Introductory Nonlinear Dynamics
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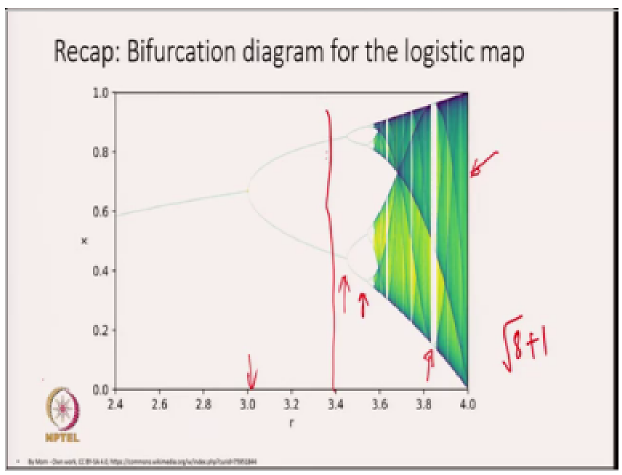
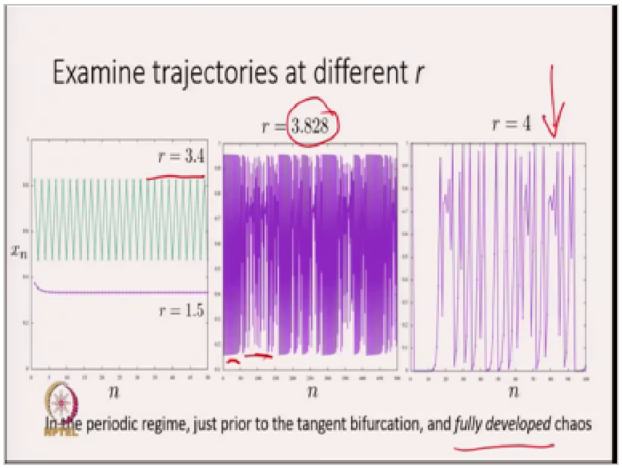
Lecture 10
Intermittency Crises.

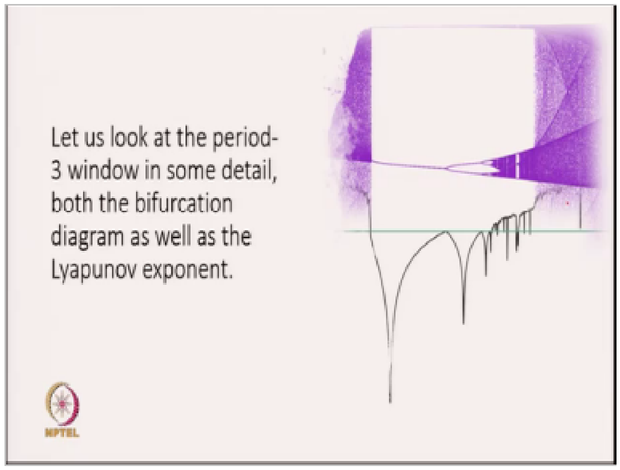
Hello. We continue our study of bifurcation phenomena and dynamics in simple non-linear systems focusing again on the logistic map as we have been looking at it in some detail over the past few lectures, but today we will look at some other phenomena in the same system. Let us recap what we studied in the last few lectures. (Refer Slide Time: 00:47)



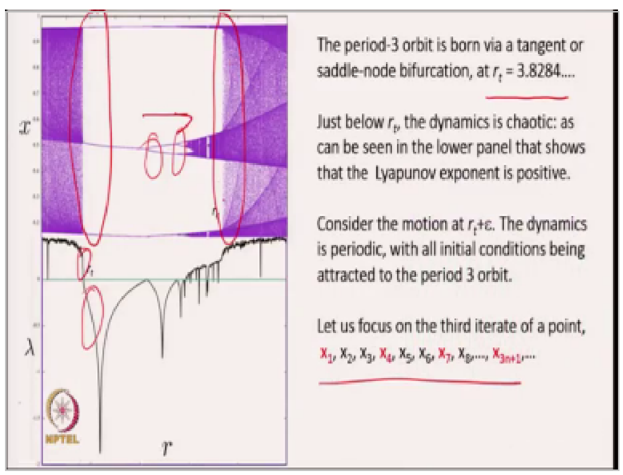
The logistic map $x_{n+1} = x_n(1 - x_n)$ or rx_n with parameter non-linearity parameter. As you vary r you find that initially period-1 orbits are stable, there is a transcritical bifurcation that happens before we get into this particular frame at which another period-1 orbit is born. And, we go along a particular locus up till the value r is equal to 3, where there is a period doubling bifurcation we get a stable period-2 orbit. There is another period doubling bifurcation over here we get a stable period-4 orbit which then leads to a period-8 orbit and so on, all the way up till r to the power r 2 to the power n; n being large numbers going eventually to aperiodic motion. There is a lot of detail in this bifurcation diagram of course. A particular attention for todays lecture will be on the period-3 window over here. And, we note from what we have learnt in the last lecture about the Sharkovsky sequence that by the time period-3 is born namely at this value of r square root of 8 plus 1, at this value of r there are all periodic orbits will have occurred by then in and we also know furthermore that at r is equal to 4, there are periodic orbits of all periods and all of them are unstable. (Refer Slide Time: 02:43)



