Introductory Nonlinear Dynamics
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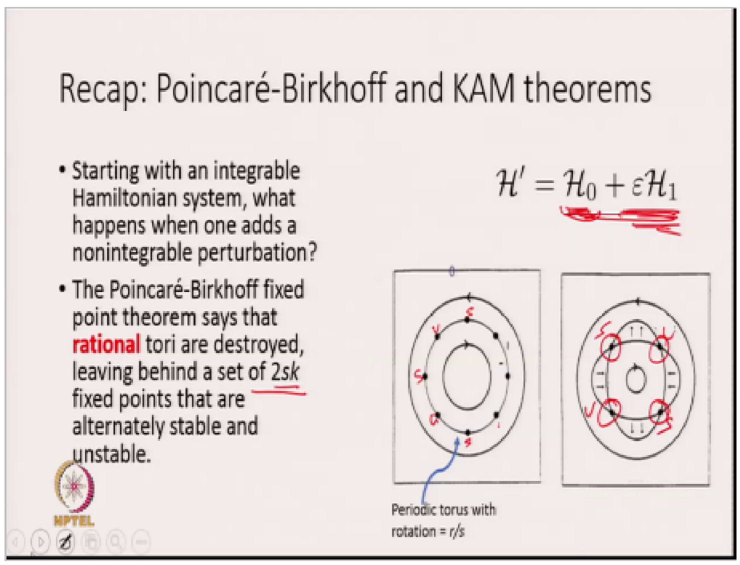
Lecture - 15
Hamiltonian Chaos 2.

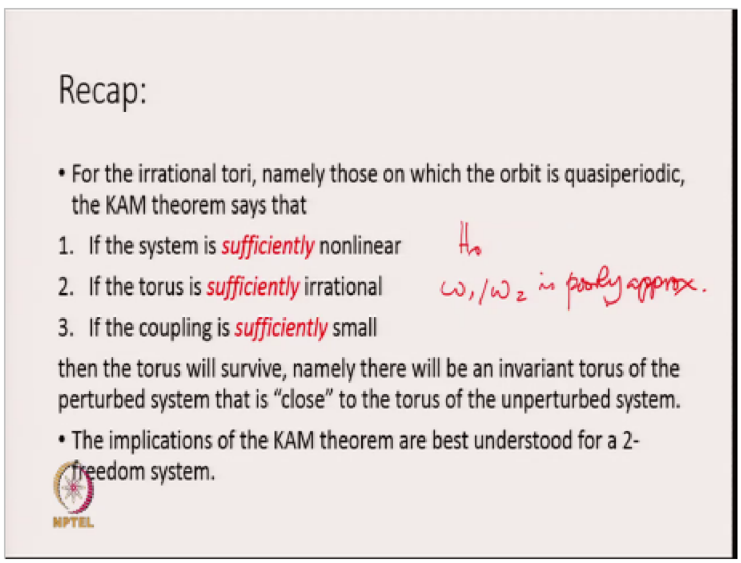
This is the second lecture on Hamiltonian Chaos and in the previous lecture we had discussed the Poincare-Birkhoff and KAM theorems. (Refer Slide Time: 00:27)



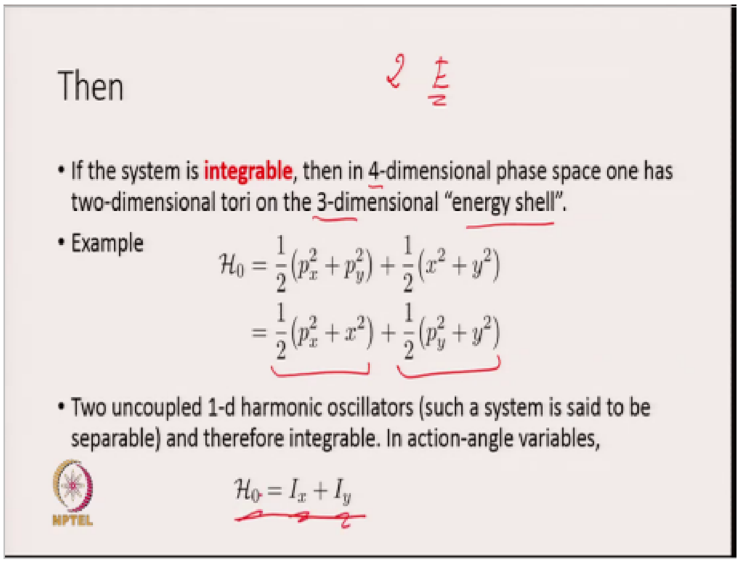
The Poincare-Birkhoff theorem and the KAM theorem addressed the fate of Hamiltonian systems when non integrable perturbations are added to an integrable part. When you have an integrable Hamiltonian and it is denoted here by H_0 then all the motion lies on tori because this is integrable and the entire phase space is covered by tori each of which is indexed by particular values of the actions.

$$\tilde{H}' = \tilde{H}_0 + \varepsilon \tilde{H}_1$$

To this integrable part if we should add a non integrable term such that the whole Hamiltonian is now non integrable, then the Poincare-Birkhoff says something about those tori on which the dynamics was periodic namely the rational tori. What the Poincare-Birkhoff theorem says is that these rational tori are destroyed and they leave behind a set of 2 times s times k fixed points that is an even number of fixed points alternately being stable and unstable. So, you have a stable fixed point followed by an unstable and so, on and so forth. There is even number of them and how many of them depends on what the what is the winding number or what are the ratios of the 2 frequencies on which correspond to the 2 different directions on this torus. The example over here and many of the examples we will talk about for 2 degree of freedom Hamiltonian systems and that means, that the sections the Poincare sections that we are examining are 2 dimensional. So, starting with a rational torus marked here once you add this non integrable perturbation this rational torus evolves in a particular way so, that all that we are left behind are these fixed points 2 of which are stay unstable and 2 of which are stable. (Refer Slide Time: 03:23)



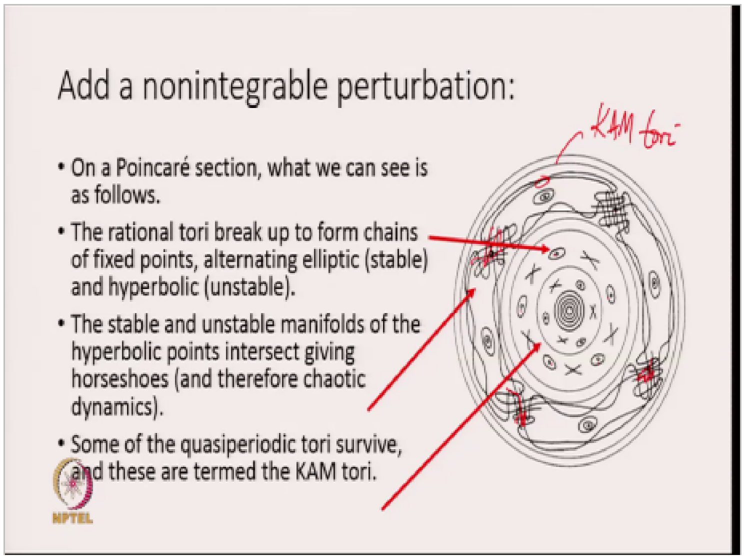
The KAM theorem addresses the quasi periodic or Tori namely those tori on which the frequency ratios are irrational and this is the vast majority of tori even in the unperturbed system what the KAM theorem says is that, if the sub the system is sufficiently non-linear that is to say H naught itself is sufficiently a non-linear system if the torus is sufficiently irrational by which we mean that the ratios of the frequencies omega 1 by omega 2 is and is sufficiently irrational or is poorly approximated by rational numbers. So, that is poorly approximated by rational numbers and if the coupling is sufficiently small, then the torus will survive in the perturbed system or in other words there will be an invariant torus of the perturbed system that is in a sense close to the torus of the unperturbed system and as the perturbation term vanishes, these will merge into each other. The implication of the KAM theorem are best understood for a 2 freedom system. So, and we will turn to that now. (Refer Slide Time: 05:03)



