Introductory Nonlinear Dynamics
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Lecture 02
Introductory, Stability, Phase Space and Invariant Sets

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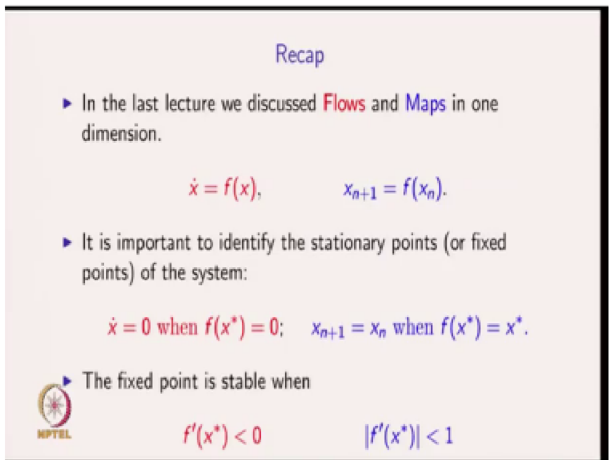
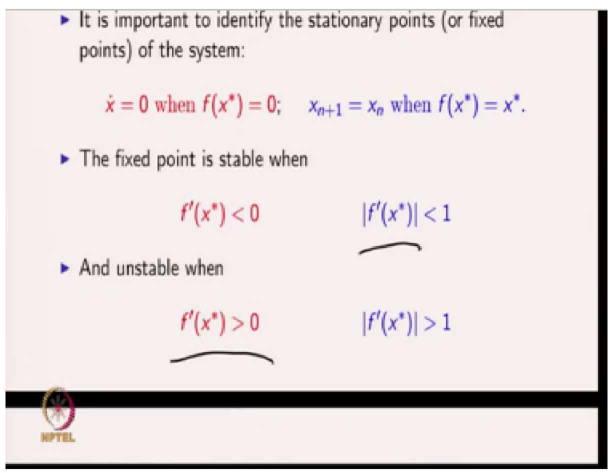
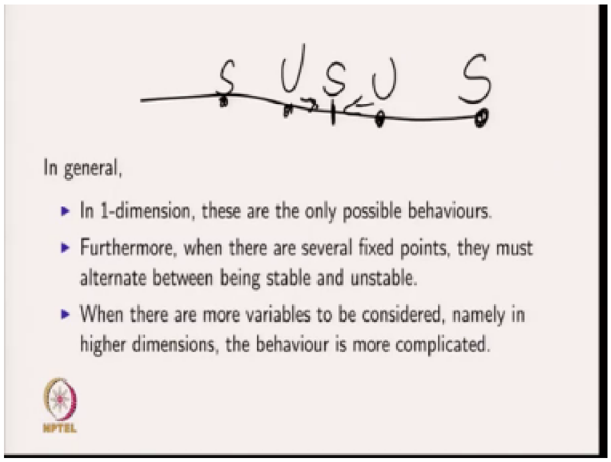


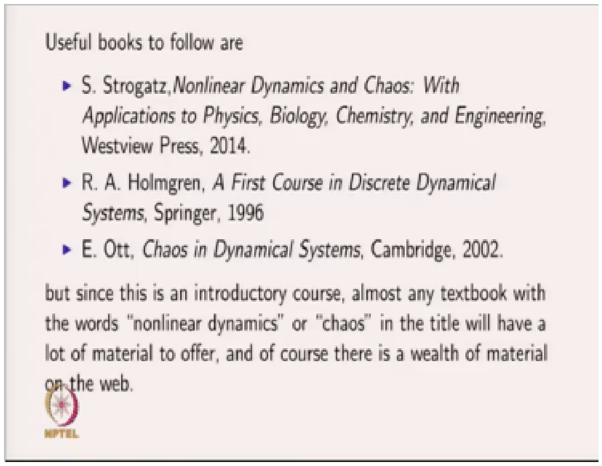
Let us recap what we did in the last lecture. Basically, this that in the last lecture it was just very very elementary stuff, where we discussed what a dynamical system is and we noted that there could be two different ways in which dynamical systems are specified either by differential equations or by iterative maps. If it is a differential equation we write down the derivative of variables dx by dt, dy by dt and so on and in one-dimension there is only a single variable, so we had $\dot{x} = f(x)$. If it was an iterative map on the other hand given the state of the system at time n, $x_{n+1} = f(x_n)$ we had a rule for going to the next time instant namely x at the time n+1 and that was given by some functional mapping. Now, what we discussed was that we had the system specified by the number of variables, here which is just one-dimension. The evolution equations were given in one of these two forms, and then we needed to analyse the system in a mixture of both qualitative and quantitative methods. In order to first figure out what the important features of the system are. We noted that it is important to identify stationary points of fixed points of the system. Stationary points are those points where essentially nothing happens. So, if $\dot{x} = 0$ the velocity is 0 then the system will not move. And this happens when the right hand side of the equation $f(x) = 0$ and f(x) being some function takes a value 0 at one or more points which we just denote by x^*. In the case of iterative mappings, nothing happens when x_n is equal to x_{n+1}, namely it is a fixed point. And this will happen if $f(x_n) = x_n$ and that is, that means, that we need to solve an equation f(x)=x, that gives us one set of roots which are termed x^* over here. So, the equations are slightly different, but not really different in principle. Now, these fixed points are stable in the following situation. If you have the differential equation namely $\dot{x} = f(x)$, then the fixed point is termed stable if the derivative of f at the point x star is negative and if it is if the derivative is positive then the fixed point is unstable. For the map on the other hand, at the fixed point if the derivative is less than 1 in modulus then the fixed point is stable and if the derivative is greater than 1 in modulus the fixed point will be unstable, all right. (Refer Slide Time: 03:55)