Now, that we have reminded ourselves of this bifurcation diagram for the logistic map let us see what varying r does. When r is equal to 1.5 trajectory of the system gives you a period-1 orbit well it is off the screen over here, but there is a period-1 orbit at r is equal to 1.5. At r is equal to 3.4 a period-2 orbit is stable and that we recognize over here 3.4 namely somewhere over here, we have this period-2 orbit that one can see. At r is equal to 4 we have an orbit that goes all over the interval this is an aperiodic orbit because there is no stable periodic orbit we know this for sure. Such dynamics which completely covers the interval in which the map is functional is often called fully developed chaos because it is over the entire line over here. At r is equal to 3.828 which is just below where period-3 is born, trajectories look very different in particular they look almost periodic for a certain period of time. Then, it looks aperiodic and chaotic, and then this sequence occurs again. And, in todays lecture we will worry about the dynamics that looks that appears like so, where does it come from and what are the implications. (Refer Slide Time: 04:25)

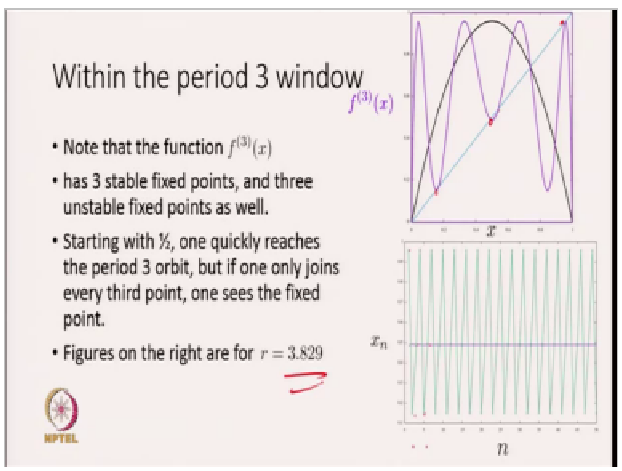


So, let us look at the period-3 window in some detail, we look at the bifurcation diagram as well as the Lyapunov exponent and we are going to also look at the densities and so on and so forth. (Refer Slide Time: 04:49)

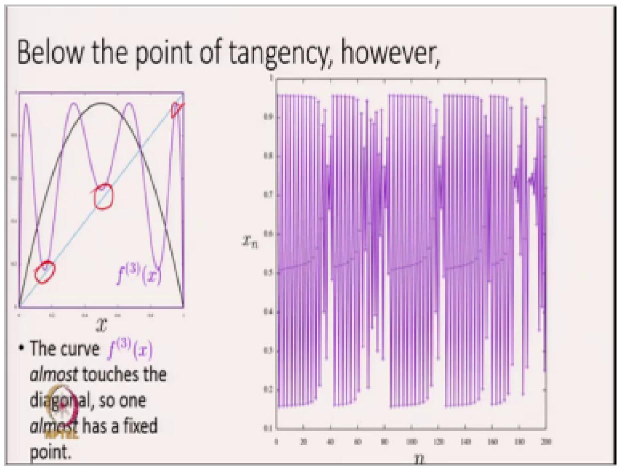


So, here schematically we have the bifurcation diagram and superimposed is the a graph of the Lyapunov exponent which is you know with axes etcetera is drawn over here. Notice that this by the tangent bifurcation occurs at r sub t which is indicated here and it is 3.8284 etcetera etcetera. Now, just below this point of tangency as you can see on the graph of the Lyapunov exponent over here, the Lyapunov exponent is positive ok. So, this says that the dynamics just below the tangent bifurcation is actually chaotic. Once you have this tangency there is a bifurcation the Lyapunov exponent becomes