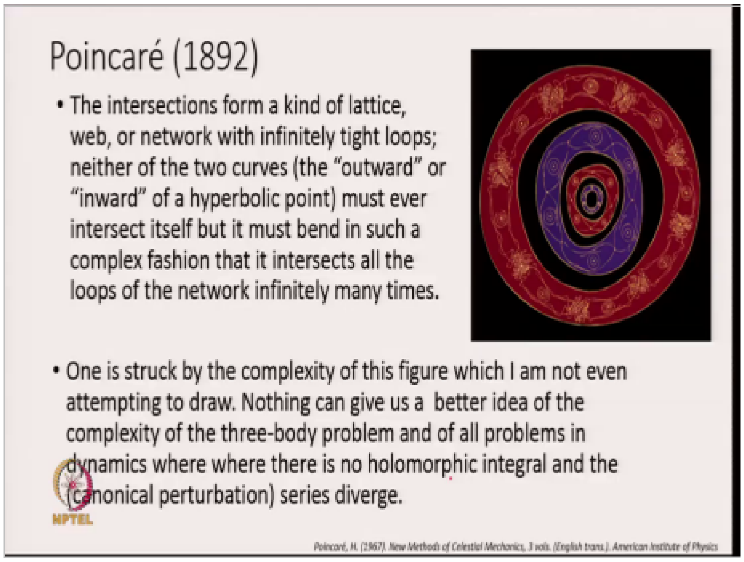
If a 2 degree of freedom system or a 2 freedom system is integrable, then note that when the degrees of freedom are 2 the phase space dimension is 4; it is a 4-dimensional phase space. If there is a Hamiltonian and you have a conserved quantity like the energy one conserved quantity in this 4 dimensions automatically constraints everything to a 3-dimensional energy shell. And in this phase space we have tori if the system is integrable a good example is given by 2 by a 2 degree of freedom harmonic oscillator. So, I write down the Hamiltonian as a

$$\tilde{H}_0 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2)$$
$$= \frac{1}{2}(p_x^2 + x^2) + \frac{1}{2}(p_y^2 + y^2)$$

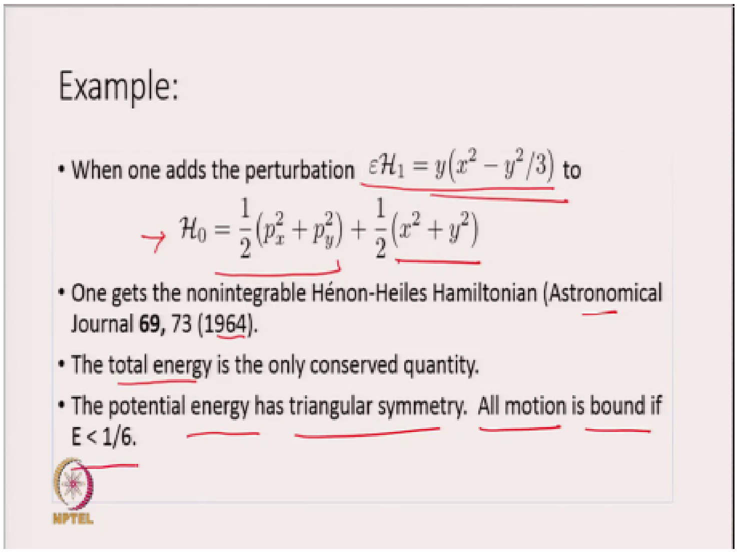
and rearranging the terms you will note that this is written as a harmonic oscillator in the x variable and the harmonic oscillator and the y variable, which are uncoupled from each other and is separable and because it is a separable system it is also integrable. In action angle variables you can easily show that in the action variable I_x and I_y, H_0 just becomes $H_0 = I_x + I_y$. (Refer Slide Time: 06:31)



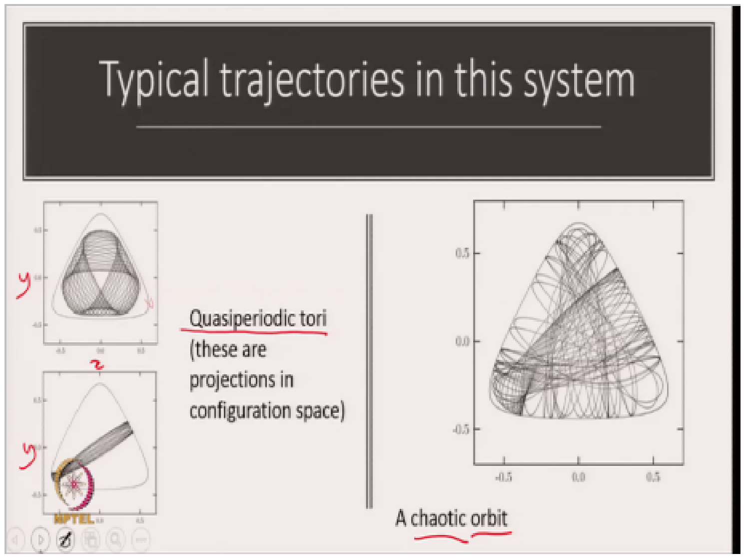
If to this we were to add a non integrable perturbation on a Poincare section what we can see is the following. The rational tori will break up to form chains of fixed points alternating elliptic and namely stable fixed points and hyperbolic or unstable fixed points you can see that over here where you have got the elliptic fixed points over here and the hyperbolic ones marked out by xs different rational tori seem to have broken up over here. So, there is another rational torus that is broken another one over here and one over here. For this outermost ring of stable and unstable fixed points, we have also drawn in the stable and unstable manifolds of the hyperbolic fixed points over here and as is typical in such systems, it is possible for the stable and the unstable manifolds of different fixed points to intersect and because they have intersected once they must intersect infinitely often giving rise to this tangle over here or over here around every hyperbolic fixed point there is a hyperbolic tangle. Some of the quasi periodic tori are surviving and these are the solid curves that you see over here recall that this is the Poincare section of the flow from the 4 dimensional system to the 3-dimensional energy shell and this is a slice across the 3 dimensional energy shell. The tori that survived are often termed as KAM tori. (Refer Slide Time: 08:27)