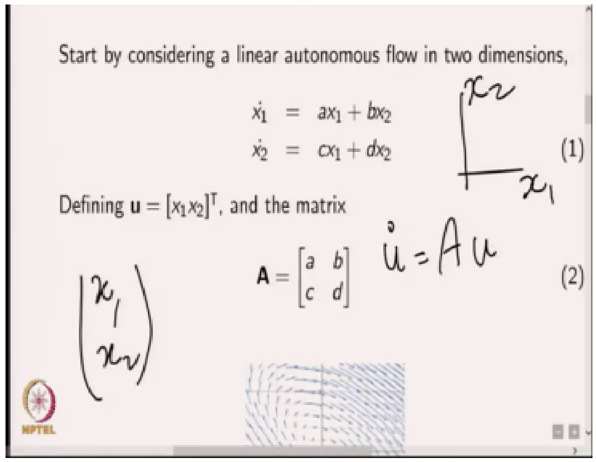
So, this more or less summarizes what we had discussed in the previous lecture. (Refer Slide Time: 04:10)



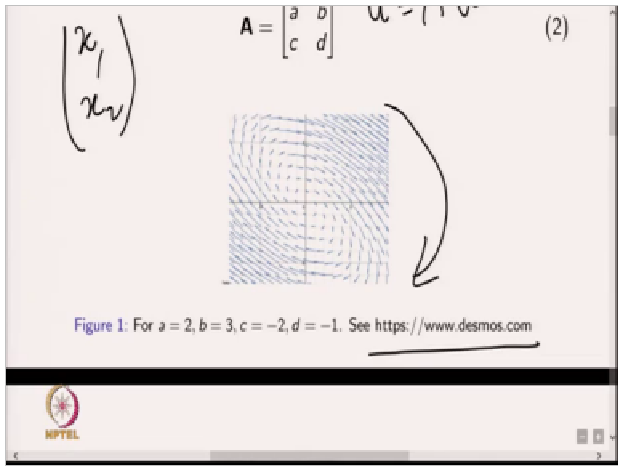
And let me just point out that in general, in one-dimension these are the only possible behaviours, namely you can have fixed points of course, you can have no fixed points at all with in some exceptional situations. But if you have a fixed point then the only possible behaviours are that it is either stable or unstable. Of course, there is this intermediate case of being neutral ah, but those are not particularly interesting and I am not going to spend time discussing them. An important geometric point over here is that when there are several fixed points because we are considering the situation in one-dimension, if I have a stable point let me call that over there and if I have an unstable point over here, if I have another point over here it has to be stable. You cannot have a situation where you have two unstable points one next to another, because as we saw in the previous lecture the vector field points away from the unstable point, it points away from this unstable point, so somewhere in between there must be a point which is stable because the flow is coming in on both sides. So, we must always have alternation between stable and unstable a fixed points. Now, this is what happens in one-dimension there is a lot more to be said and we will come back and talk about one-dimensional systems in great detail next week. But when there are more variables to be considered namely in higher dimensions the behaviour is somewhat more complicated and we will turn to that shortly. (Refer Slide Time: 06:10)



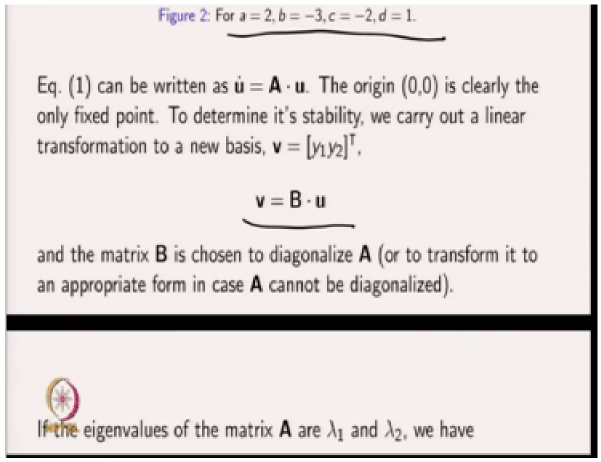
Before, I turn to that though let me point out that for this course I have already specified that will be largely following these books. Beautiful book on Nonlinear Dynamics is by Steven Strogatz and that is the title over there. There is an edition available over here. The very nice slender little book on mathematics by Holmgren, which is titled a first course in Discrete Dynamical Systems. And then of course, there is the classic book by Edward Ott, Chaos in Dynamical Systems and there are several editions of this book. I should point out that this is an introductory course. And almost any textbook with the words nonlinear dynamics or chaos in the title particularly, will have a lot of material to offer and any book can be followed. So, to speak they are not going to be at variance with one another, where they will differ really is in the order in which the topics are presented, but the material is the same. And there is a huge amount of stuff that is available on the internet, starting with Wikipedia, Scholarpedia, there any number of sites where people have got material which they wherein they discuss these concepts of non-linear dynamics. Now, in this course, I am going to be going through in a somewhat non-linear fashion through these various topics. And my personal choice is to try to understand what happens in an around, in systems around the name around name fixed points. So, in one-dimension as we saw the fixed points are very simple and can be classified very trivially. In two-dimensions and this is symptomatic of what is going to happen in higher dimensions. The situation is a little more complicated. (Refer Slide Time: 08:26)