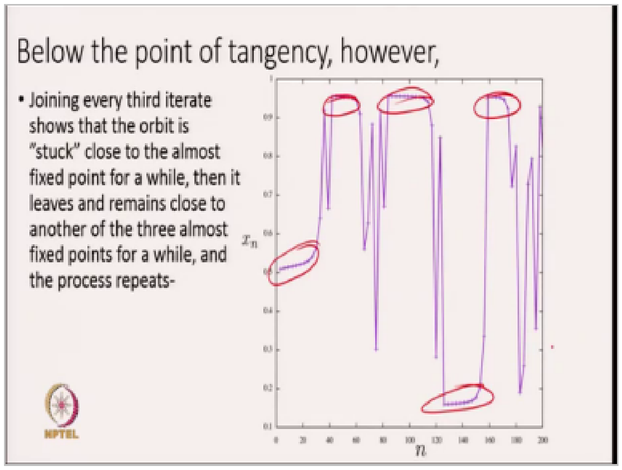
0 at the bifurcation and we know that because we know that the slope of the map is equal to one in modulus and therefore, the Lyapunov exponent is 0 and subsequently a period-3 orbit is born. The period-3 orbit is born at through a tangency. So, there is a stable periodic orbit of period-3 and an unstable periodic orbit of period-3 both born at the same time. And, so, if we look at the dynamics let us say over here this is aperiodic orbit and it will be at you know the almost all the initial conditions are going to be attracted to this period-3 orbit. Now, the period-3 orbit is if I write it out over here x 1, x 2, x 3, x 4 is equal to x 1, x 5 is equal to x 2, x 6 is equal to x 3 and so on and so forth ok. So, now, let me just take any arbitrary orbit, and then look at the third iterate of every point. Within this period-3 window that is this window over here that you know in which we can see there is basically period-3 behaviour. (Refer Slide Time: 06:55)



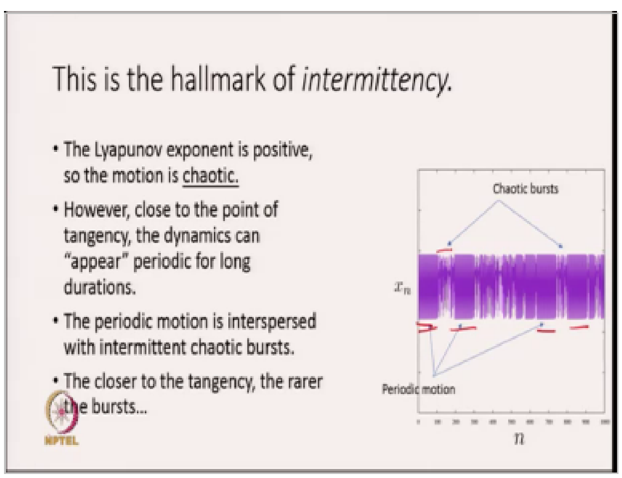
Note that this function $f^3(x)$ has three stable fixed points and three unstable fixed points as well. So, over here the slope is bigger than 1, over here the slope is less than 1. So, this is part of the stable periodic orbit and we can you know check over here and this will also be a part of it and we will this point over here. So, those three form the stable periodic orbit of period-3. The other three points of intersection form the unstable periodic orbits of period-3. Now, if you start with the point half that is the map maximum one quickly reaches the period-3 orbit, but if one only joins every third iterate you can see the fixed point as I have done over here the green orbit over here is the period-3 orbit. So, first point, second point, third point, first point, second point, third point etcetera etcetera. Now, if I join only every third point I have got this point joined to that join to that and you can see that that now just looks like a fixed point. So, the figure of on this side is for the value of r=3.829 which is above the bifurcation point. (Refer Slide Time: 08:18)



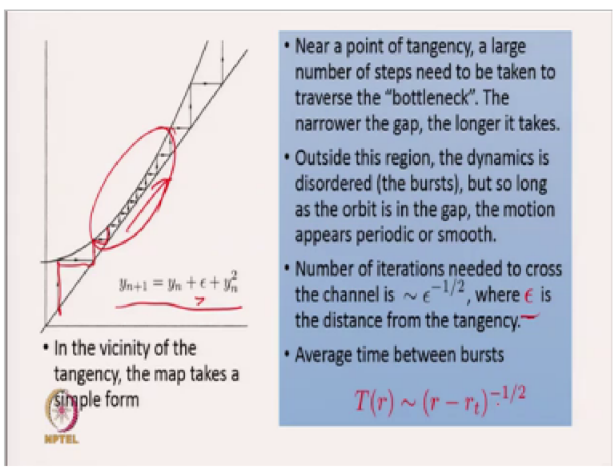
Now, below the point of tangency however, the map looks something like this, there are no fixed points. But, this curve is appearing almost to touch the diagonal little less almost over here and a little more almost over there, but what one can see is that at below the point of tangency the curve comes very close to the diagonal so, there is an almost fixed point over there. And, what does an orbit look like? Well, here is an image of the orbit at that same value of r which is a little below 3.8284 etcetera. And, if I just were to join all the points I get this image and if I were to join only every third point then I get this image. (Refer Slide Time: 09:16)



So, you can see that for a certain period of iterations it stays very close to this fixed point then suddenly it escapes and goes close to this fixed point and then it escapes again and it comes back out here and then again it comes back to that fixed point and then to this fixed point and that fixed point and so on and so forth. So, if you join every third iterate rather than joining every iterate you see that the orbit is stuck very close to this almost fixed point for a while then it leaves and it goes close to some other of the almost fixed points and so on and so forth. And, you can see that this behavior is a consequence of the fact that we are not yet at the point of tangency, but we are just below it. (Refer Slide Time: 10:11)



