Introductory Nonlinear Dynamics
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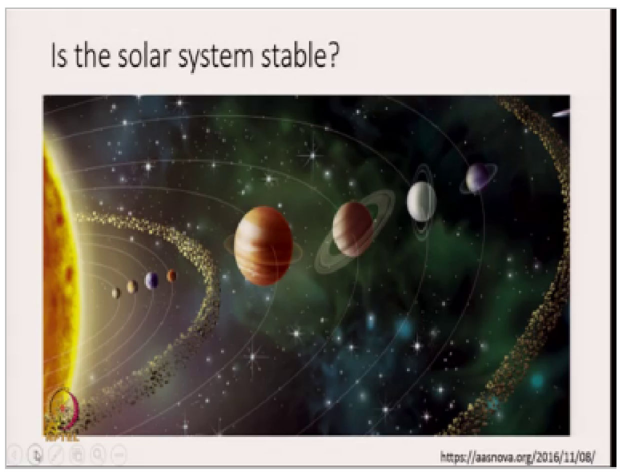
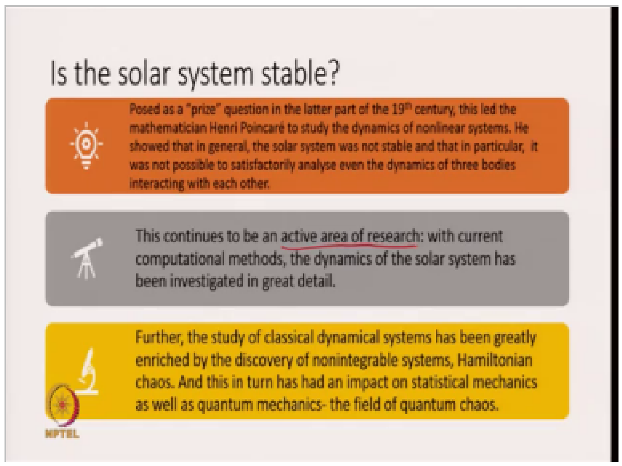
Lecture 14
Hamiltonian Chaos 1.

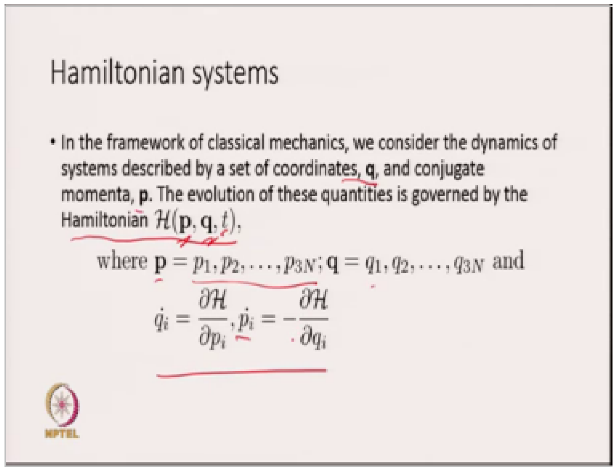
In the final two lectures of this Introductory course on Non-linear Dynamics we will consider Hamiltonian Chaos. (Refer Slide Time: 00:27)



The field of chaos in some sense actually started with a discussion of Hamiltonian systems through a question as dramatic as this namely, is the solar system stable? This was the topic of a prize question that was propagated in Europe in the late 1800s. And, it was a prize offered by the King of Sweden for the mathematician or physicist, astronomer who could satisfactorily answer the question is the solar system stable. What this means is that rather more involved discussion about what is meant by stability and so on. But, suffice it to say that this question spurred the great revolution in celestial mechanics, in classical mechanics, in many branches of mathematics and of course, it spawned in some sense this entire field of chaos and non-linearity, but that took quite some time to happen. The prize was eventually won by Poincare, the mathematician the French mathematician and in a sense he established that the solar system was not stable. But, it was not stable in an interesting way; I am not going to discuss the stability of the solar system per se because this is a rather more involved topic something on which research is still going on. But, I would leave it to you also to think little about it questions like why we have only 8 or 9 planets orbiting the sun? How come all of them are in almost in the same plane? Why is there a set of orbit objects that lie between the Earth and Mars in Jupiter the asteroid belt? Why is there such a thing like that? What does looking at the structure of Saturn tell us a little about our own solar system? These are all questions that really belong in a of course, on classical mechanics or celestial mechanics extremely interesting topics. (Refer Slide Time: 03:16)



Nevertheless, I want to discuss what this question did mostly, because it continues to be an active area of research. So, as I just mentioned this was posed as a prize question, but it continues to be an active area of research till today, because there are very practical implications. Every day there are satellites that are shot into space, the most recently the Chandrayaan 2 has gone up from India. And, it uses a lot of mechanics in order to be able to land on the Moon which will be happening hopefully within the next few days. But, more fundamentally the answer to this question started an area of research which has really enriched different aspects of physics. In particular it has had an impact on statistical mechanics, it has had an impact on quantum mechanics, the field of quantum chaos and there are implications for very current developments in things like quantum entanglement and so on. What I would like to do in the next two lectures is really to discuss what the sort of an introduction to this particular area of mechanics from the viewpoint of non-linear science. (Refer Slide Time: 04:45)