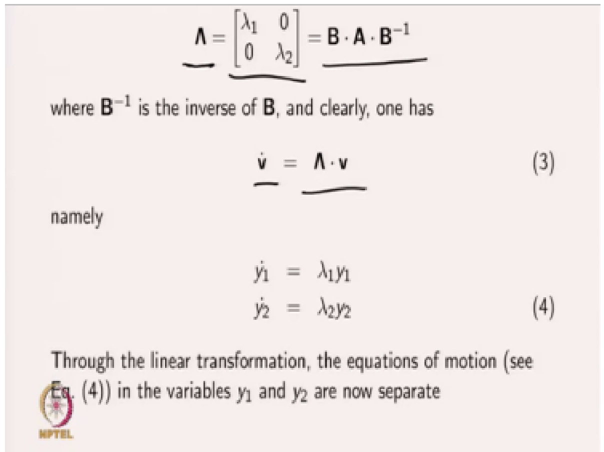
So, let us start by considering a linear system in two-dimensions. And let it be autonomous that is to say time is not going to appear on the right hand side and the two variables that I am going to denote for the axis are x 1 and x 2 this is mostly for notational convenience and occasionally, I will slip into calling it the x axis or the y axis, but hopefully you should be able to figure out what I mean by the context. So, the two axes I said are x 1 and x 2. So, this is the x 1 axis and this is the x 2 axis and this linear autonomous flow in two-dimensions is just the following set of coupled first order differential equations linear. Because it is a linear system I can define a vector this is the variables x 1 and x 2 that is my vector u. And if I take the matrix of coefficients a b c and d, then, I can rewrite this equation as u dot is equal to A times u, that is to say this equation can be written down as u dot is equal to Au, where these are the two vectors oops sorry some mistake over there, all right. (Refer Slide Time: 09:36)



Now, depending on the values of a b c and d the right hand side of this equation, this gives us a vector field and that can also be sketched like we saw in the one-dimensional case. So, here is an example for a equals 2, b equals 3, c equals minus 2 and d equals minus 1, we have the following vector field. Namely, what we do is to go back to these equations and at any point x 1, x 2, you draw the vector in the x 1 direction as that and the vector in the x 2 direction as that and the resulting vector is where is pointing as it is. And here we see automatically that this fixed point, the fixed point is at the origin 0 0 this fixed point has an interesting character. In the sense of you can see that there is a circulation around the fixed point and that is what the flow the flow shows. Now, I drew this particular vector field using this online resource desmos dot com and you can just go in there and enter your equations of motion, if that is you can enter the you know enter the right hand side in two-dimensions and it will give you an image of what the flow is like, all right. (Refer Slide Time: 11:37)



If I change a b c and d, and for these values of a b c and d, the flow is somewhat different and here you can see that there are arrows pointing inwards along this direction and outwards along this direction. Now, the whole notion of stability therefore, needs to be looked at a little more subtly over here in two-dimensions. The origin is clearly the only fixed point from these two equations. You can see that the only solution which is going to give us both of them equal to 0 is 0 0 and to determine its stability we proceed as follows. First, let us carry out a linear transformation to a new basis v which I will index by the two variables y 1 and y 2, and the transformation between u and v is given by this linear transformation over here. And the matrix B being chosen to diagonalize A. In case A cannot be diagonalized as a standard appropriate form in which we would like the matrix B to be sorry in the in case A cannot be diagonalized, B can be chosen to bring A into some standard form, but in the simplest case the matrix A can be diagonalized and we choose B as the matrix that will diagonalize it in the following way. So, $BAB^{-1} = \lambda$, right. (Refer Slide Time: 13:14)

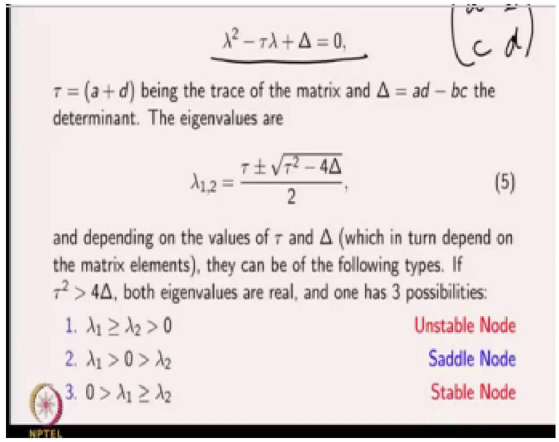


Now, if the eigenvalues of the matrix A are λ_1 and λ_2 then the matrix the diagonal matrix lambda is just given by this particular diagonal matrix. And if you just follow the algebra over here you can clearly see that v dot is equal to lambda times v. Writing this in terms of the variables you find that

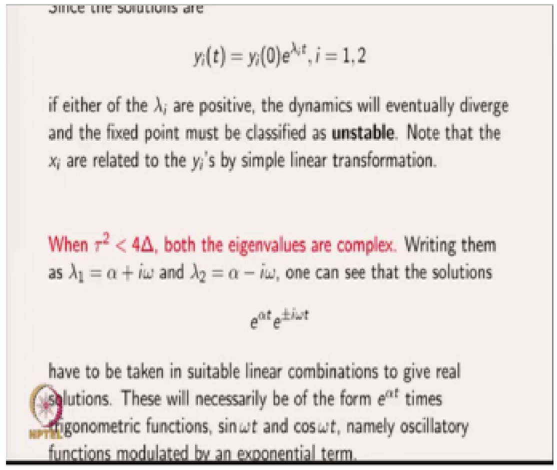
$$\dot{y_1} = \lambda_1 y1$$

$$\dot{y_2} = \lambda_2 y_2$$

that is by this linear transformation we have separated the two variables and got much simpler equations which are effectively just one-dimensional equations, all right. Now, we have already done the analysis for one-dimension in the last time and therefore, the stability or instability of this particular system is very trivial for us to determine. This is going to be stable if λ_1 is negative and unstable if λ_1 is positive, likewise this will be stable if $\lambda_2$is negative and unstable if λ_2 is positive, all right. (Refer Slide Time: 14:38)