This is the hallmark of what is called intermittency in the dynamics. And, the name probably comes from the fact that intermittency especially in fluid flow refers to the fact that you have something that is smooth and laminar for a while and then suddenly there is a little you know sort of turbulent motion, and then again the smooth motion for a while, and then again some whirlpools and eddies and so on in real fluid flows. Some of you may have seen this in flows on a river or some you know similar locations. But, the word intermittency now should mean whenever you have something that looks very periodic smooth lamina for a while and it is interrupted by something which is not of the same period and is has got some other kind of characteristic. The motion is chaotic because the Lyapunov exponent for such motion is positive as we have just seen, but close to the point of tangency near this almost fixed point, the dynamics could appear periodic for very long durations. And, a typical image of an intermittent flow is like what I have shown over here, you have got something that looks periodic for fairly long periods of time and in between there are these chaotic bursts. The closer we are to the tangency note that at the tangency we have a periodic orbit so, it is going to be periodic orbit forever, but close to this tangency these chaotic bursts become rarer and rarer and less and less frequent alright. The reason for this can be seen by looking at the map in some detail. (Refer Slide Time: 12:18)



In the vicinity of this tangency the map as one can show which is almost going to be tangential to it to this diagonal line takes a particularly simple form. In particular it has this so called normal form that

$$y_{n+1} = y_n + \varepsilon + y_n^2$$

. When epsilon is equal to 0, you will see that the fixed point is at y n is equal to 0, but the moment epsilon is different from 0 this curve is no longer tangential to this is no longer tangential to this diagonal line. And, any orbit that comes in over here let us say following our procedure for drawing these orbits you go from the diagonal to the curve to the diagonal to the curve and so on, but in order to you see the orbit is now going to stay or take a very large number of steps in this region because the bottleneck over here is very very narrow. And, how many steps will it take? Well, that is the amount of time that the period the orbit will look almost periodic because it is very close to this fixed point that is going to be created over here. So, it will look almost periodic for as many steps as it takes to escape from this bottleneck. Now, the narrower the gap you can clearly see that it is going to take many many more steps and when the gap is 0, it never escapes from that fixed point namely, it is on a periodic orbit. Outside this region the dynamics does whatever it does, but near this point of tangency it is just going to take a lot of very very small steps to get out. And, the number of iterations needed to cross this channel is of the order of 1 upon epsilon the 1 upon the square root of epsilon where epsilon is the distance from the point of tangency. Now, the point so, therefore, the average time between the bursts is going to go as

$$T(r) \sim (r - r_t)^{-/2}$$

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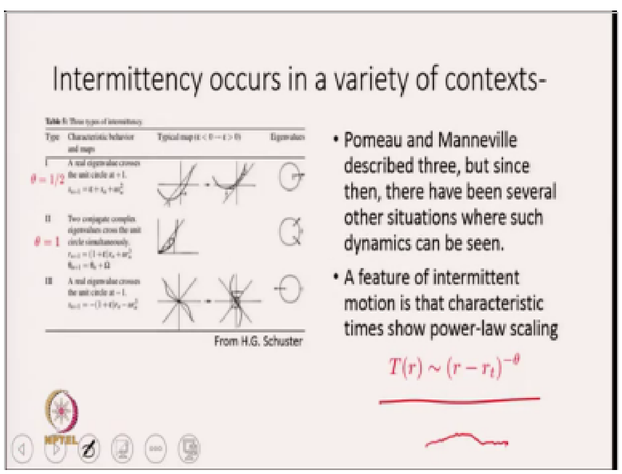
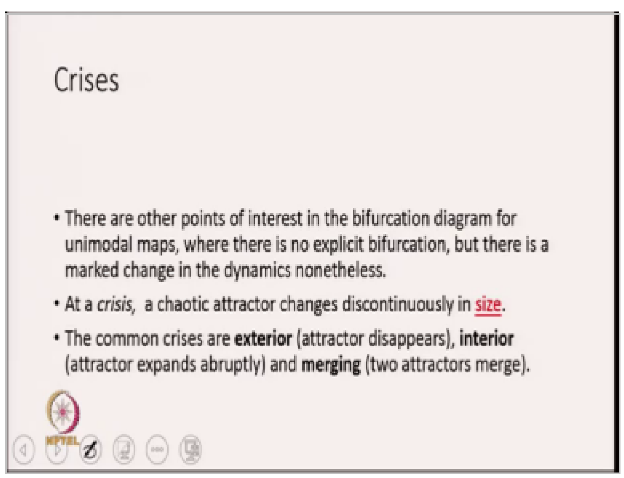
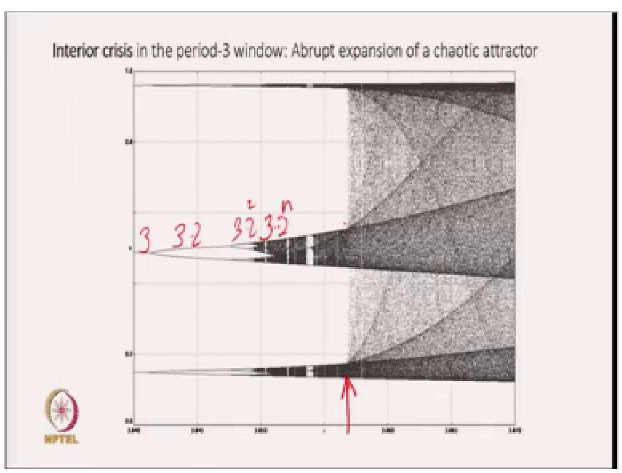


