 0 we were already at a fixed point and it is not a limit cycle. So, this kind of excludes the whole class of systems, all gradient systems from having any limit cycles because of this condition. (Refer Slide Time: 21:20)



The Lyapunov function and Lyapunov is the name that you will come across quite often in a course in on chaos theory or on or non-linear dynamics or dynamical systems. So, if the Lyapunov function was introduced by the Russian mathematician Lyapunov and basically it is the following; if there is a fixed point of the system let us call that x star, a Lyapunov function is a function V(x) such that at the fixed point $V(x^*) = 0$. Now for all other points in the domain of this dynamical system V of x is positive and its derivative is negative. So, in some sense it looks like some kind of a little parabola or paraboloid around this fixed point. If you can find such a function which is well defined in your dynamical system, then x star is a globally asymptotically stable is globally asymptotically stable fixed point and this system cannot have any cycles it cannot even have a closed orbit because everything is going is falling downwards. So, to speak everything is just flowing down, because this derivative is negative. Now finding Lyapunov functions is not as simple as all that and in fact, Strogatz says that you know some kind of divine inspiration is probably needed, but more or less quadratic type functions, whenever people have been successful in finding Lyapunov functions they have had this general shape ok. (Refer Slide Time: 23:29)



Now, we just take a brief break over here to recall to remind everyone about the formula for change of variables in an integral. In one dimension if you have this integral G of y dy limits being a and b. If you want to change from y to the variable y to the variable x where you define y is f of x and x as G of y, then this is the standard form for the change of variable. In two dimensions if I have got the variables x and y on the one hand and u and V on the other hand and x and y are defined like. So, as and u and V are defined sort of in terms of x and y and these kind this through this relationship, then any integral f of x y taken over the region in the x y plane so, it is a it is an area integral over here. When I transform to the u and V variables of course, I have got to change x and y in terms of the arguments over here, but also importantly I have to add the Jacobian of the transformation namely this determinant. Now, this is standard calculus so, I am not going to actually sit and try to derive it out, but I just want to remind you about this for the following discussion. (Refer Slide Time: 25:09)