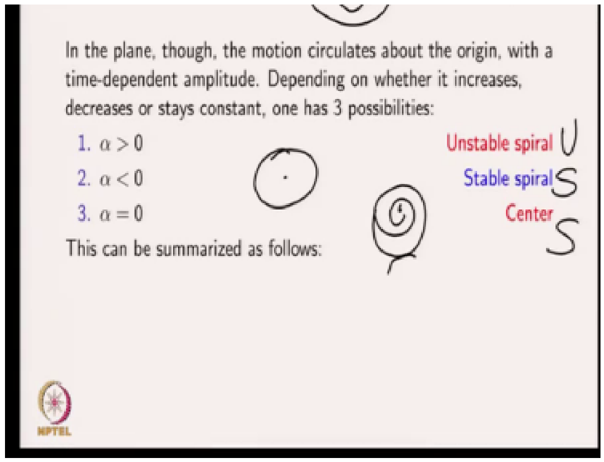
Now, these eigenvalues are the roots of a quadratic equation and again given the fact that this is a 2 by 2 matrix and fairly simple to solve. We can see that the quadratic equation which you need to solve is just lambda squared minus tau times lambda plus delta, where tau is the trace or the sum of the diagonal elements. Recall that the matrix A is a b c and d, all right. So, that the sum of the diagonal elements is the trace and the determinant is delta and the eigenvalues are given by the following equation. And now, I mean the difference from one-dimension begins to be seen over here, because depending on the values of tau and delta which of course depend on the matrix elements themselves there can be several possible cases. To start with if tau and is if tau squared is greater than 4 delta, then the discriminant is positive, the square root will give us two real numbers and depending on various possibilities we can either have both the eigenvalues as positive and real, you can have both the eigenvalues as negative and real and in one case you can have one of them positive and one of them negative. These 3 cases and I will come back to this are termed unstable node, saddle node and stable node. The node refers to the fact that 0 is the fixed point and the stable or unstable tells you, well stable or unstable or the saddle tells you what is the behaviour in the neighbourhood of the node. (Refer Slide Time: 16:47)



Now, since the solutions that we have in this particular case the equation itself is a trivial one-dimensional equation and the solution is

$$y_i(t) = y_1(0)e^{\lambda_i t}$$

. If either of the λ_i is positive then the dynamics will eventually diverge because then you will have a function that grows in time. So, if either of the λ_i are positive, then the dynamics will eventually diverge and the fixed points must be classified as unstable because any motion in the neighbourhood of the fixed point will eventually go away. Now, the x sub is are related to the y sub is by a simple linear transformation and therefore, if it is unstable in the variables written down in the y coordinate system, it is going to be unstable in the x coordinate system as well. Now, when tau squared is less than 4 delta, then both the eigenvalues are complex and writing them as lambda 1 is alpha plus i omega and lambda 2 is alpha minus i omega, one can see that the solutions are going to be of the form e to the alpha e to the plus or minus i omega. Now, these have to be taken in suitable linear combinations to give real solutions, because after all the solutions of the equations are real. And therefore, these combinations that are going to give us real solutions out of e to the i omega t have to be trigonometric functions sin and cosine omega t. And these are being multiplied by e to the alpha t. So, we have an exponential times an oscillatory term, all right. So, basically we find solutions that are going to oscillate in time, but the amplitude can be variable. (Refer Slide Time: 19:03)



Now, since the motion is in the plane x1, x2 or y1, y2, the motion will circulate about the origin with a time dependent amplitude which depends on the alpha. One again has 3 possibilities, alpha positive in which case its spiralling outwards because its moving you know the amplitude if this is the fixed point things are moving outwards. If alpha is negative then around the fixed point we have stuff that is moving inward, and if alpha is equal to 0, then it neither grows nor, it neither grows nor shrinks and we have what is called a centre. In this previous case of the stable unstable or saddle nodes. In the case of the unstable node sorry the in the y 1 direction it is increasing, in the y 2 direction it is also moving away and so this is the kind of behaviour which goes with it being an unstable node. In the case of both negative eigenvalues we have a situation where the motion is coming inwards in the y 1 direction and it is coming inwards in the y 2 direction as well. The interesting case over here is of the of is the case where you have one positive and one negative eigenvalue and that gives us the following, that in the y 1 direction the motion is moving outwards because lambda 1 is positive and in the y 2 direction lambda 2 is negative. So, things are moving inwards. And this is the reason why it is called a saddle because you have one direction which is moving outwards and one direction which is moving inwards, sort of like the saddle on which one sits on a horse, ok. So, given this particular scenario depending on the value of tau and delta we can have not 2, but 6 different behaviours. And these 6 can be separated out as, this is of course, unstable the saddle has one unstable direction and therefore, it is just unstable, the stable node is stable, all right. And here we have the unstable which is a spiral which would be classified as unstable obviously, or the stable spiral is one of the stable ones and the centre is also stable in the sense that it is not unstable. So, out of the 6 types of fixed points or the 6 types of stationary points that we can have in a two-dimensional case. These can be represented in the following way. There are 3 of them which are stable in 3 of them which are unstable and they occur as follows. (Refer Slide Time: 22:34)