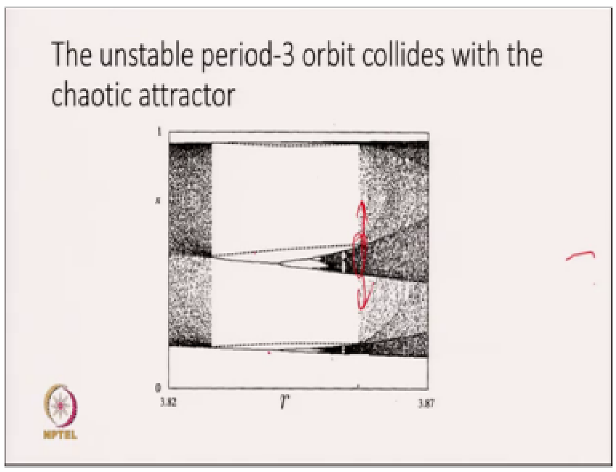
And, this characterizes and the intermittent dynamics. Now, what I have described over here is the intermittent dynamics near a point of tangency and that is described by it was first described explicitly by Pomeau and Manneville in the early 1980s and they characterized it as one of three possible types of intermittency that can occur. The first type which is you can see just slightly shrunken version of the same one the same image that I have shown earlier on. They the characteristic behavior is that a real eigenvalue crosses the unit circle at plus 1. There are two other types that they describe type 2 and type 3 which are also described you know discussed in this particular image. The important part is that intermittent motion always shows a characteristic power-law kind of scaling and this time between bursts so to speak is you know from the point of view of the distance from the bifurcation point it goes as some negative power theta in the case of type one intermittency which is the one we have just discussed, the exponent theta is half and for these other two types the exponent theta is equal to 1. Now, there are other types of intermittency that people have discussed, but all of them by enlarge will have this form namely a namely a power-law scaling at this point of intermittency and the dynamics to some approximation looks like this basically you have motion of one kind interspersed with motion of the other kind and at the point of bifurcation you have all motion of the one kind. So, you can always distinguish between the two types of motion that you see over here contributing to the intermittency. (Refer Slide Time: 17:18)



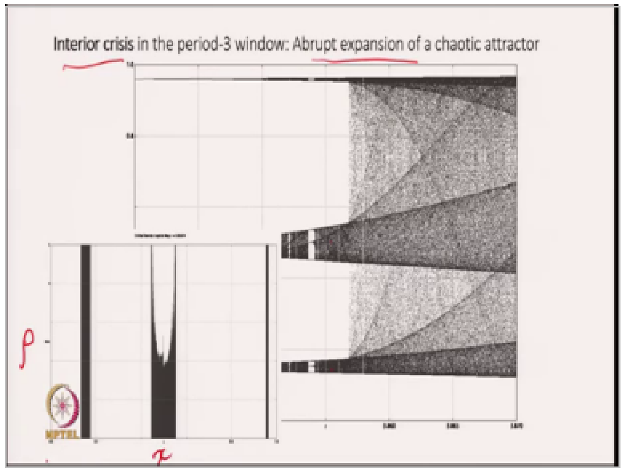
Another scenario where intermittency happens is at what are called crises. Now, crises are special points in the bifurcation diagram as I am going to show you now for modal maps, but this is quite common in many many kinds of systems. There is no explicit bifurcation per se. So, for example, the Lyapunov exponent does not actually pass through a zero although there is a marked change in the dynamics before and after this crisis point. With a kind of crises that I am going to describe in the next few slides the basic feature of this of this so called this phenomenon this changed this dynamical transition is that a chaotic attractor changes discontinuously in size. Crises were described you know formally by Ott and by the Maryland Group in the early 1980s. And, there are basically three kinds that they discuss. The exterior crisis where a chaotic attractor hits a particular point like a periodic point and actually disappears, there is an interior crises where an attractor expands abruptly, and a merging crisis where two attractors merge to become one larger attractor. Now, in the logistic map in the bifurcation diagram, there are examples of two of these kinds of crises and I will discuss that just now. We started out by saying that we would look at the period-3 window, and let me just remind you that the period-3 window we started looking at we examined this part of the period-3 window as far as the tangency in the tangent bifurcation and the intermittency was concerned. Now, we are going to look at this part of the period-3 window and I would just like to point out that the period-3 window also shows period doubling bifurcations. In particular, over here you see that the period-3 orbit splits at this point through a period doubling bifurcation to a period-6 orbit and then there is a period-12 orbit and so on and so forth. So, the period doubling cascade happens over here as well. However, we are going to however, we are going to look at this part of the diagram and where we see that there is a crisis occurring. (Refer Slide Time: 20:13)