Hamiltonian systems called so, because the they are sort of the way in which we understand a lot of classical mechanics is through a Hamiltonian function which is usually written as a function of momentum variables, position variables and perhaps the time. The dynamics of a system which is given by a set of these coordinates, these could be physical coordinates or they could be a generalized coordinates, their conjugate momenta note in **p**. If I have N particles then the momenta take the values p_1 to p_{3N} and the positions or the coordinate values typically go from q_1 all the way up till q_{3N}, each particle let us say having 3 positions and 3 momenta variables. And, these variables evolve according to the rather simple and very elegant set of equations namely Hamiltonian $\tilde{H}(\mathbf{p}, \mathbf{q}, t)$ where $\mathbf{p} = p_1, p_2, ..., p_{3N}$; $\mathbf{q} = q_1, q_2, ..., q_{3N}$ and

$$\dot{q}_i = \frac{\partial \tilde{H}}{\partial p_i}, \dot{p}_i = -\frac{\partial \tilde{H}}{\partial q_i}$$

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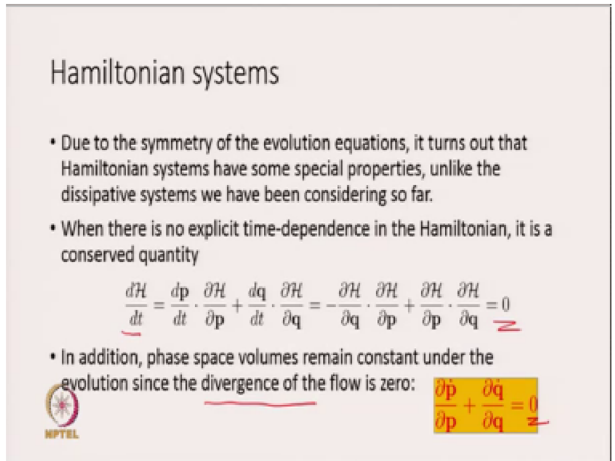
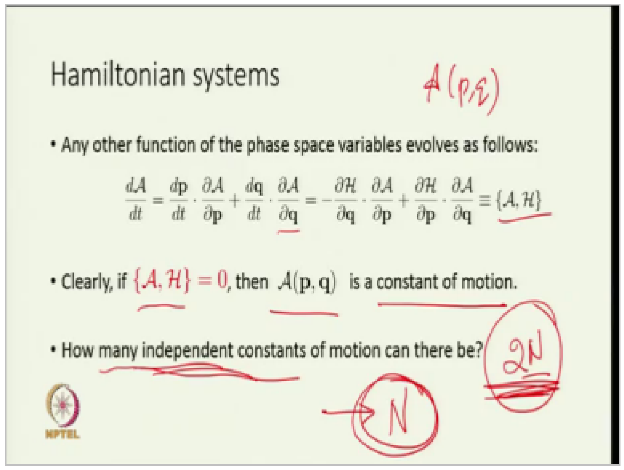


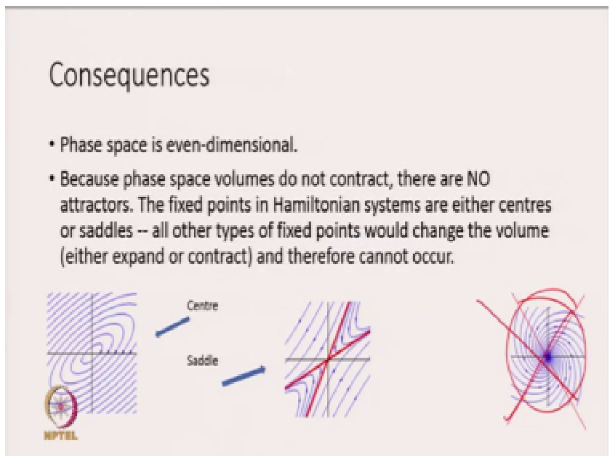
Now, because of the symmetry of these evolution equations, it turns out that Hamiltonian systems have some very special properties unlike the dissipative systems that we have been considering so far. In many cases there is no explicit time dependence in the Hamiltonian. So, the Hamiltonian itself whatever value it has is as a conserved quantity, it is constant in time because you can figure out the $\frac{d\tilde{H}}{dt}$ will because of the symmetry of these evolution equations turn out to be exactly equal to 0, therefore H is a conserved quantity.In addition and an even more important conservation law over here is that under the evolution phase space volumes will remain constant. And, this is so because as we have discussed in an earlier lecture a volume in phase space changes in time according to the divergence of the flow equations or according to the value of the determinant of the Jacobian matrix in the case of discrete, discrete systems. Now, for a Hamiltonian system the divergence of the flow which I can symbolically write as $\frac{\partial \dot{p}}{\partial p} + \frac{\partial \dot{q}}{\partial q}$. Because, of the fact then the $\dot{q}$ comes with an with a positive sign over here, but the $\dot{p}$ comes with a negative sign over here. This implies that the divergence of this vector field must be equal to 0. This says that volumes in phase space stay constant. (Refer Slide Time: 08:01)