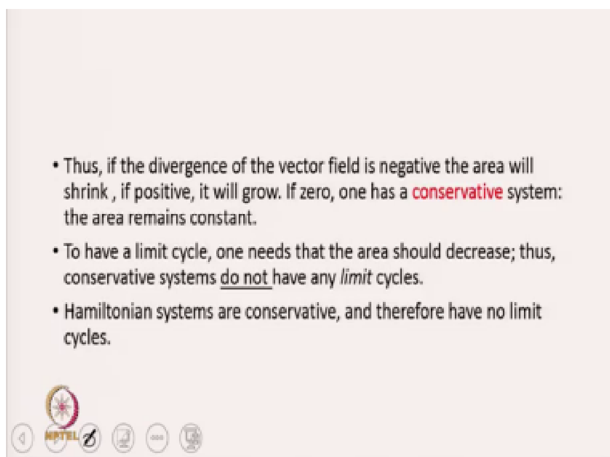
If I have got a flow

$$\dot{x_1} = f_1(x_1, x_2)$$

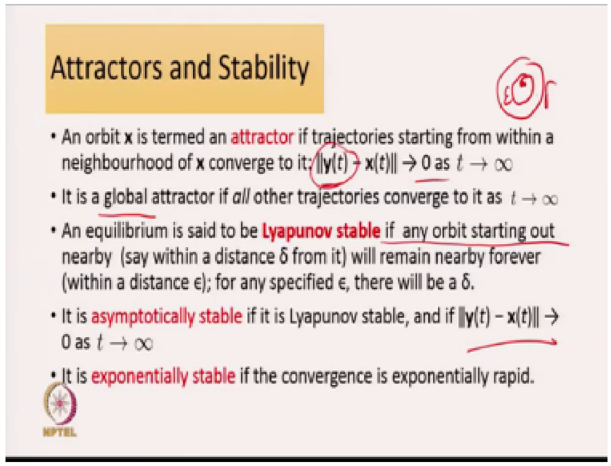
,

$$\dot{x_2} = f_2(x_1, x_2)$$

the usual flow that we have been considering it is just a simple non-linear flow. How does any area change alright? So, I want to take some unit area and ask as my system obeys these following equations, how does the area change? Now, in order to do that I can just I can do the analysis as follows, at a later time x1 at time t plus delta t this is just going to be approximately I mean to leading order, because I am not going to take all the terms in the expansion. It is x 1 plus delta t times x 1 dot plus of course, you would have delta t square by 2 factorial x double dot and so on and so forth, you would have all those are the terms. But I am not considering any of those and let me just take this to be the simplest one the first term over here. So, similarly x 2 at a time t plus delta t is just x 2 plus delta t times f 2 of x 1 and x 2, or I can rewrite this as in terms of the old coordinates x 1 and x 2 and the new coordinates x 1 prime and x 2 prime, I can write this as the following linear equation. Now, if I am going to change you know so, I have got an area and I have got the flow equations going moving and moving me in some direction. So, as I change what happens to an unit area? We have seen this formula for the change of area except now x and sorry u and V and x and y are all the same really except now we have to add this namely the Jacobian of this transformation. And this Jacobian you can easily calculate it is written as so, the new variables in terms of the old variables taking that Jacobian gives me a factor of 1 plus delta t times partial of f 1 with x1 plus partial of f 2 with respect to x 2. I am sorry about the mistake over there, but I will correct that subsequently, plus of course, higher order terms which I am going to ignore, because I am doing it to leading order. Now, therefore, this volume A or an area A will transform to a new area A prime by this multiplicative factor. A prime is just A plus A times delta t times partial of f 1 with x 1 and partial of f 2 with respect to x 2. Or rewriting this by moving this over onto this side and dividing by delta t taking limits ecetera, the rate of change of area in the plane is given by A times the divergence of this vector field f which is specified by f 1 and f 2. Now, this is a very important formula and it generalizes in any number of dimensions where you could just write down that dV by dt excuse me will be given by the divergence of this vector field times V. And this says that the area or the volume more generally the area changes at an exponential rate where this exponential factor is given by the divergence of this vector field. (Refer Slide Time: 29:34)



If the divergence is negative this means that the area will shrink. If the divergence is positive the area will expand and in case one has a conservative system the area remains constant. Namely the divergence of this vector field is 0. To have an attractor or to have a limit cycle one needs that area should decrease; namely you want some kind of contraction to keep happening therefore, conservative systems do not have any limit cycles. Hamiltonian systems are conservative, and they are very large and important class of systems that we are interested in dynamics and therefore, in Hamiltonian systems there are no limit cycles either. (Refer Slide Time: 30:29)



Some final discussion on attraction and stability, these two concepts are slightly different from one another. You can have an attractor which is not necessarily stable in certain technical senses that we will discuss momentarily and you could have stable systems which are not necessarily attractors. So, in orbit x and this orbit could be a point, it could be a limit cycle. An orbit x is termed an attractor if trajectories which start from within some neighborhood of x converge to it. So, if y is in the neighborhood of x, as time goes to infinity, y of t minus x of t will go to 0 and that tells you that x is an attractor and this is true for all y lying inside some neighborhood. If all the trajectories converge to this attractor regardless of where you start from, then the attractor is global otherwise it is a local attractor. Now, as far as stability is concerned, let us just discuss the stability of an equilibrium point. An equilibrium point is said to be Lyapunov stable or stable in the sense of Lyapunov, if any orbit starting out nearby let us say within a distance of delta from this particular equilibrium point will remain nearby forever within some distance epsilon. And for any specified epsilon there will be some delta. So, we have a fixed point and here is some region of radius delta, all orbits that start within delta will stay within will eventually get to a region around this fixed point within a distance of epsilon and will always stay inside epsilon. So, it is not as if all the orbits attract if this excuse me it is not as if all the orbits will go to the point x, as in the case of an attractor, but it is also stable in the sense that it will always be close by. In case it does go to 0, then its termed asymptotically stable and if the rate of convergence is exponentially rapid it is termed exponentially stable. So, this brings me to the end of this particular lecture, where we have discussed notions of stability of attraction and seen certain examples of what kind of behavior is possible in the plane. Almost everything that I have said today is particular to the plane except for a few things which will come back in subsequent lectures. So, in the plane we can have only two kinds of behavior; we can either have I took as a behavior asymptotically which are attractive ah, namely you can either have an at a fixed point or you can have a limit cycle. You can have cycles you can have orbits that are not closed, you can have all sorts of other things. But basically, the two-dimensional systems do not have any more complicated kind of dynamical behavior. In order for there to be something more complicated we need to increase the dimension and that we will do in subsequent lectures of this course. Thank you.