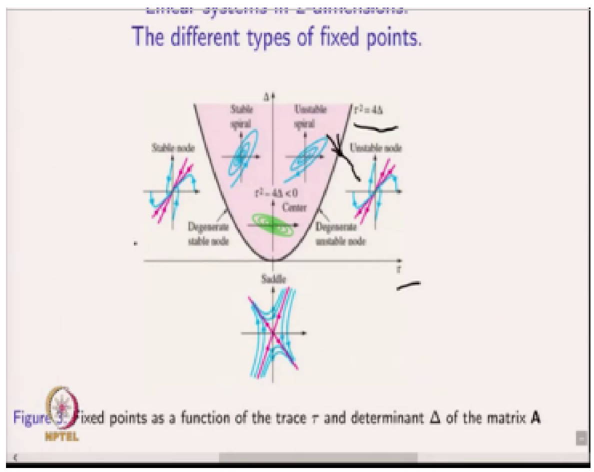
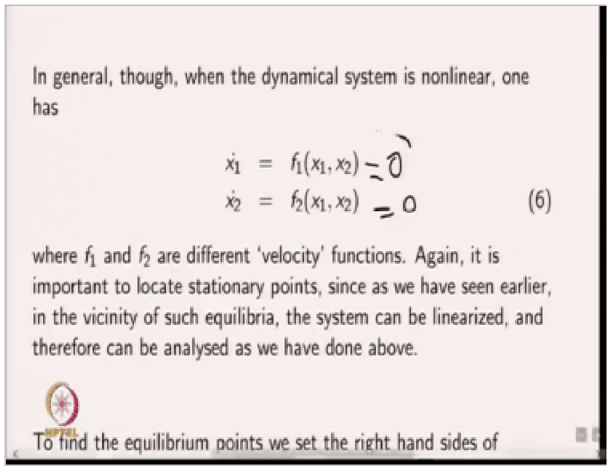
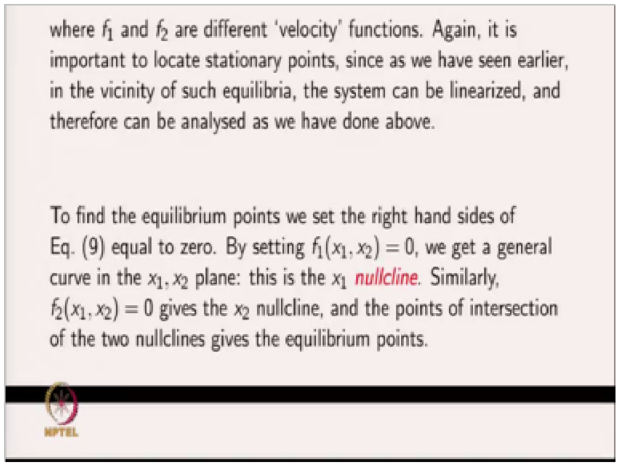


Figure 3: Fixed points as a function of the trace τ and determinant Δ of the matrix **A**

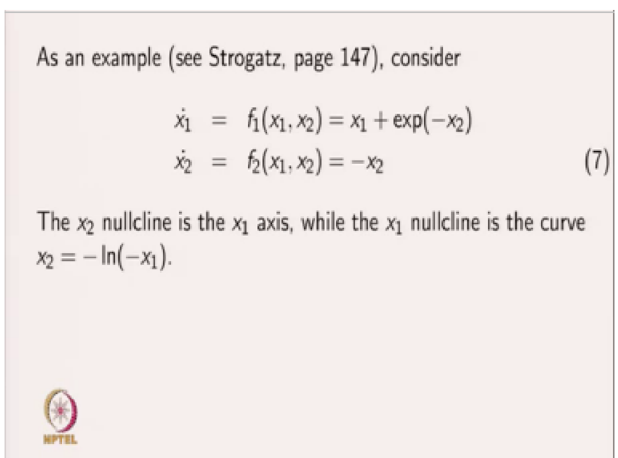
Here is the sort of a phase diagram if you like of the system as a function of tau and delta. The line tau squared equals 4 delta, that particular line is this parabola over here and that separates the real from the complex eigenvalues. In the real case if tau, you know for positive tau and below this particular line we have both the eigenvalues positive, on this side you have both the eigenvalues negative and therefore, you have the stable node on one side and the unstable nodes on the other side. And below this line, so for all negative delta you have saddle points. Within this parabolic region along the line tau equal to 0, you have centres because to have a centre if there is the real part of it must be 0, and you can see that the real part of the eigenvalue, the real part of the eigenvalue is always tau. So, if tau is equal to 0, we have a centre and if delta is positive, then you have the complex the purely imaginary roots and on this side you have the unstable spiral and on this side you have the stable spiral. Along the line you have a transition from the unstable spiral to the unstable node and on this particular line tau squared is equal to 4 delta, occasionally you will have exceptional behaviour. These are the degenerate stable nodes and the degenerate unstable nodes. In the same way of course, being on this line tau equal to 0 is also exceptional. So, in a sense centres are also exceptional behaviour, all right. So, this summarizes what we know about fixed points in two-dimensions and they can be of these 6 following types, but this is purely for the linear system. How does this analysis move on when we have non-linear systems? (Refer Slide Time: 25:21)



When you have this general dynamical system which is non-linear, the right hand side is specified by a function sum f1(x1,x2) and f2(x1,x2) and these are different velocity functions and they will depend on x1 and x2 in maybe complicated ways. The analysis however, proceeds in the same way as we did in the one-dimensional case, where we considered non-linear velocity functions. It is important to locate stationary points and it is important for the following reason that in the vicinity of such equilibria the system can be linearized, and then we can analyse it in the way in which we have done earlier. (Refer Slide Time: 26:12)



So, we start by linearizing by in order to find the equilibrium points. We start by setting the right hand side of these equations equal to 0, to find those points where the entire vector field will vanish. When we set f(x1,x2)=0 and this gives us an equation, where the variables x1 and x2 are functionally dependent upon each other that may specify a curve in the plane in this particular case it will be a curve and this is termed the this is called an nullcline because it has a values which is strictly 0 along this particular curve. Similarly, if you take f2(x1,x2) equals 0 this gives us the x2 nullcline and the points of intersection of the two nullclines will give us the equilibrium points of this system. So, the intersection of the two nullclines will give us the equilibrium points. (Refer Slide Time: 27:22)