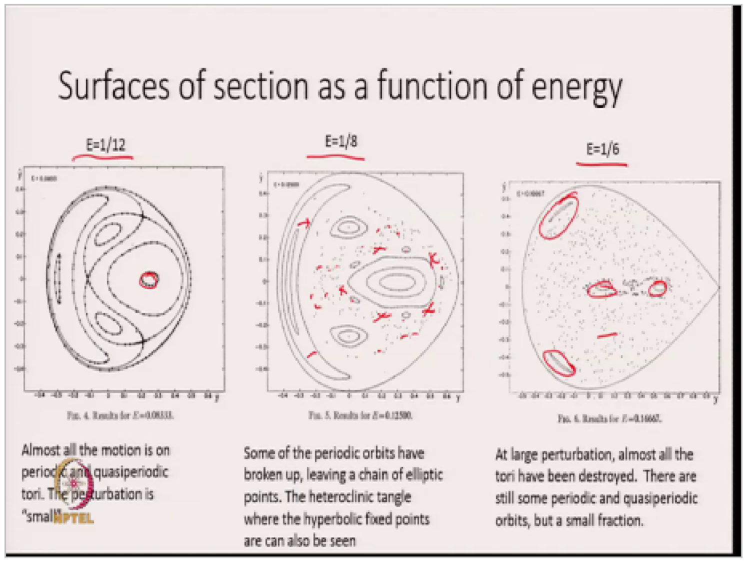
Poincare himself in .. 1892 was aware of this image even though he could not quite draw it he writes in the new methods of celestial mechanics it is a translation, he says the intersections form a kind of lattice web or network with infinitely tight loops neither of the 2 curves this unstable manifold of the hyperbolic point or the stable manifold must ever intersect itself but it must bend in such a complex fashion, that it intersects all the loops of the network infinitely many times. One is struck by the complexity of this figure which I am not even attempting to draw nothing can give us a better idea of the complexity of the three-body problem and of all the problems in dynamics, where there is no holomorphic integral and the canonical perturbation series diverge. The absence of a holomorphic integral over here essentially refers to the fact that the system is non integrable. (Refer Slide Time: 09:45)



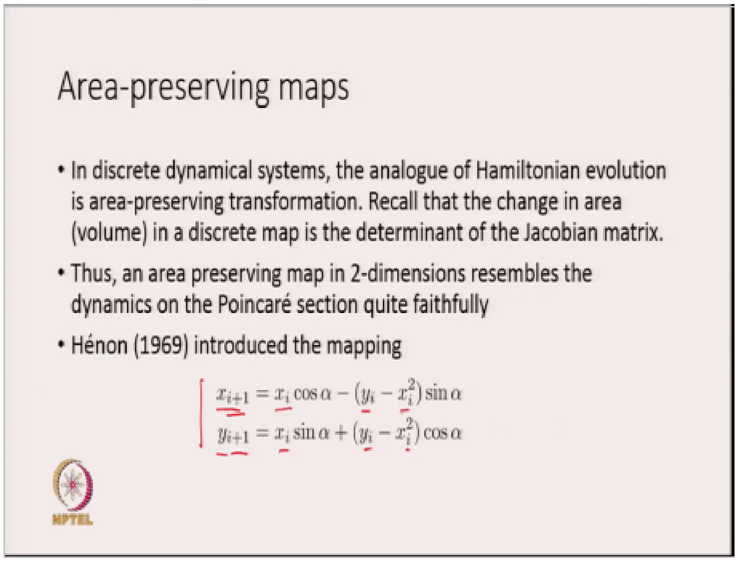
So, to the system of 2 harmonic oscillators that we have started with. If we now add this particular perturbation which is $\varepsilon\tilde{H}_1 = y(x^2 - y^2/3)$, one gets a very famous nonintegrable Hamiltonian system that was first written down by Mitchell Henon and Carl Heiles in the Astronomical Journal in 1964. Very shortly after Lorenz study of the Lorenz attractor in 19 in 1963. This Hamiltonian system has in many ways played the same role as the Lorenz system in that the Lorenz system plays in the study of dissipative systems and dissipative chaos the Henon-Heiles Hamiltonian plays a very similar role in Hamiltonian chaos. For this system there is only one truly conserved quantity and that is the total energy. Short calculation will show you that the potential energy which is these terms x square plus y square and this additional perturbation term all this is purely potential and all this is purely kinetic. This potential energy has triangular symmetry and all the motion is bound if the energy is $E < 1/6$. (Refer Slide Time: 11:17)



In 1964 Henon and Heiles Made calculations and they were able to see orbits in this particular system and they found 2 different kinds of orbits. One was a set of quasi periodic tori and here are images of this quasi periodic tori with the potential energy surface drawn over here. So, this is in the xy plane, here are images of tori and you can see the very smooth a very well behaved well ordered curves over here, what is distinguishing about the tori are these caustic curves where the trajectories all focus along certain points and this is very typical of motion on a torus, note that this is a motion on a torus in a 3-dimensional space, which is being projected now onto 2. But what you see these caustic curves are very characteristic and are a way of identifying toroidal motion. In contrast you have a motion that seems to be near a torus for a little while and then it just seems to escape and go all over the place and this is a an orbit, which is termed chaotic because you can compute the Lyapunov of exponent using the methods that we have discussed earlier and we find that you have a positive Lyapunov of exponent but again you have chaotic motion in this Hamiltonian system. (Refer Slide Time: 13:01)



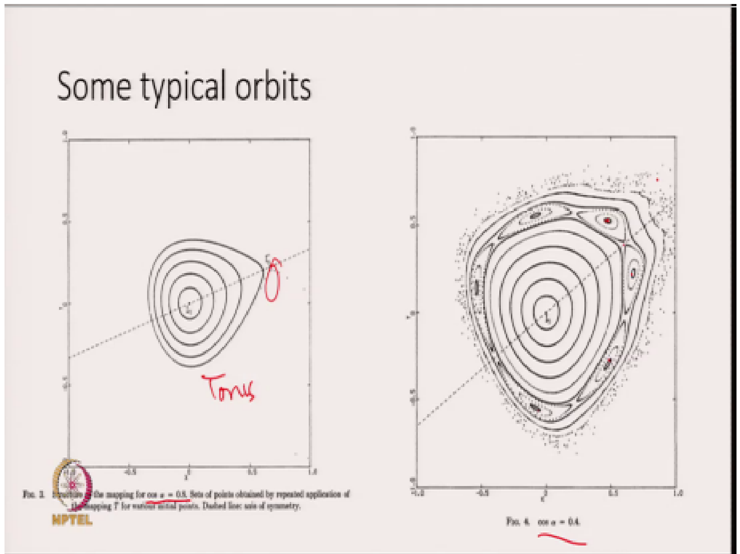
If you were to compute surfaces of section as a function of energy and these are images taken from Henon-Heiles original paper. Again recall that we are doing a surface of section of a 2 freedom system, 4-dimensional phase space, 3-dimensional energy shell and you are taking a surface of section to bring the whole image of the motion onto 2-dimensions. When you have tori all over the place, then when you cut through this set of tori you basically find nested tori one inside the other and at very low energy, E is 1 by 12 you find that the motion is completely covered by tori everywhere, the image of the tori on the Poincare section is a closed curve. So, wherever you look you find closed curves at a higher energy of 1 by 8, you find that now some of the rational tori have broken up leaving behind the stable and the unstable points. Surrounding the stable points are also these tori these small islands that you see over here and the unstable points would have been somewhere over here, but because of the hetero clinic tangle that we saw in the earlier picture you do not see those hyperbolic fixed points anymore and what you have is just the image of chaos all over the place. You have the remnants of KAM tori you see some of them over here. As you increase the energy even higher and this is essentially the highest energy for which all the energy all the motion is bound, you find that the phase space is really largely covered by chaotic orbits that is points on an orbit that do not follow any curve although there are some KAM tori you can see them over here and here and here, but by and large the entire motion is dominated by chaos, chaotic dynamics and what you see is chaos everywhere, chaos in a limited region over here and mostly tori everywhere over here. Integrating equations of motion in Hamiltonian systems is not particularly difficult because you know the Hamiltons equations for the system. (Refer Slide Time: 15:57)