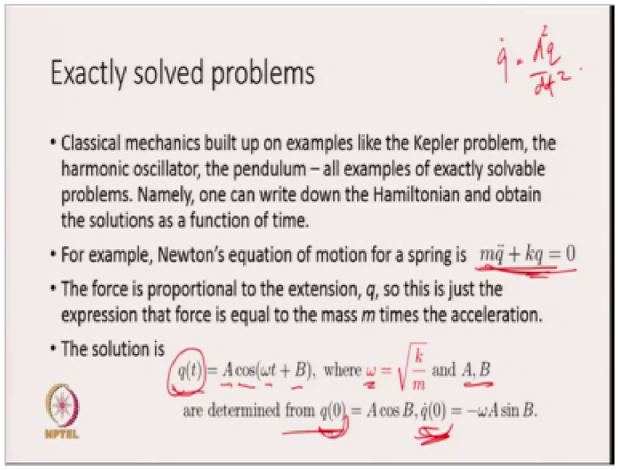
If there is any other function of the phase space variables, let us say we call it A and A is a function of p and q then the way in which this evolves is as follows

$$\frac{d\tilde{A}}{dt} = \frac{d\mathbf{p}}{dt}.\frac{\partial \tilde{A}}{dp} + \frac{d\mathbf{q}}{dt}.\frac{\partial \tilde{A}}{dq}$$

. Again by the symmetry of these equations, we see that this is the partial of A with respect to p, times the partial of H with respect to q negative plus partial of A with respect to q partial of A with of H with respect to p. This is called the Poisson bracket of A and H and is written in this particular notation as curly bracket A comma H. Clearly now, if (A,H) is equal to 0 then $\frac{dA}{dt}$ would be equal to 0 and A would be a constant of the motion or a conserved quantity. It is always of interest to know what are the constants of motion for a given dynamical system. And, an important consideration is how many independent constants of motion can there be, because of the symmetry of the Hamiltonian system and the way in which these equations are set up there are some important consequences. First of all the number of independent constants of the motion turns out to be equal to the number of dimensions of the coordinate space namely N. So, if I have got an N dimensional system my phase space is of dimension 2N and the total number of independent constants is always is equal to N means that the motion of the system occurs on a subspace of dimension N. Each constant of motion reduces the dimensionality of the motion by 1. And so, if there are no constants of motion your the phase, the motion in phase space takes place in a 2 N dimensional space, with each constant of the motion the dimensionality comes down by 1. So, when there are N independent constants of the motion your the dynamics occurs on an N dimensional subspace of the 2 N dimensional phase space. There could be even more constants in fact, if there are 2 N constants then the system is completely constrained and will only exist at a particular point. The case when there are N independent constants is of great interest in dynamics in classical mechanics partly, because this is in a sense a complete condition for integrability; the system is reduced from a N dimensional system to N 1 dimensional systems. (Refer Slide Time: 11:39)



Now, we also realize that because there is always coordinates and momenta phase space is always an even dimensional space. And, because phase space volumes do not contract, there are no attractors in Hamiltonian systems. The fixed points in Hamiltonian systems can either be centres or saddles. They cannot be nodes, they cannot be spirals all other types of fixed points would change the volume either expanding or contracting phase space and therefore, they cannot occur in Hamiltonian systems. So, in Hamiltonian systems we either have centres namely a fixed point with a circulation around it or a saddle namely a fixed point with one expanding direction and one contracting direction matching sorry cancelling each other out completely. And, any other kind of fixed points such as a spiral etcetera is simply not permitted. (Refer Slide Time: 12:44)



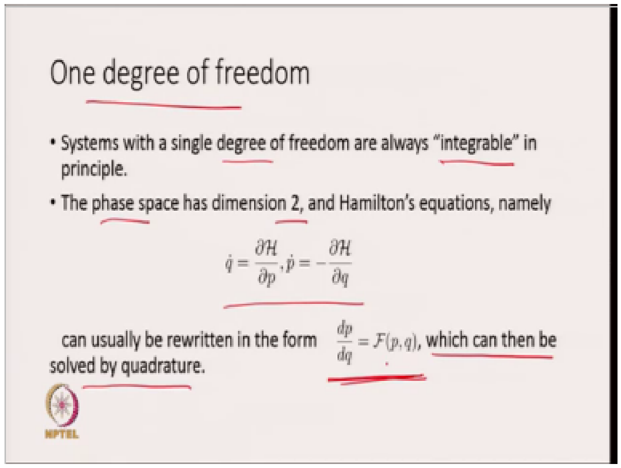
Now, classical mechanics is built up historically and you know ended by its very structure it was built by examples like the Kepler problem or the harmonic oscillator or the pendulum, all of which are instances of exactly solvable models. Now, what do we mean by an exact solution? Well, in this context what we mean is that we can write down the Hamiltonian and obtain the solutions as a function of time. Now, Newtons equation of motion for a spring let us say one of the earliest examples of such systems that was introduced historically. But, also one of the earliest examples that are typically introduced in a classroom is the usual force is equal to mass into acceleration statement of Newtons law. So, that can be written down as $m\ddot{q}$, this $\ddot{q}$ over here is just the $\frac{d^2q}{dt^2}$ or it is the acceleration; if I take q is to be a position plus kq is equal to 0 where, the force is just proportional to the distance that the spring is extended.