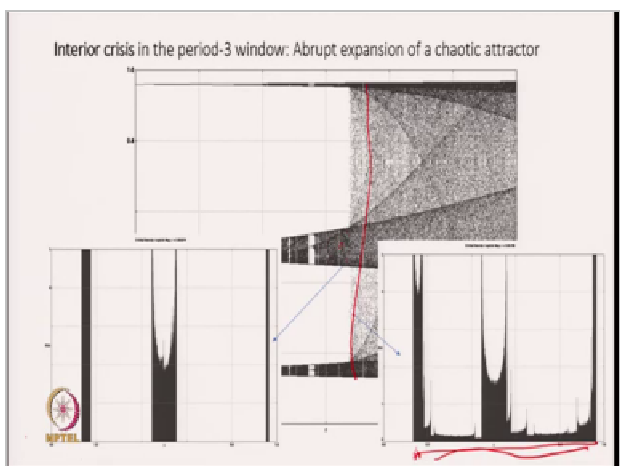
Now, this crisis point is very visible at this location where you notice that there is an the chaotic attractor which has three branches here is the period doubling leading to the you know this was an orbit of period-3, then period orbit 3 times 2 then this is 3 times 2 squared and so on and so forth all the way up to 3 times 2 to the n; n going to infinity. So, we have the period doubling cascade over there and yes that cascade is also geometrically characterized by the same two Feigenbaum constant that we discussed a couple of lectures ago. But, let us look at this particular point over here at this point we find that this chaotic attractor which was initially just consisting of these three portions over here suddenly becomes it suddenly occupies this entire period on this entire portion of the phase space. (Refer Slide Time: 21:25)



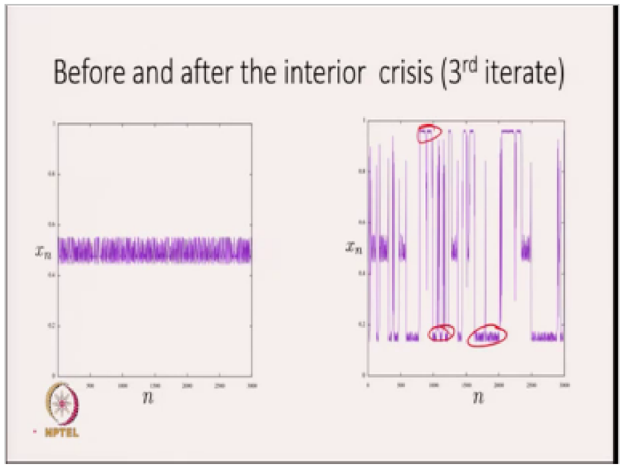
Basically what has happened is that this when the at the point of tangency recall that there was a period a stable period-3 orbit and an unstable period-3 orbit. When this unstable period-3 orbit collides with this chaotic attractor at this point over here there is a sudden expansion in the width of the attractor leading to this at this particular point of the crisis. (Refer Slide Time: 21:59)



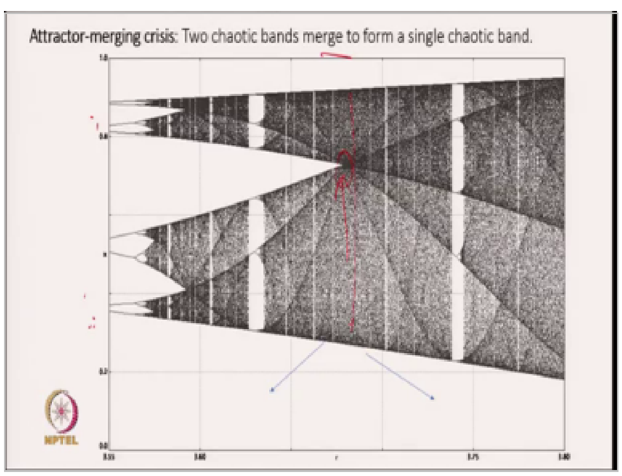
So, the interior crisis in the period-3 window is the abrupt expansion of the chaotic attractor. If I looked at the orbit on this side of the crisis the density now this is plotting the density of the orbit on the in the phase space. So, this invariant density is concentrated in three regions that is one over here, one in the middle and one on the top that is over there. So, these three parts can contribute to this density on this side of the crisis. (Refer Slide Time: 22:40)