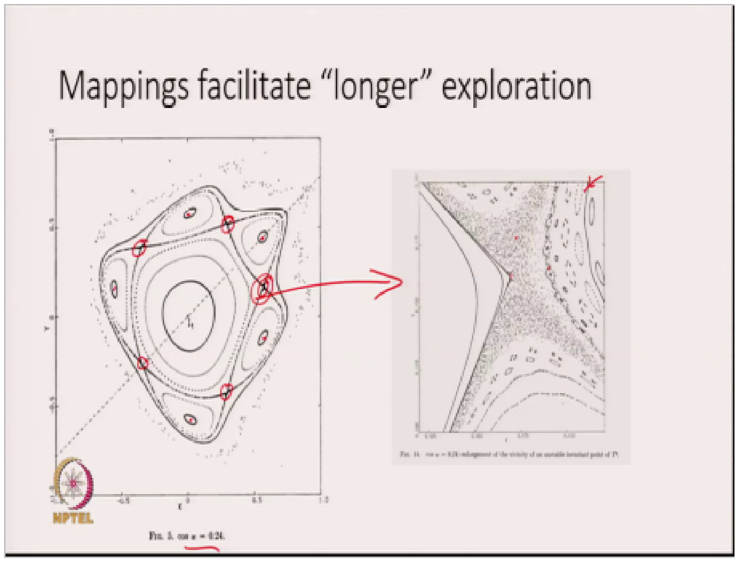
But it is possible to integrate these equations for much longer times, if we work with mappings rather than differential equations. In discrete dynamical systems and a lot has been studied in this area, the analog of Hamiltonian evolution is area preserving transformation. Recall that in 2 dimensions early in one of the lectures of this course I have shown and discussed that the change in area in a discrete map is essentially the determinant of the Jacobean matrix. So, area preserving maps in 2 dimensions resemble the dynamics on the Poincare sections of this 2 freedom system quite faithfully. Mitchell Henon again in 1969 and actually also in the Henon-Heiles paper introduced mappings to mimic Hamiltonian dynamics and here is one such map which shows some features of the dynamics particularly well, this is a 2 freedom map. So, you have

$$x_{i+1} = x_i cos\alpha - (y_i - x_i^2) sin\alpha$$
$$y_{i+1} = x_i sin\alpha - (y_i - x_i^2) cos\alpha$$

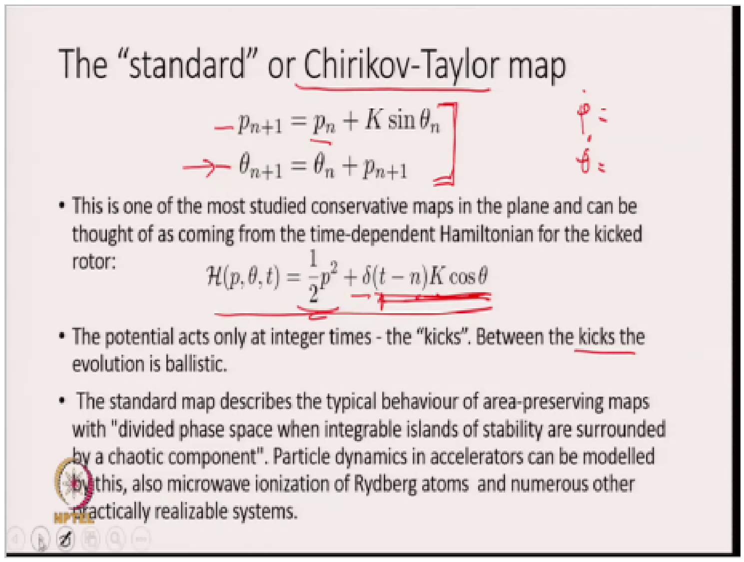
iteration through this particular relationship and you can easily verify that the Jacobean of the transformation has determinant 1. (Refer Slide Time: 17:45)



This dynamics is very interesting and very similar to the Henon-Heiles surface of section that we saw for different values of alpha ah. So, here is the case where cosine of alpha is equal to 0.8 and you find that all the points in this region are lying on smooth closed curves very much like the tori torus motion that we had in integrable systems. The mapping is not integrable by any stretch because there you can see that there are orbits that move away but certainly you can see the image of toroidal motion over here. As you change alpha and here the changes to cosine alpha is equal to 0.4, you find this familiar image of stable and unstable points alternating on a ring. So, here you have a an elliptic fixed point, a hyperbolic fixed point, an elliptic hyperbolic elliptic hyperbolic and so, on and so forth. Then there are orbits that escape to infinity and that is a feature of this map. Very interestingly you can also see the effect of the homo of the hetero clinic tangle. (Refer Slide Time: 19:01)



Here is an image of the mapping for cosine alpha is equal 2.24 and here you find that the elliptic fixed points and you will see them really at the center of these islands over here and the hyperbolic fixed points which you see over here. Now the hyperbolic fixed point you cannot see it itself because it is an unstable fixed point, but you can see the effect of the heteroclinic tangle if you zoom and this I think is the zoom of this particular point over here yeah of this point over here this get zoomed out and what you see is that, the image of this heteroclinic point is somewhere over there that that location over there and then every point over here you see is really an image of this heteroclinic tangle. You can see KAM tori, you can see other chains of islands and you can see that between these chains of islands also there are heteroclinic orbits and the tangle over here. So, there is tangles everywhere, this is very much an image of what has happened to the rational torus that were surrounding this point, having broken up and then it has resulted in all this chaotic dynamics. (Refer Slide Time: 20:33)



One can study these maps many of these maps, but the standard or the Chirikov-Taylor map has played a very important role in a lot of discussion on Hamiltonian chaos.