$$m\ddot{q} + kq = 0$$

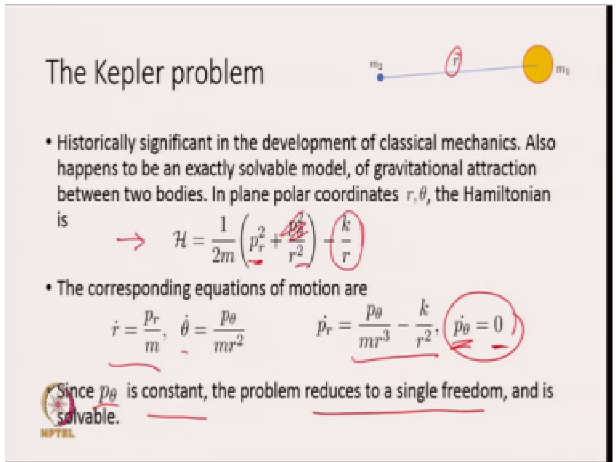
So, this is a statement that mass into the acceleration is equal to the force and the solution for this differential equation can be easily obtained. And, that solution says that

$$q(t) = Acos(\omega t + B)$$

where omega the frequency of oscillation over $\omega = \sqrt{k/m}$.And, A and B are constants which are determined from the initial conditions and the initial conditions are what is the value of $q(0)$ and what is the value of $\dot{q}$. Now, this is a way in which we consider a system to be solved namely given the initial conditions, if I specify what is $q(0)$ and $\dot{q}(0)$, I can easily find out what is q at any later time. So, this is one way of thinking of a of a problem as being a solved problem. (Refer Slide Time: 15:35)



It turns out that systems where there is a single degree of freedom are always integrable or always solvable in this sense in principle. The reason for that is that the phase space, if there is a single degree of freedom there is one position variable and one momentum variable. So, the phase space has got dimension 2 and Hamiltons equations are just these simple equations and you can usually rewrite them in a nice form of dp by dq. And, this can be integrated or which can be solved by some procedure of quadrature, alright. So, this can be solved as an integral, there is it is a very straightforward way of solving it. So, all one freedom one degree of freedom or one freedom problems are in principle solvable. Even if you may not know precisely how to solve this differential equation; you know in terms of the algebra of it, there is nothing intrinsically unsolvable about such a system. (Refer Slide Time: 16:51)



An early example in fact, the one which sort of laid the foundations of classical mechanics in some sense is the problem is the Kepler problem that is the problem of the attraction of the Earth to the Sun. And, the way in which it is solved in classical mechanics textbooks is to write down the equations of motion. But, I am going to just you know go into a slightly different presentation of the problem over here because, I am not interested in the solution so, much as to show why it is considered or can be considered to be a solvable problem. This refers to two particles moving in three-dimensional space. So, there are many there are 6 coordinates, 3 for one particle and 3 for the other. Can always transform to a frame where the distance between the two of them forms one coordinate and that is this distance r. And, because there is a conserved quantity in this system known as the angular momentum, we can write down the Hamiltonian for this problem as

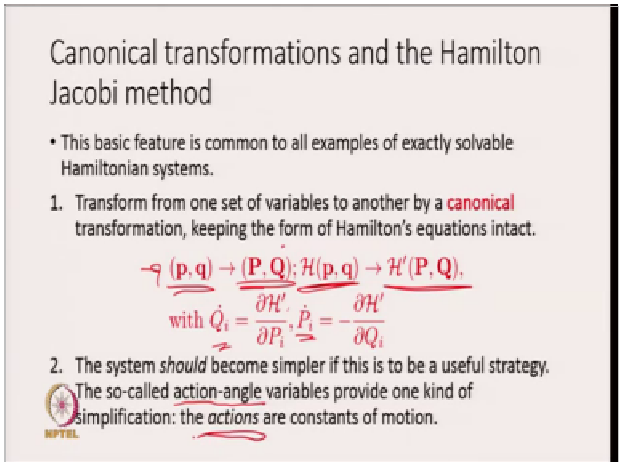
$$\tilde{H} = \frac{1}{2m}(p_r^2 + P_\theta^2/r^2) - \frac{k}{r}$$

. This is your gravitational attraction and this is the equation of motion for the radial and the angular coordinates. Now, given a Hamiltonian like this the corresponding equations of motion are very simple, they are

$$\dot{r} = \frac{p_r}{m}, \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{p_r} = \frac{p_\theta}{mr^3} - \frac{k}{r^2}, \dot{p_\theta} = 0$$

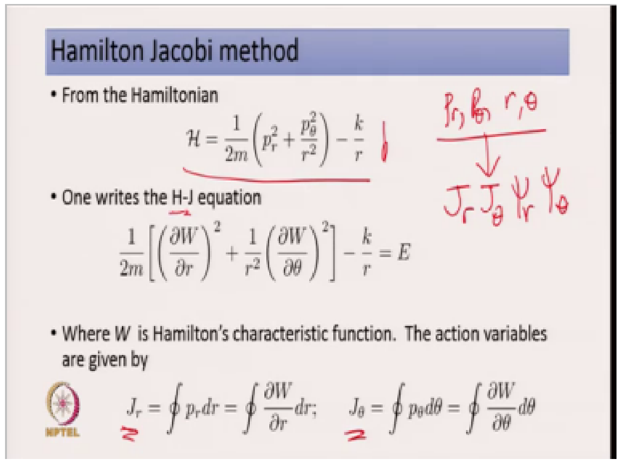
, theta does not appear anywhere else in this equation. Now, since p_θ is a constant because of this quantity, this just can be replaced by a constant over here. The problem reduces to a single freedom and as I have just pointed out, all one freedom problems are solvable and this too was very beautifully solved by Newton to start with. (Refer Slide Time: 19:29)