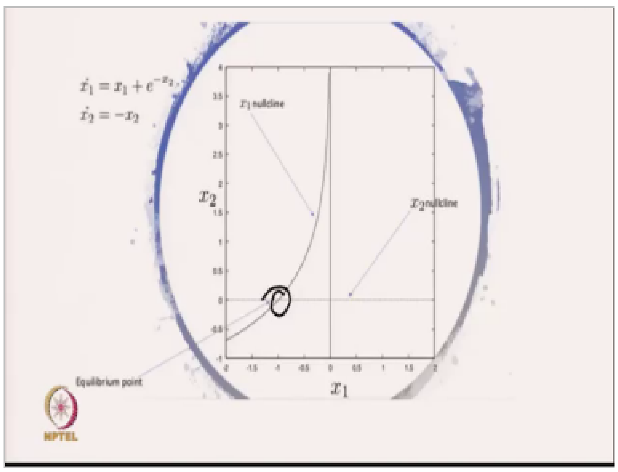


And here is a very nice example from Strogatz, ok. It is one of many possible examples, but if you have access to the book, on page 147 the following rather simple system is given.

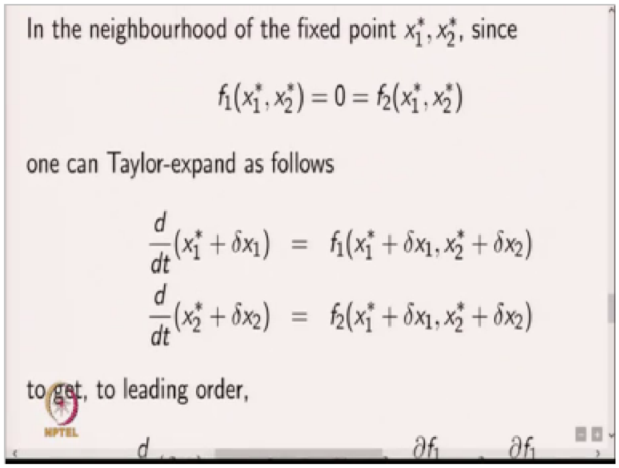
$$\dot{x_1} = f_1(x_1, x_2) = x_1 + exp(-x_2)$$

$$\dot{x_2} = f_2(x_1, f_2) = x_2$$

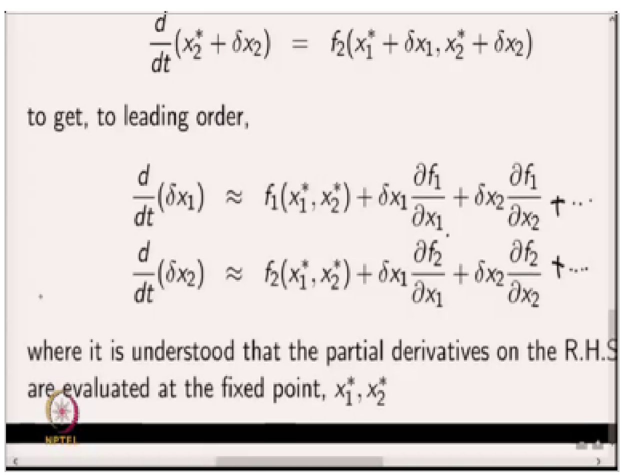
. So, the x 2 nullcline simple enough is just the x 1 axis because it says x 2 is equal to 0. The x 1 nullcline is this curve x, from here you can figure out x 2 in terms of x 1 is just minus the logarithm of minus x 1. Therefore, this curve is not defined in the positive x 1 on that positive part of the axis and you can draw it rather simply over here. (Refer Slide Time: 28:17)



So, that is your x 1 nullcline and here is the x 2 nullcline which is just the x 2, the x 1 axis, and the intersection of these two curves that point gives us the equilibrium point for this system. (Refer Slide Time: 28:43)



What is the advantage of the equilibrium point? Now, in it the neighbourhood of the equilibrium point since f1(x1*,x2*) and f2(x1*,x2*) star are both equal to 0, you can take a Taylor expansion and try to see what the differential equation that we were considering the this differential equation, how does this transform under or how does on linearize around this particular fixed point. So, this is fairly straightforward. I look at small displacements from x1*, let me call that delta x1 and a small displacement from x2* called that delta2. (Refer Slide Time: 29:33)

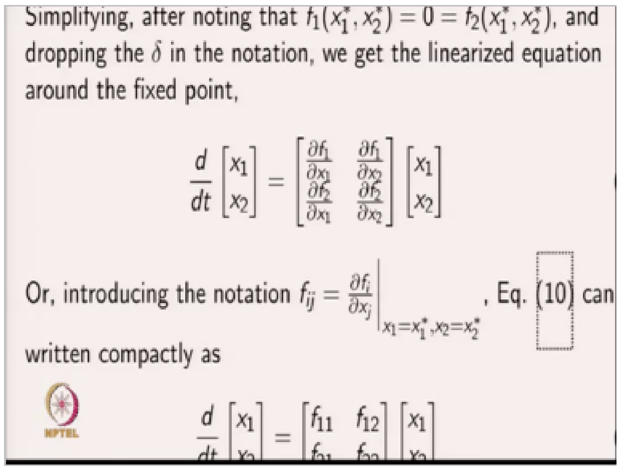


And two leading order now, since x1* and x2* are just points, they do not have derivatives. So, two leading order we get

$$\frac{d}{dt}(x_1^* + \delta x_1) = f_1(x_1^* + \delta x_1, x_2^* + \delta x_2$$

$$\frac{d}{dt}(x_2^* + \delta x_2) = f_2(x_1^* + \delta x_1, x_2^* + \delta x_2$$