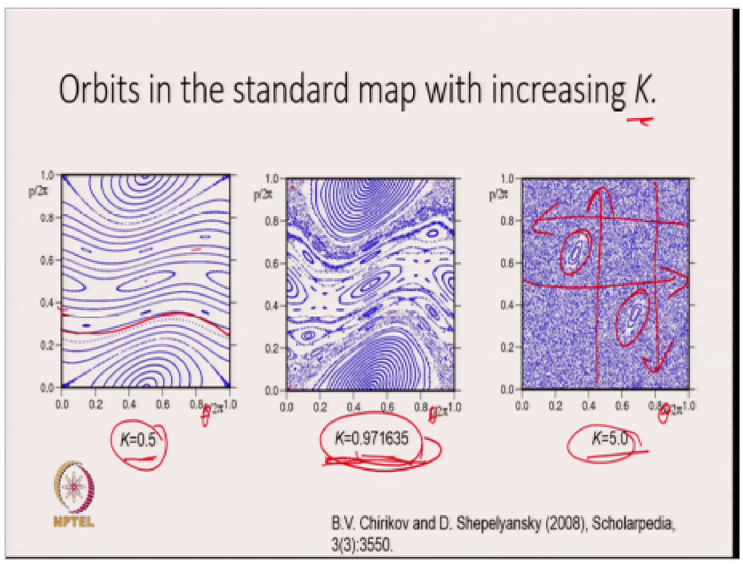
$$p_{n+1} = p_n + Ksin(\Theta_n)$$

$$\Theta_{n+1} = \Theta_n + p_{n+1}$$

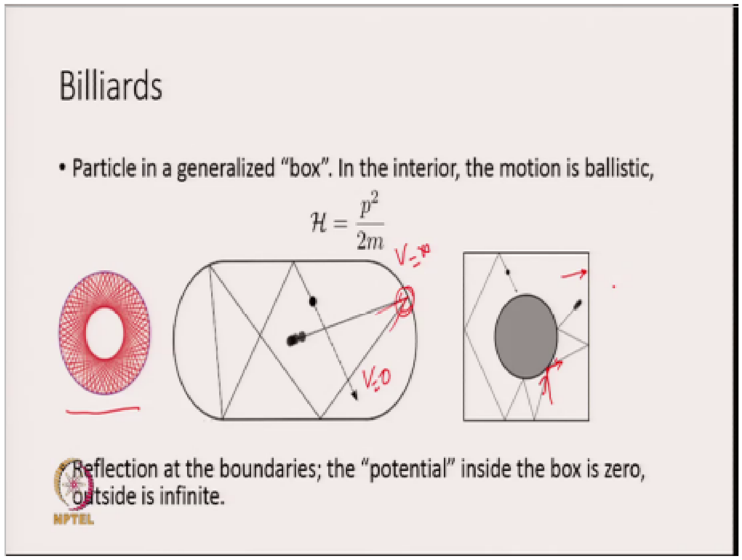
This is a another 2-dimensional mapping in loosely speaking action angle variables except that now we have got you know this is like a momentum variable and this is an angle. So, this actually can be shown to come from a Hamiltonian system which is a time dependent Hamiltonian system and here the ok.

$$\tilde{H}(p.\theta, t) = \frac{1}{2}p^2 + \delta(t-n)Kcos(\theta)$$

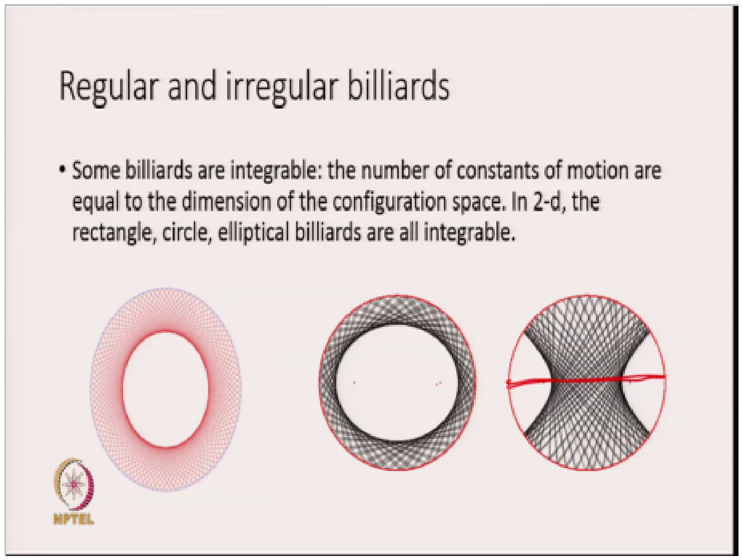
So, the potential energy which is $Kcos(\theta)$ that actually acts only at times at integer times. So, whenever the time is equal to n; n is 0, 1, 2, 3 etcetera integers, this potential this potential energy acts up because of the delta function and between the kicks the motion is purely ballistic it is just a half p squared, the Hamiltonian is a half p squared. So, you just have got straight motion. If you were to write down the corresponding Hamiltonian equations for p dot and for theta dot from this Hamiltonian, you would and then you proceeded to discretize it the following equations emerge, this is a mapping now and this mapping is actually a fairly faithful representation of this dynamics because the potential energy acts only at integer times. So, the momentum in a sense can only change at integer times and the angle evolves smoothly between the kicks. So, there is no so, that is this evolution between the kicks the evolution is ballistic. Now, this standard map describes typical behavior of area preserving maps with a divided phase space I am going to come to that in the next slide, and it is of great relevance to particle dynamics and accelerators also to microwave ionization of Rydberg atoms numerous are the very practical realizable experimental systems. The dynamics can be of course, you know in integrated very simply you start with the whole set of different initial conditions and you just evolve them under this map and what you see is as a function of k the dynamics the image of the dynamics is very different. (Refer Slide Time: 23:31)