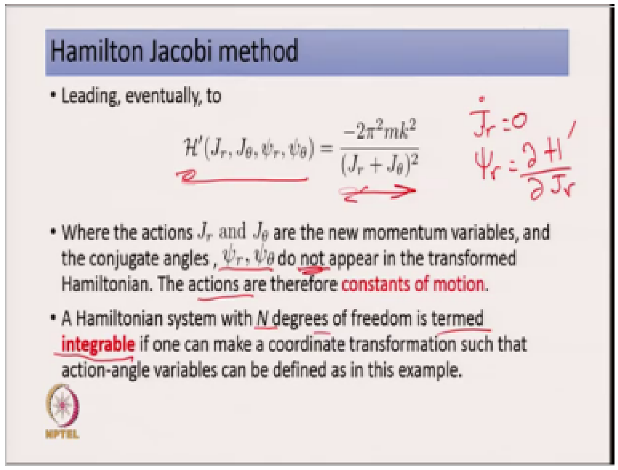
This kind of feature is what is common to all examples of exactly solvable Hamiltonian systems. You start with expressing your problem in (x,y,z) or any other set of coordinates, you transform to a new set of variables in which the Hamiltonian or your problem statement itself is somewhat simpler. And, once you have that you can transform in many ways, but typically what when tries to find is a; so, called canonical transformation. The nice part about a canonical transformation is that it keeps the structure of Hamiltons equations intact; namely if I go from some variables (p,q) to some other variables (P,Q) and my Hamiltonian H(p,q) transforms to some new Hamiltonian H'(P,Q). I would like to keep my equations for

$$\dot{Q}_i = \frac{\partial \tilde{H}'}{\partial P_i}, \dot{P}_i = -\frac{\partial \tilde{H}'}{\partial Q}$$

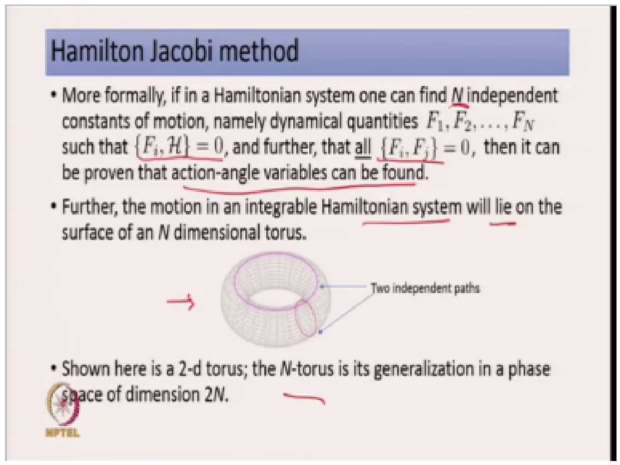
. Now, the aim of this transformation to go from one set of variables to another would be pointless unless the system itself became somewhat simpler. And, there has been you know several centuries of discussion and mathematics and development on this and it turns out that there are a type of variable, the set of variables which in many cases of a simplification. These are called action angle variables and they are you know they are extremely important, because it turns out that when you can find such variables then your Hamiltonian becomes very simple. And, there are new constants of motion which are the actions right. (Refer Slide Time: 21:52)



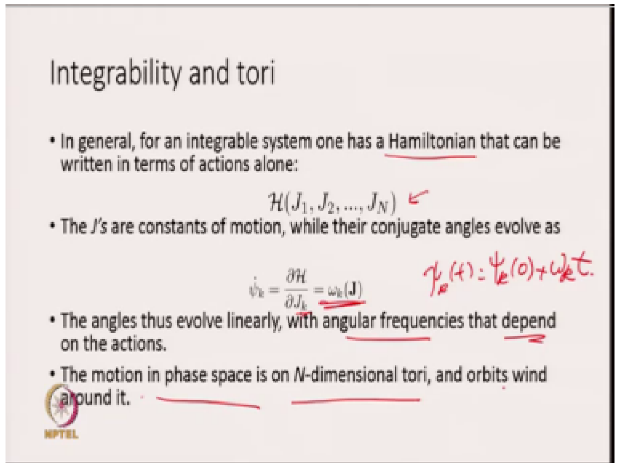
Now, the way in which this has been done again the few centuries of development that have gone into this and this is typically the subject of a sort of a longer course. But, let me just try to introduce the flavour of why we look at these things. Given this Hamiltonian that we just wrote down earlier, there is a new kind of equation known as the Hamilton Jacobi equation that comes from this equation by invoking a canonical transformation. When you have this particular canonical transformation, it turns out that this permits you to define the so, called action variables. And, the action variables that I will define in this particular case are the variable J_r and J_θ. And, there are you know we have gone from the variables $p_r, p_{\ theta}$, r and θ that is we have started with these four variables. And, then we have gone to another set of four variables through a canonical transformation and these are the variables J_r and J_θ and corresponding angle variables ψ_r and ψ_θ. So, you would use you know use the machinery of classical mechanics to define these particular variables, define the transformation. (Refer Slide Time: 23:35)