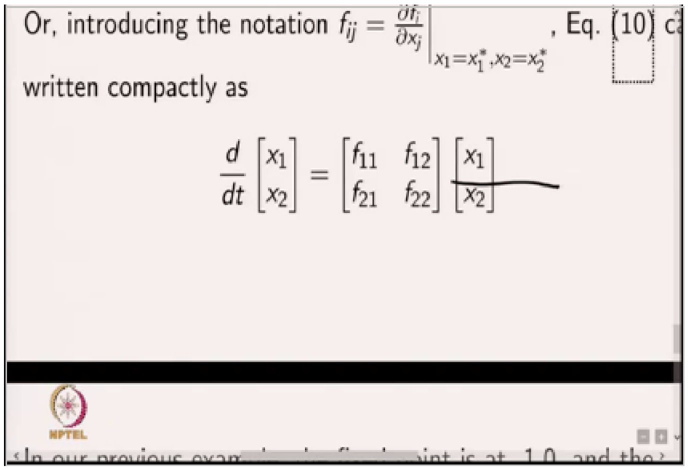
These partial derivatives, from Taylors theorem they have to be evaluated at this fixed point x 1 star x 2 star. (Refer Slide Time: 30:24)



Now, simplifying notation, noticing that both these terms are going to be 0 and the delta is of course, superfluous in this notation, so we get the linearized equation around this fixed point as d by dt of x 1, x 2 as this matrix of partial derivatives. And this matrix of partial derivatives multiplying the vector x 1, x 2 or even further simplifying our notation writing

$$f_{ij} = \frac{\partial f_i}{\partial x_j}$$

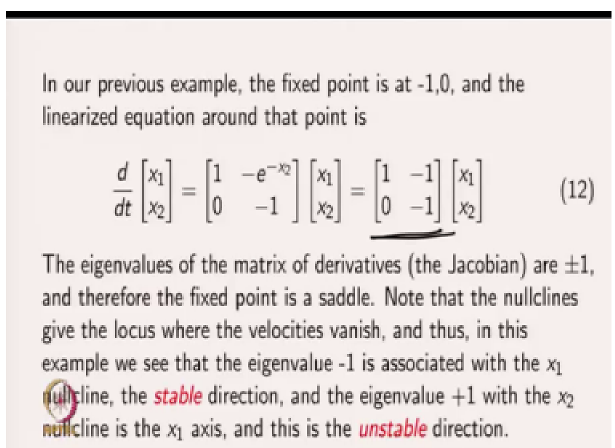
. So, f_{ij} is the partial of f_i with x_j evaluated at the fixed point. (Refer Slide Time: 31:21)



We can rewrite this entire equation compactly as

$$\frac{d}{dt}\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array}\right] \cdot \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

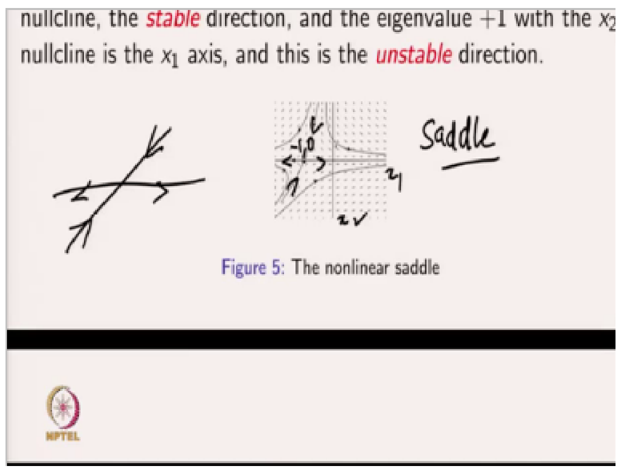
Notice that this is exactly our linear equation in two-dimensions that we started todays lecture with. (Refer Slide Time: 31:41)



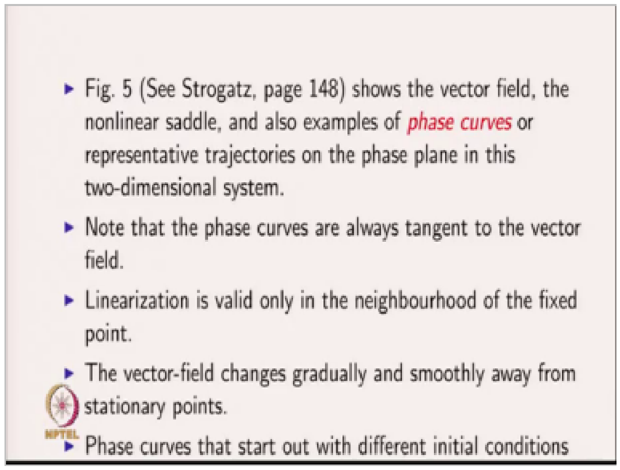
In the previous example, the fixed point was at -1 and 0 and the linearized equation around that point is obviously, given by

$$\frac{d}{dt}\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{cc} 1 & -e^{-x_2} \\ 0 & -1 \end{array}\right] \cdot \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] == \left[\begin{array}{cc} 1 & -1 \\ 0 & -1 \end{array}\right] \cdot \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