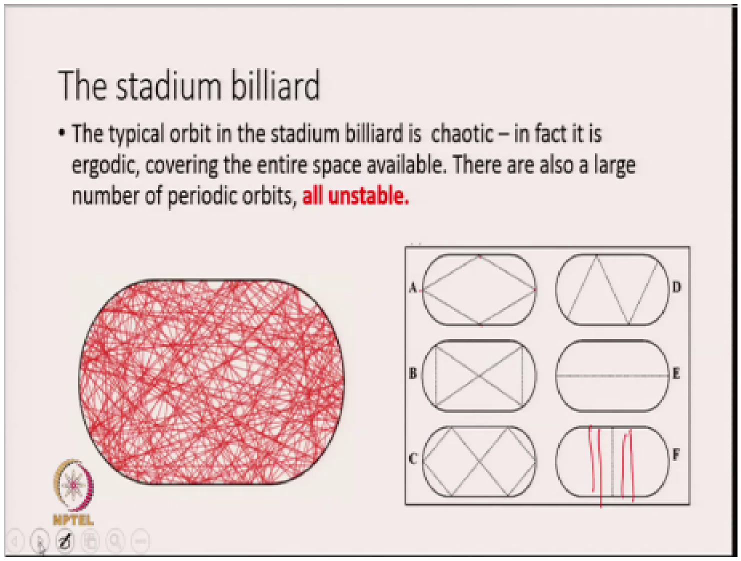
This map is defined modulo 2 pi. So, all the variables are just divided by 2 pi in this diagram over here ah. So, everything this is theta going from 0 to 1 right and this the images themselves are taken from Chirikovs paper. For low k, k is equal to a half you find that the if I start with an initial condition over here, it will trace out a smooth curve that goes that traverses the square from one side to the other. Most of the curves are smooth joining one on you know joining it back onto itself occasionally, you have these islands that you can see over here these are remnants of rational tori and one must remember that k not equal 0 is immediately a non integrable system over here, because if you go back and look at it. If k is equal to 0 this term does not exist and p n plus 1 is a conserved quantity because p n plus 1 is just equal to p n. As you increase k to this special value, you find that this is the point where orbits are of they have this special feature that an orbit starting over here and let us say remains within this particular band and orbits that start over here remain within this particular region even though they are completely chaotic, they do not traverse this square from top to bottom. So, this is the divided phase space that we refer to in the case of the standard map. And the reason is that there is a curve which is traversing the square from left to right which in a sense prevents traversal of orbits from bottom to top. At this value of k the last such curve just disappears. Notice that this is a curve that is traversing from left to right as is this as is this as all these. So, for small k you have a very large number of curves that traverse from left to right and prevent top to bottom motion. At k equals 0.97 whatever you find that the last such curve has just disappeared and for larger k now orbits can move freely from top to bottom or from left to right, they can just move everywhere every which way because there are no curves that bound the motion to remain in any particular region. They still are KAM tori even at the higher it at higher nonlinearities and this is a feature of many many chaotic systems ok. The standard map has been very important because it is possible to calculate this last value of k analytically this is in a sense in a helping people understand exactly how widespread chaos comes to be what are the what are the things that go into the breaking of tori and how the KAM theory works in practice. Another class of systems that has helped in the understanding of chaotic dynamics in Hamiltonian systems particularly has been billiards. (Refer Slide Time: 27:33)



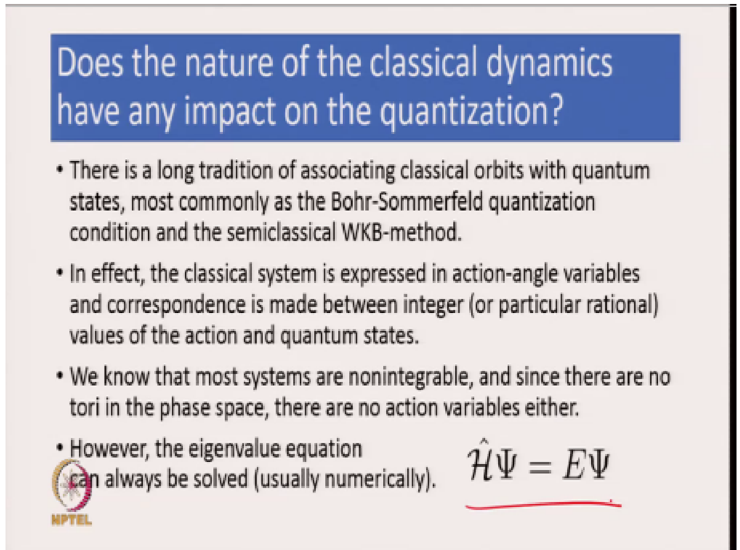
Now, billiards are a generalization of the particle in the box problem, where basically you have a particle that is moving freely inside an enclosure and when it comes to the edge of the enclosure, you find that it gets scattered from the edge of the enclosure with the angle of incidence equal to the angle of reflection all right. So, the potential energy inside the box is equal to 0 and the potential outside is infinity and some billiards have been studied in great detail helping us to understand the difference between chaotic and non-chaotic dynamics. What has shown over here is a circle billiards. So, you have a particle inside a circular enclosure this is a particle inside a square except that there is a scattering a scatterer in the middle of this square. So, the particle not only reflects from the boundaries of the of this enclosure but also it reflects from the from this scattering element over here right. This is known as the scene I billiard and this is in the shape of a stadium and so, it is called the stadium billiard. (Refer Slide Time: 29:01)



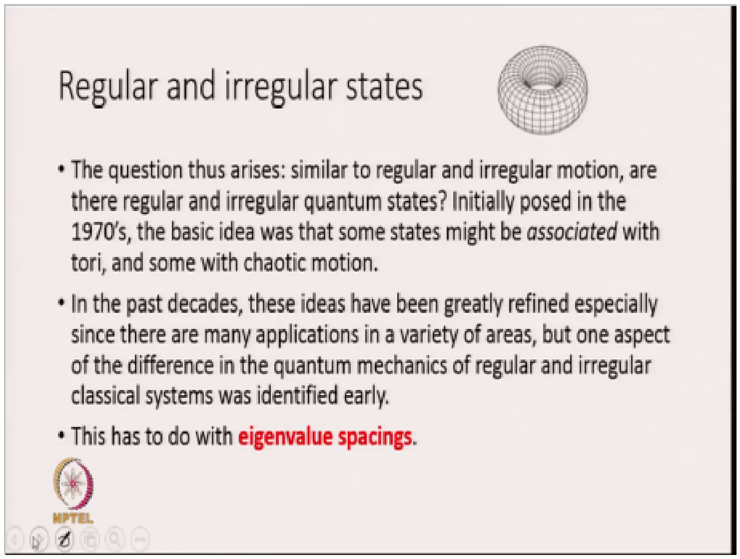
Now, billiards can be regular or irregular in particular some billiards are integrable and the number of constants of motion are equal to the dimension of the configuration space. Now in 2 dimensions some familiar billiards are the rectangle billiard which also is a real physical billiard when you play the game, a circle billiard and the elliptic billiard these all turn out to be integrable. What I have shown over here are orbits inside a circular billiard and you can see that the circular billiard now the caustics of the motion that is to say the shape which is formed by the envelope of the trajectories, the caustic of the motion is another circle. Likewise, for an ellipse there are some orbits that form the that have a caustic of an ellipse, but there are also other orbits many other kinds of orbits in particular there are orbits whose caustics form a hyperbola and these 2 points over here are the foresight of this particular ellipse there are orbits in addition that will just bounce back and forth along that particular point over there. So, there are many different kinds of orbits that are possible. In contrast to the integrable billiards there are non integrable billiards as well and the stadium is an archetypical example. (Refer Slide Time: 30:39)