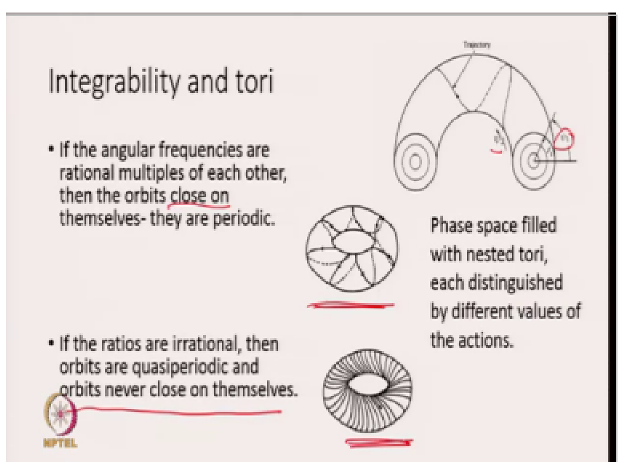
And, when you do you get a new Hamiltonian function, this new Hamiltonian function, a function now of J_r,J_θ ,ψ_r and ψ_θ, this turns out to have a very elegant form, negative of some constants up there divided by J_r+J_θ squared. These are the two new momentum variables and note that the conjugate angles ψ_r and ψ_θ do not appear in the transformed Hamiltonian. Because, they do not appear in the transformed Hamiltonian, the actions are constants of motion. Now, I have not actually written down the solutions for r and theta and all of which can be done very simply and form the subject of many lectures in classical mechanics. But, this is another way of considering that the system is solvable; namely I am able to transform my system from one set of initial coordinates via canonical transformations two a set of new variables wherein, I have only constants of the motion. The corresponding variations are the Hamiltons equations over here are the $\dot{J}_r = 0$, it is a constant. And, ψ_r will be given as the partial of H prime with respect to J_r. It is a straightforward equations that one can derive over here. Now, when you can do this for an N degree of freedom system, if you can do this you will find; obviously, N constants of motion. And, if you can find these N constants of motion then the system is termed integrable or in this language it is a way of saying that the system is completely solvable. Every system in a classical mechanics that is known to be solvable has this essential structure the same structure. (Refer Slide Time: 26:07)



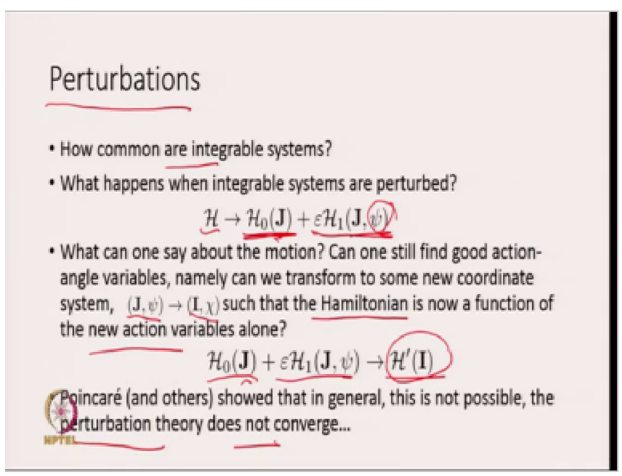
Now, it turns out that more formally, if in a Hamiltonian system, one can find N independent constants of motion, namely dynamical quantities F_1, F_2 etcetera all the way up till F sub N such that they are all constants of the motion. Namely, the Poisson bracket with the Hamiltonian vanish and further if not only do the Possion brackets with the Hamiltonian vanish, but their Poisson brackets mutually vanish. Then it can be proven that action angle variables can be found in principle. Furthermore, the motion in an integrable Hamiltonian system will lie on the surface of an N dimensional torus. And, an N dimensional torus is an N dimensional generalization of this object. This is an example of a 2-dimensional torus and I have just projected it onto a sheet of paper. So, you can think of this torus as forming the surface of a bicycle tube, the inner tube of a bicycle or a doughnut or a vada or whatever object you can think of this as representing. The N torus is a generalization of this and remember that the N torus will live in a phase space of dimension 2N. (Refer Slide Time: 27:47)



Now, in general in an integrable system one has a Hamiltonian that can be written finally, in terms of action variables alone because you can find these N constants of motion which are the actions. The conjugate angles because the conjugate variables are given as all angular variables; this is given as the partial of the Hamiltonian with respect to the action variable. And, this is a number it is a frequency, because the angular variation is just $\psi_k(t) = \psi_k(0) + \psi_k t$. It is a straightforward linear evolution. The angles evolve linearly, with angular frequencies that now depend on the actions. The motion in phase space though is on the surface of an N-dimensional torus and the orbits will be winding around this torus, because that is how orbits do. (Refer Slide Time: 28:53)