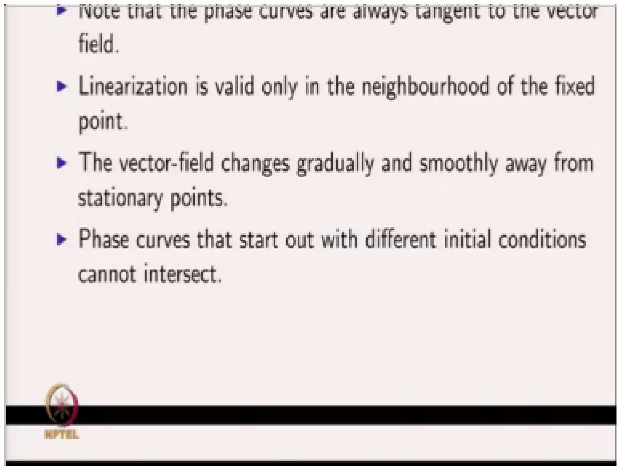
, you get this very simple matrix. Now, the eigenvalues of this matrix of derivatives which is termed the Jacobian, they are plus and minus 1 and one of them is positive one of them is negative therefore, the fixed point is a saddle. The nullclines give the curves along which the velocity is vanish and thus we can see in this example that the eigenvalue minus 1 is associated with the x 1 nullcline which is the stable direction. (Refer Slide Time: 32:53)



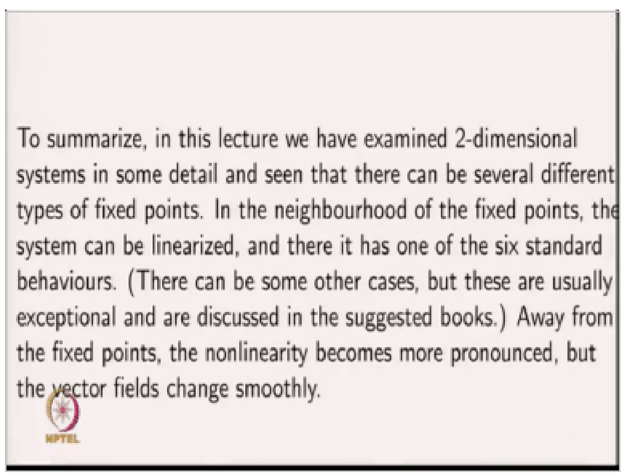
And the eigenvalue plus 1 is associated with the x 2 nullcline which is the unstable direction and this is what the flow looks like. So, expanding it a little over here, you can see that you know here is the fixed point. This is the variable x 1 and this is x 2, here is the point minus 1, 0 and this is the unstable direction, and this is the stable direction, and this is obviously, a saddle. In the linear system, the saddle would have just been a set of straight lines with one unstable direction and one stable direction. Because the system is non-linear we can see that this you know this stable direction actually curves and the further you go away from the fixed point, the further you know the region of linearization is such, the further you go away from that place where the linear linearization is a good approximation, you are going to be able to see the curvature in the nullclines. (Refer Slide Time: 34:10)



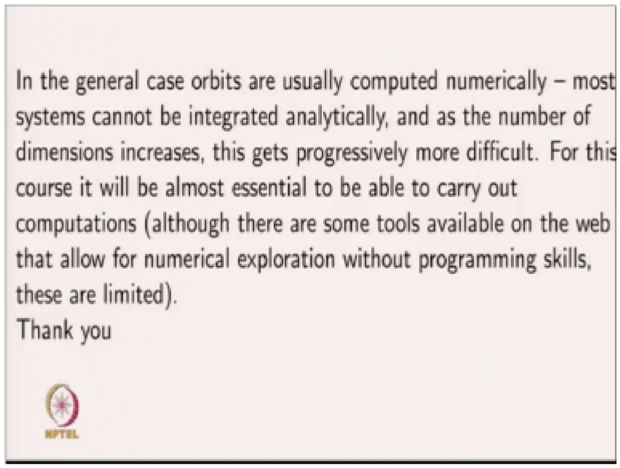
Now, this image is taken from Strogatzs book. And it shows the vector field the non-linear saddle and in addition to the vector fields which are these arrows that you see over here and the non-linear saddle you also see some examples of phase curves, namely what would happen if I started at a particular point and moved on and integrated the equations of motion. Now, these phase curves are representative trajectories on the phase plane in this two-dimensional system. It is interesting to note and you can verify it for yourself that the phase curves are always tangent to the vector field, ok. I have already pointed out that linearization is only valid in the neighbourhood of the fixed point and further away from the fixed point these you know the linear approximation will break down and the lines could become non-linear. Nevertheless, there is an important point to remember that it is only at these stationary points that this behaviour which is describable can change in any drastic manner. The vector field itself can only change smoothly and gradually as you move outwards and any abrupt change in motion will have to will require additional stationary points. (Refer Slide Time: 35:50)



A point that I should you know note, is that phase curves that start out with different initial conditions cannot intersect in any number of dimensions. So, if I start with a point over here and I move on this particular curve, sorry if I start with a point over here and move along this particular curve start with another point this behaviour is not possible. So, through any point in the phase space there is one and only one trajectory, ok. (Refer Slide Time: 36:32)



So, to summarize, in this lecture we have examined two-dimensional systems in some detail. We have looked at flows and we have seen that there can be several kinds of fixed points. In the neighbourhood of the fixed points the system can be linearized and there it has one of the 6 standard behaviours. I have already pointed out that there can be some other cases which are the marginal ones, these are exceptional and are discussed in the suggestive books. Away from the fixed points the nonlinearity becomes more pronounced, but the vector fields themselves will change smoothly. (Refer Slide Time: 37:14)



In the general case, orbits are usually computed numerically. Most systems cannot be integrated analytically and as the number of dimensions increase this will get progressively more and more difficult. For this course it will be essential in order to do any of the homeworks that you should be able to carry out computations and although there are some tools available on the web that allow for numerical exploration without programming skills, these are limited. So, you would be well advised to pick up one, anyone computing language. That is about it for today. Thank you.