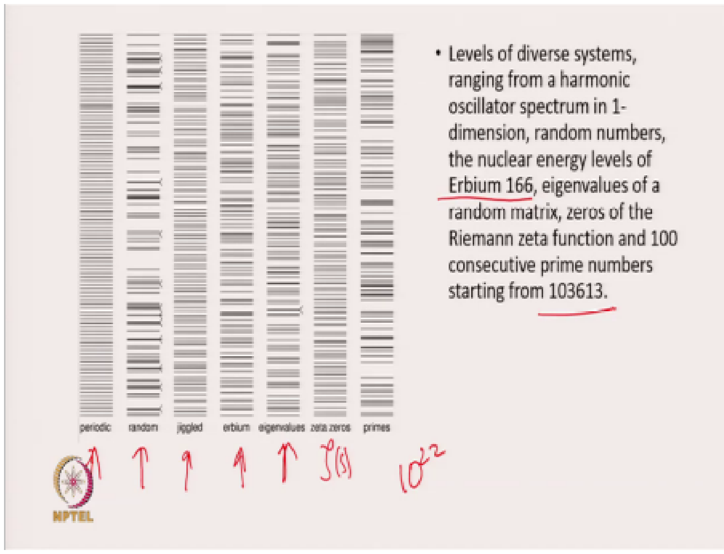
A typical orbit in the stadium billiard is completely chaotic, it just goes all over the place. In fact, if you give enough time it is also ergodic it will go over the entire space available. And this is in sharp contrast to let us say something like the circular the circular billiard because wait; however, long you want the central region is never going to be filled in by any orbit. In addition to a chaotic orbit such as the one that is shown over here, we also have a host of periodic orbits for example, there is one that bounces back and forth between the midpoints over here there are an infinite number of orbits that will bounce back and forth between these walls. In fact, they have a name they are called the bouncing ball modes. Then you have an orbit that will just bounce from the midpoint of this side to this side to this side to this side like a diamond over here and an orbit that looks like a z and a figure of 8 and so, on and so, forth. These are all periodic orbits because they retrace themselves nevertheless all of them are unstable and this is a characteristic feature of these kinds of systems all right. (Refer Slide Time: 32:05)



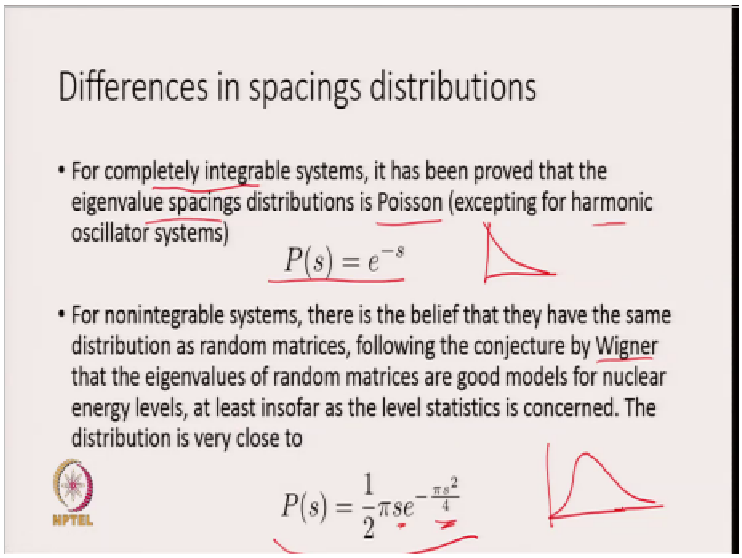
This brings me to the topic that I want to discuss at the very end of this introductory set of lectures, because a lot of physics deals with classical dynamics of course, but more relevant question to ask is to do with the quantum mechanics of such systems. Does the nature of the classical dynamics have any impact on the equivalent or the related quantum phenomena on the quantization? Now part of the reason for asking a question like this is historical because there is a long tradition of associating classical orbits with quantum states. In fact, in most early treatments or the earliest kind of treatment that the typical student is exposed to is Bohr-Sommerfeld quantization where a specific kind of orbit is associated to the quantum state or further down the line when you see the semi classical WKB-method that is also a way of associating classical and quantum mechanics. What is done in those cases is that the classical system is effectively expressed in action angle variables and correspondence is made between integer values of the action or specific half integer values or any other there is some other cases but correspondence is made between the integer values of the action and specific quantum states. Now, in order for that to happen you must have action angle variables and that must mean that you have tori but we know that most systems are non integrable this is the discussion we have had in the last 2 lectures. And if there are few tori or no tori in the phase space then there are no action variables that you could construct easily anyhow. However, you can take a completely different view of quantum mechanics and say that the quantum mechanics, the quantum Schrodinger equation over here can always be solved even if $\hat{\hat{H}}\Psi = E\Psi$ it is a numerical solution you can always solve the quantum equation over here and get the associated eigenvalues and eigenfunctions. (Refer Slide Time: 34:47)



Now, this led in the 1970s to the question are there regular and irregular states? Namely we know that there is regular and irregular motion in Hamiltonian systems. The regular motion is associated with tori the irregular motion the chaotic motion does not have tori is there some connection between states that are associated with tori and are there other kinds of states that are associated somehow with chaotic motion? Now this was an early discussion of the problem, but over the years there is been a lot of work in this area and these ideas have been greatly refined especially because there are a whole lot of applications. But one aspect of the difference between quantum mechanics of regular systems and irregular systems was identified very early and this has to do with eigenvalues spacings. (Refer Slide Time: 35:57)



Looking at the spacings of many quantities has been an area which has been if interest and here is a here is an image from which I picked up from the net from somewhere which shows a lot of different different levels ok. Over here is just this bunch of lines that are equally spaced these could be the energy levels of a harmonic oscillator for example, equally spaced energy levels. If one just had purely random levels then this is what a typical realization of random levels looks like. If I take the periodic spacing over here the harmonic oscillator and then just jiggle it a little add a little noise you get something that looks like. So, begins to look like a barcode, but right next to it is the end eigenvalue spectrum the levels of a radioactive nucleus Erbium 166 these are the nuclear energy levels of Erbium 166. Here are a set of eigenvalues of a matrix which is got completely random entries it is a symmetric matrix but it is got completely random entries and though in one half of it and here are the eigenvalues written down. So, this is the Riemann zeta function and here are the zeroes of the Riemann zeta function starting from the 10 to the 22th zero upwards. So, you can see how they are spaced and here finally, are a 100 consecutive prime numbers starting from 103613 and that is what the spacing looks like. Of course, these are all normalized such that you have a 100 levels in each of these columns but you can see that the kind of system and the kind of levels seemed somehow to be related or rather it is very characteristic. Here is a harmonic oscillator, absolutely equal random and various levels of correlation between these different kinds of levels within the same system right. Where do chaotic and non chaotic systems fall in this in this classification? And note that chaos or lack of you know regularity or integrability and chaos terms that come from classical dynamics and we are asking the question of what happens to the quantum mechanics of such systems. (Refer Slide Time: 39:03)



Well attention was drawn largely by Michael Berry and E.M Percival to the spacings distribution in those days and it was proved that for completely integrable systems, the eigenvalue spacing distribution is Poisson leave out harmonic oscillator systems because they are exceptional

$$P(s) = e^{-s}$$

but if you take a typical integrable system you would find that the eigenvalues have got a Poisson distribution which is basically just a decaying exponential and the highest probability is to have spacing of 0 or levels are clustered very closely together. For nonintegrable systems on the other hand, there is a belief that they have the same distribution as random matrices in some sense the complexity of the classical dynamics shows up in than matrix elements that are completely random and this goes back to an observation a conjecture by Eugene Wigner on the that the eigenvalues of random matrices should provide a good model for nuclear energy levels at least as far as the statistics of the levels were concerned. Wigner surmise that the distribution would be something like this, this is basically a Gaussian but it is shifted it is sort of it is multiplied by s. So, this actually is a distribution that looks something like so, ok. So, you have a distribution which is Poisson or just a decaying exponential or a Wigner distribution and the conjecture was that for integrable systems one would find Poisson for chaotic systems you would find the you would find the Wigner distribution.