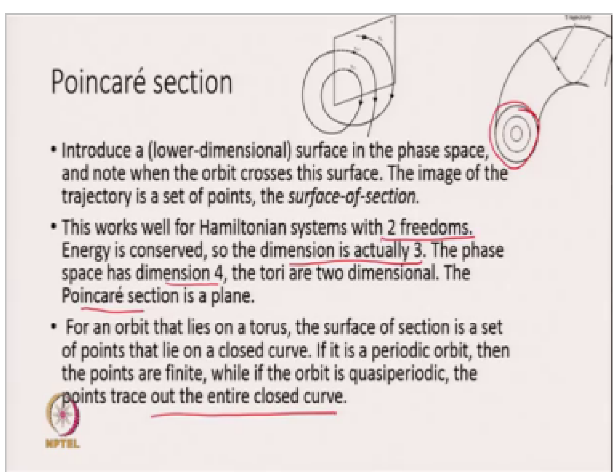
In for a two freedom system for example here you have got your two dimensional torus, your orbit is moving around in it you know around this particular torus as you can see. The angular frequency in direction 1 is psi 1, in direction 2 is psi 2 and it moves around the surface of this torus. There are two possibilities for how this orbit winds itself around the torus. If the angular frequencies are rational multiples of one another then, a little argument will convince you that, if the frequency in one direction is n times let us say the frequency in the other direction, then after some time the orbit must close on itself. So, this is a schematic where you have this orbit which is the frequencies in one direction are rationally related to the frequencies in the other direction and you have a periodic orbit that closes on itself. On the other hand if the ratios of the frequencies is irrational, then the orbits will never close on themselves and the orbit will uniformly and smoothly just cover the entire torus. So, the you have a distinction over here of either a rational relation between the frequencies or an irrational relationship between the frequencies. So, all over phase space in an integrable system you have only tori and these tori all have different frequencies of the angular variation around them. So, orbits are moving around on these tori and the entire phase space is covered by tori with all these different orbits going around on them. (Refer Slide Time: 31:22)



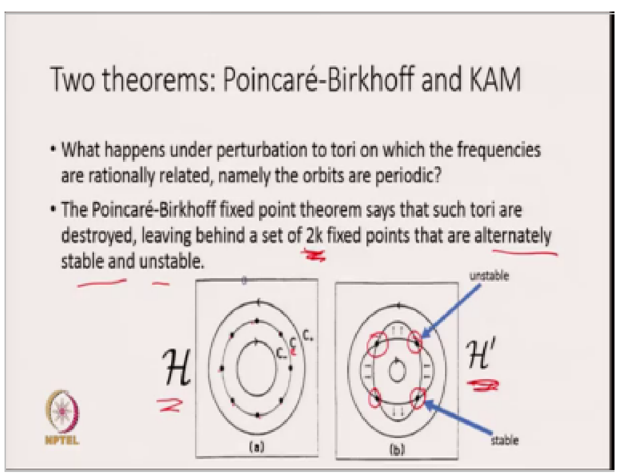
Now, this would be fine if all systems that we knew off are all systems that could exist were all integrable because, if they were all integrable then you could always find action angle variables and so on and so forth. It turns out that things are not always so simple and we have to consider the effect of perturbations. So, how common are integrable systems? It turns out that they are not common at all, they are exceptional, but we build up our knowledge of the real world on idealizations and integrable systems in many cases are idealized models of classical mechanics. So, the natural question is what happens when integrable systems are perturbed or mathematically; if you started out with an integrable system H_0 which you can write completely in terms of action variables, no angles there. But, now you add a perturbation, where the perturbation could involve angles. If you have a new Hamiltonian which is a function of not just the action variables, but also has a perturbation that includes the angles, what can you say about the motion? What can one you know what can one say about it, in the context of integrability namely can one still find some other good action angle variables?. Can you transform to yet another new coordinate system instead of J_ψ, then we go to I_χ it is a new variables such that the transformed Hamiltonian is a function of the new action variables alone? Namely, I go from

$$\tilde{H} \rightarrow \tilde{H_0}(J) + \varepsilon \tilde{H}_1(J, \psi)$$

to a new set of variables I and chi set meant transformed Hamiltonian is only a function of these new action variables. Enough work has gone into showing that this is not possible in general. And, Poincare a most famously showed that the perturbation theory actually diverges in most cases. And, we do not have you do not have the possibility of finding new action variables regardless of the type of perturbations that you that one may apply. (Refer Slide Time: 34:04)