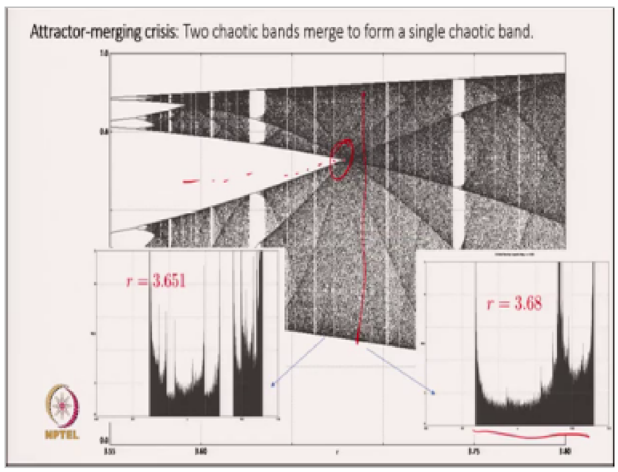
On the other side of the crisis, let us say if I took a point somewhere over here, the density still has features that you see in the earlier diagram, but as you can see now it is continuous. It actually extends entirely over this entire region from there till. So, the crisis is marked by a discontinuous change if you like in the support of the invariant density. (Refer Slide Time: 23:13)



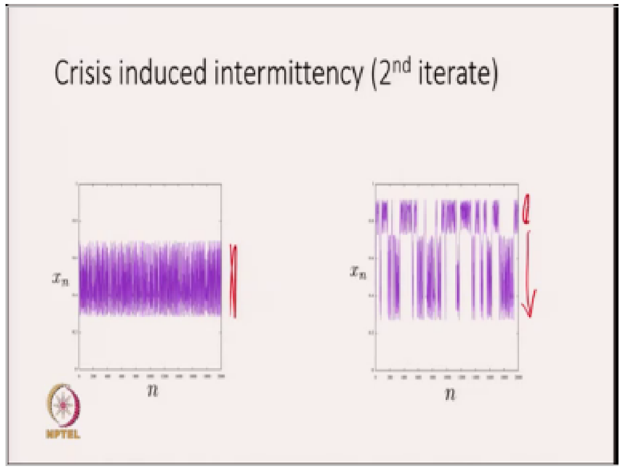
If I looked at the dynamics before and after the interior crisis and now, I would just look at the third iterate. When I am on this side every third iterate goes on to a chaotic orbit that lives in this particular region over here and it looks like so. This is a completely chaotic orbit, but it is confined to that in a band. After the interior crisis has happened that chaotic orbit escapes from that region although like any intermittent motion it stays close to this almost chaotic attractor over here moves to this part for awhile comes down to this part for awhile. And, so, this kind of motion is also intermittent except that now we have intermittency going between two different kinds of chaotic attractors so much for the interior crisis. (Refer Slide Time: 24:26)



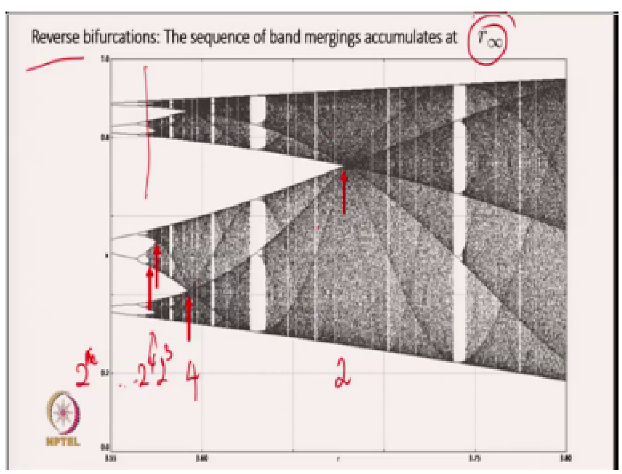
Another kind of crisis also occurs in the in the logistic map. There are a lot of features in this logistic map as one can see and here we have two chaotic bands merging to form a single chaotic band. Recall again in the bifurcation diagram that you have a period-2 orbit which then becomes a period-4 orbit, but the points of the period-4 orbit are always alternating between the upper and the lower branch, period-8 all the powers of 2 oscillating between the upper and the lower branch all the way down here where the points alternate between the upper and the lower branch. After this point over here you notice that there is no distinction between the upper and the lower branch. These two branches these two chaotic bands have merged to form a single chaotic band. (Refer Slide Time: 25:29)



And, if I were to look at the invariant density before and after this merging crisis it looks like so. Before the merging crisis I have two well separated regions of support for this invariant density, and after the merging crisis it is still remembers the old the way in which the density is distributed, but now it is actually continuous over the entire region where it is where the orbit lives. So, these two chaotic bands now merge to form a single chaotic band at this particular point and there is an unstable periodic orbit which actually hits the chaotic attractor at that particular point. So, the merging crisis the attractor merging crisis can also be discussed in terms or you should properly be discussed in terms of the collision of an unstable periodic orbit with a chaotic attractor. (Refer Slide Time: 26:33)