$$P(s) = \frac{1}{2}\pi s e^{-\frac{\pi s^2}{4}}$$

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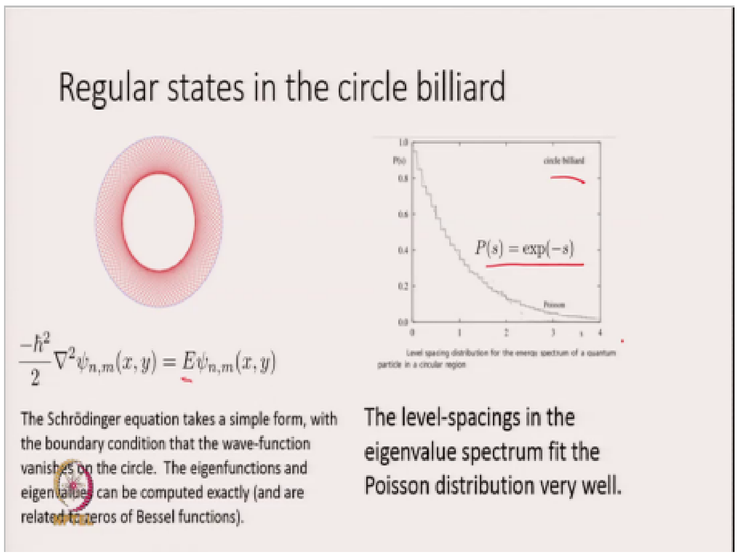
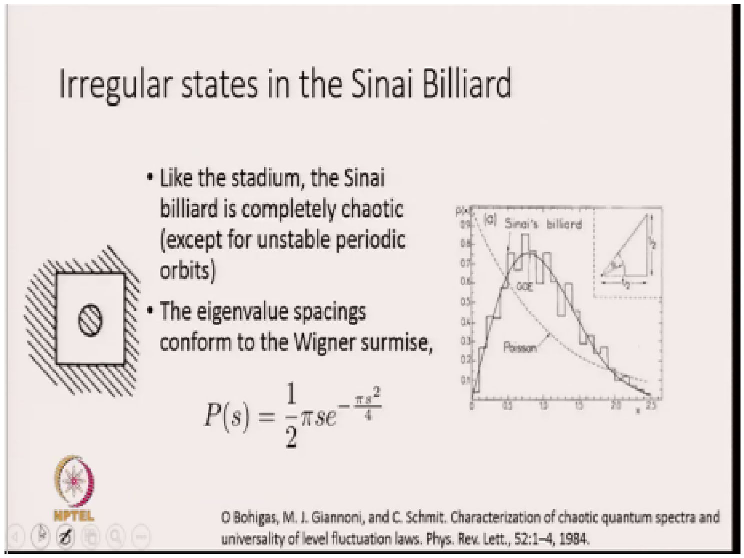


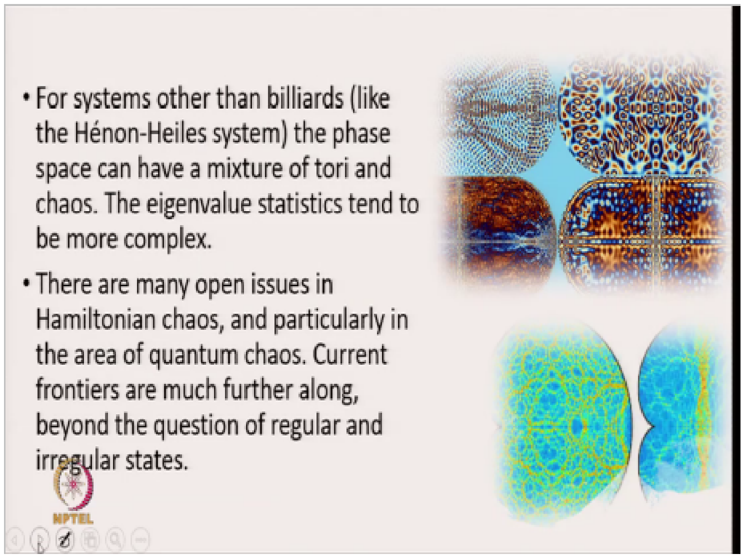
Well if you take a circle billiard it is an integrable billiard, you can write down the Schrodinger equation and that is it is in 2 dimensions.

$$\frac{-h^2}{2}\bigtriangledown^2 \psi_{n,m}(x,y) = E\psi_{n,m}(x,y)$$

So, you just have the this is the kinetic energy or and this is here is the this is the eigenvalue equation and the boundary condition of course, is that the wave functions have got to vanish on the outer circle over there and this is a well studied problem and it can be solved exactly; the eigenfunctions and the eigenvalues can be computed exactly and they are related to zeros of the Bessel function now if you calculate about you know 1000 such levels in the circle billiard and you form a distribution you here is a histogram of the distribution it follows the Poisson distribution very well. So, here you find this is an experimental result if you like, here are a hundred thousand levels of the circular billiard and it follows the Poisson distribution rather well. (Refer Slide Time: 42:23)



If you take the Sinai billiard as Bohigas Giannoni and Schmit did in 1984 then you can again solve the Schrodinger equation in this particular enclosure here there are 2 boundary conditions to be followed, the wave functions have to vanish on the square and they have to vanish on the boundary of the inner deflector this is a completely chaotic system there are no periodic orbits that are stable and so, there is no question of having some mixtures of tori and chaos. So, the eigenvalue spacings for this they really conform to the Wigner surmise which you see over here is this particular curve that has been drawn. And again you find a histogram that has been drawn the Poisson distribution is this exponential curve that you see over here and the actual distribution is nothing like it bohigas and company did not actually solve it in this enclosure they solved it in a quarter of this enclosure which is essentially the same features of chaos in all the orbits. So, it seems that the eigenvalue distributions the spacings distributions follow these 2 extremes of these 2 extremes. (Refer Slide Time: 43:47)



In these of the Poisson distribution on the one hand and the Wigner distribution on the other hand for systems other than billiards like Henon-Heiles the phase space can have a mixture of tori and chaos and the statistics tend to be more complex but it is quite understood you know in which way there are more complex. Now, this is a subject which has really progressed way beyond where it started out in the 70s and 80s and people have looked at the eigenfunctions the nature of the eigenfunctions in different kinds of chaotic billiards here are just some images taken from the web on that there are many many open issues in Hamiltonian chaos and particularly in the area of quantum chaos, and the current frontiers are much further along than looking at eigenvalue statistics although this continues to be like many other problems something of abiding interest. Thank you.