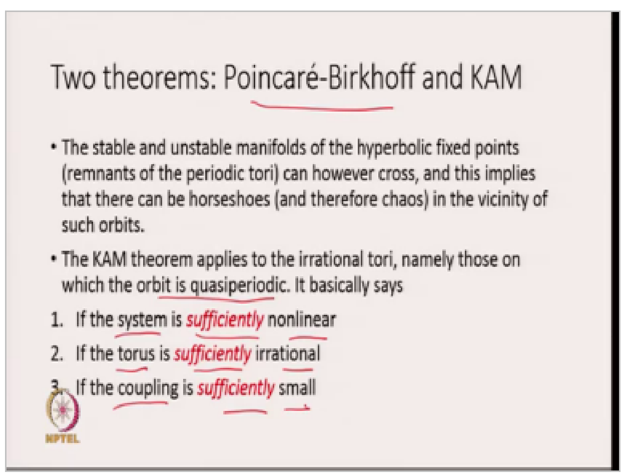


In order to make things somewhat more visually simple Poincare a introduced a technique which is now known as the Poincare section or the Poincare surface of section method. In order to ask what is the nature or what is the geometry of this of the orbits in phase space, just in passing one should note that Poincare also invented topology during the course of his studies and so, this was all integral to that depending on. So, what Poincare said was or the technique that in, but in Poincare introduced this technique of the surface of section to ask what is the geometry of orbits in phase space. So, if in the phase space you take a lower dimensional slice and ask where does the orbit cross the slice, then you have a lower dimensional representation of the higher dimensional motion. So, this is you know, this itself is a technique known as the surface of section method. And, this would allow you to have a visualization of higher dimensional objects in a lower dimension. In particular when the Hamiltonian system has 2 freedoms, then the phase space has dimension 4. If you are working at a constant energy the dimension of the phase space is actually 3. And, then if you slice through this 3-dimensional phase space with a plane, you find the Poincare section which gives you the image of the orbit on the plane. Now, you can imagine cutting a bicycle tube or a doughnut with a knife, when you cut it with a knife you get a circular cross section. So, for an orbit that lies on a torus the surface of section is a set of points that is going to lie on a closed curve. If it is a periodic orbit then the number of points will be finite whereas, if the orbit is quasi periodic then the points on the section will trace out the entire closed curve and pretty soon we are going to see examples of that. (Refer Slide Time: 36:45)

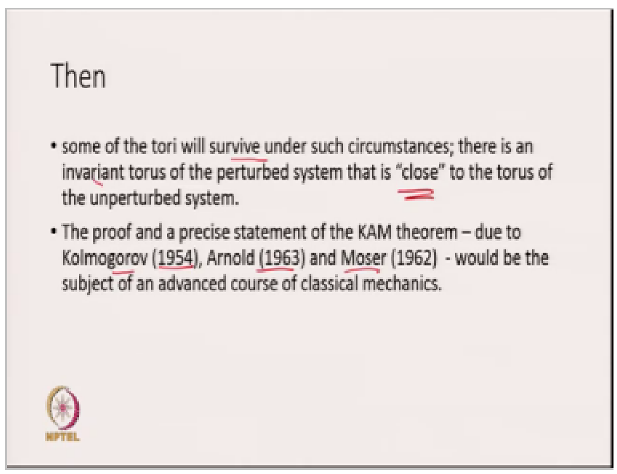


So, what happens to orbits? What happens to systems under perturbation?

Just to briefly remind you, we are starting with a Hamiltonian system which is integrable. And, in this integrable Hamiltonian system we have orbits that only lie on tori. What we would like to know is that when I add a perturbation H prime and go to a new Hamiltonian H prime, what happens to these orbits? What happens to the tori? And, it turns out the two extremely important theorems tell us what happens to tori in integrable systems, when you add a non-linear and a non-integrable perturbation. I am just going to state these theorems not even attempt a proof, I am just going to tell you what the impact of these theorems are. The Poincare-Birkhoff fixed point theorem addresses tori which are rational. So, you take a torus which I have marked here as c and we are seeing its image on the Poincare section. So, what you have are a set of points that you find over here representing the rational ratio of the frequencies in the two directions and the system. What the Poincare-Birkhoff fixed point theorem says, that when you add a perturbation then the rational tori are destroyed. And, they leave behind a set of 2k fixed points an even number of fixed points and these fixed points are alternately stable and unstable; namely you will find a centre, a hyperbolic fixed point another centre, another hyperbolic fixed point and so on and so forth, ok. The total number of fixed points will always be even and they will be alternately stable and unstable and this is the fate of all rational tori. So, under perturbation the rational tori do not live, they are destroyed and they will give rise to this alternate set of fixed points. (Refer Slide Time: 39:29)



The KAM theorem addresses the quasi periodic orbits on the irrational tori, although they are to just complete the discussion of the Poincare-Birkhoff fixed point theorem; note that you have alternating centres, hyperbolic fixed point centres, hyperbolic fixed points. The stable and the unstable manifolds of the hyperbolic fixed points which born out of the destroyed periodic tori, these can cross and because when they cross you have horseshoes and there you have chaos and so on and so forth. We know that the fate of periodic orbits of periodic tori, these rational tori is to give rise to elliptic fixed points as well as possibly a hetero clinic tangle. The KAM theorem on the other hand addresses the irrational tori on which the orbit is quasi periodic. And, the basic statement of the quasi of this KAM theorem is that if the system is sufficiently non-linear, if the torus in question is sufficiently rational that is to say the ratio of frequencies and the two directions is a sufficiently irrational number. (Refer Slide Time: 41:06)



And, if the coupling is sufficiently small then the KAM theorem says that some of the tori will survive under such circumstances. Namely, there is an invariant torus of the perturbed system that is in some sense close to the torus of the unperturbed system. These words in red sufficiently non-linear, sufficiently irrational and sufficiently small are mathematically very precisely defined in the statement and the proof of this theorem. This theorem itself was proved over a 10-12 year period; it was stated by Kolmogorov at the international mathematics, the IMU meeting. The theorem itself was stated by Kolmogorov in 1954 and it was proved by his student Arnold in 1963 for a classical dynamical system and which had been proven for a set of maps by Moser in 1962. Its actual statement and the proof etcetera would be the subject of an advanced course in classical mechanics. But, it is an extremely important theorem and it governs the behavior of Hamiltonian dynamical systems under perturbation. So, to recap over here, what we have discussed is the fact that in integrable systems, we have only tori. When you add a perturbation which does not preserve the integrity, then it is possible that some of these tori continue to survive in the perturbed system. But, in addition you have a wealth of new behavior and this is what we will take up in the next lecture.