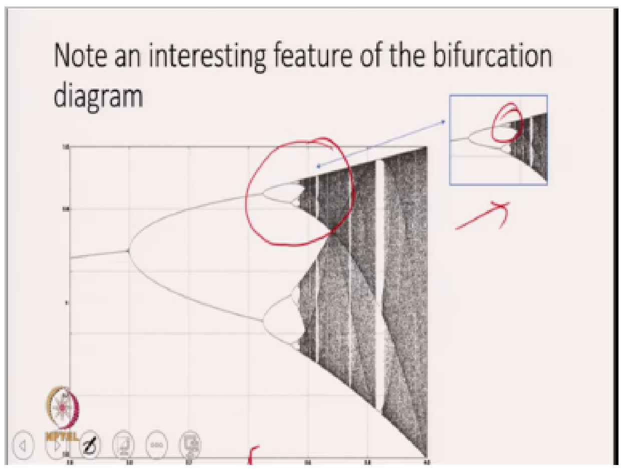


The crisis induced intermittency if I now look at the second iterate looks something like this. Every second iterate in the, you know prior to this merging crisis is on this same branch of the attractor and you see that it is confined to just that branch of the attractor. After the crisis for some time it is on this branch, sometime it is on that branch and so on and so forth. So, you see that this intermittent dynamics now takes you between two somewhat different kinds of motion and they also are characterized by power-law distributions. (Refer Slide Time: 27:13)

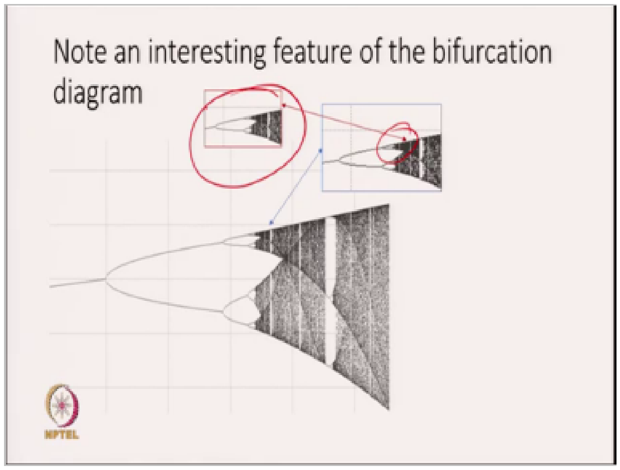


I should also like to point out that there are many band merging that happen. Here is the case where 2 bands merge to 1, here is the case where 4 bands merged to 2, and of case where 8 bands merge to 4, and 16 bands merge to

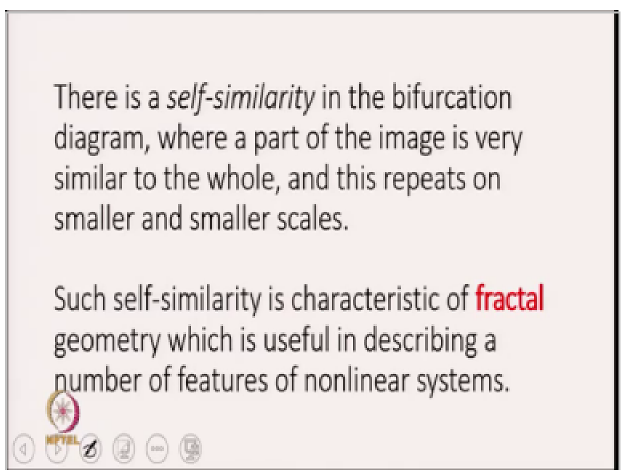
8. As a matter of fact if this is where the 2 bands merge, if this is where the 4 bands merge, this is where the 8 bands merged and the 16 bands merged. All these will now or they will be similar series of band mergings in what are called reverse bifurcations or noisy periodicity by Lawrence. And, they all this goes to 2 to the n and as n goes to infinity these all will accumulate at the same value of r namely r infinity. So, we have this very interesting sequence of forward bifurcations accumulating at r infinity and reverse bifurcations also accumulating at r infinity and in a very nice satisfying kind of geometric picture. Again, these verifications will also be characterized by the same values of delta and alpha as the forward bifurcations are. (Refer Slide Time: 28:39)



Now, there is an interesting feature the bifurcation diagram that I would like to draw your attention to. What you see over here is the bifurcation diagram from 2.8 to 4 in r, and if I were to just focus on a small portion of this if I look at it over here this looks pretty much like the entire bifurcation diagram. I need not stop over here I can go in and look at that portion. (Refer Slide Time: 29:15)



And, that portion now looks again like the old diagram and this portion of the blow up again looks pretty much like the diagram itself. So, we see that this geometric picture over here repeats almost exactly not exact not perfectly, but almost in smaller and smaller scales as we look at different portions of this of this figure. (Refer Slide Time: 29:50)



So, there is a self-similarity in this bifurcation diagram, a part of the image is very similar to the whole image and this keeps repeating on smaller and smaller scales. Now, such self-similarity is characteristic of what is called fractal geometry. This is a term introduced by Benoit Mandelbrot and this is useful in describing a number of features of non-linear systems and this is what we will turn to in the next lecture. Thank you.