Introductory Nonlinear Dynamics

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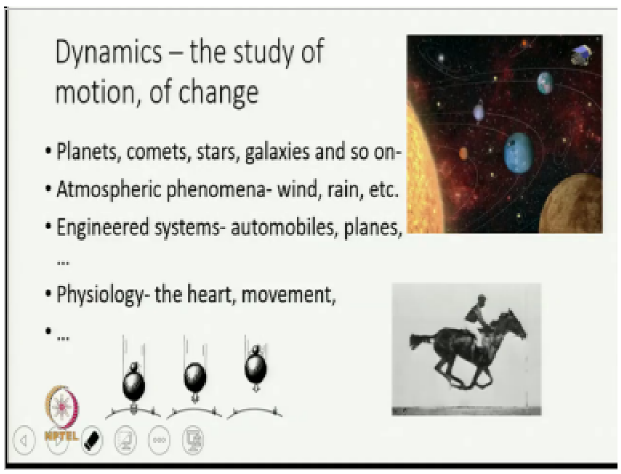
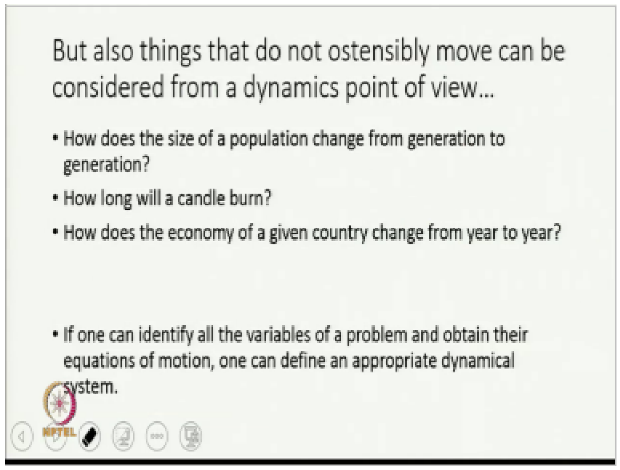
Lecture - 1

Introductory, Stability, Phase space Invariant sets

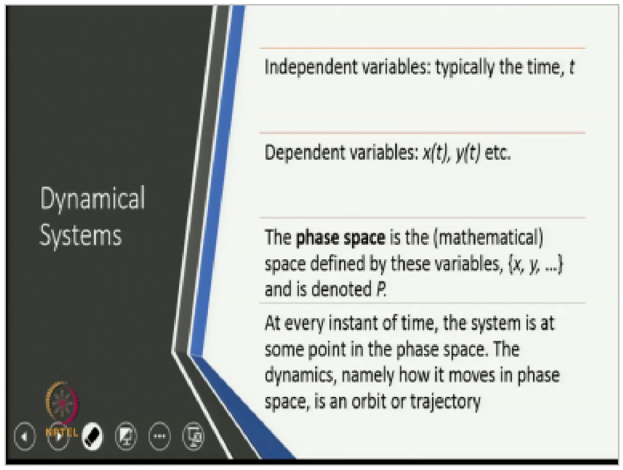
Hello and welcome to this Introductory course on Non-linear Dynamics. (Refer Slide Time: 00:27)



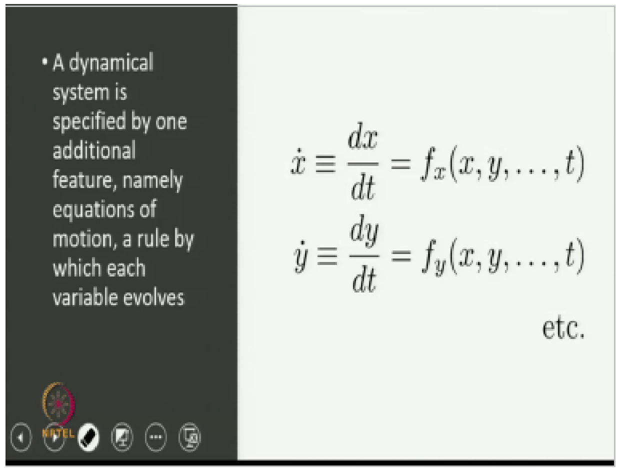
Before we start with the mathematics and so, on of this particular course, I would like to point out that dynamics the study of motion the study of change has been something that has fascinated mankind for many many centuries. From the earliest times we have been interested in the motion of planets, comets, stars, galaxies and so on. We have been observing the interstellar space, we have been seeing eclipses and there is always been an interest in trying to understand how this motion comes about. Similarly there is all manners of atmospheric phenomena that we are familiar with wind, rain, things move things that are engineered move like automobiles or planes or whatever and of course, there are motions that we see human beings other organisms carry out and within the human body, we are aware that the heart moves and so, on and the heart beats. So, a lot of phenomena that we are interested in are dynamical namely things the state of the system keeps changing as a function of time whether it is the Galileos experiment of dropped balls from the Tower of Pisa or the beating of once heart, the way in which motion happens the way in which these systems evolve is and has been of interest for a while. (Refer Slide Time: 02:03)



At the same time there are things that do not move as such but can be considered from a dynamics point differentiation if view. If you take a population in a country or in a city how does the size of a population change from let us say from one generation to another. How long will a candle burn, how does the economy of a given country change from year to year. Now one knows that there are different aspects that will go into answering any of these questions, the size of the population may change because of migration, it may change because the birth rate is high or that the birth rate is low. A candle may burn differently depending on what kind of material is there, what its shape is and the economy of a given country depends on a lot of complex different variables. Nevertheless all these kinds of questions or the questions posed over here namely what is the dynamics of planets or at the atmospheric phenomena or of the moving horse or whatever, all these can be approached from a unified kind of formalism. Basically if one can identify all the variables of a problem and obtain their equations of motion one can define an appropriate dynamical system which we can then proceed to study. (Refer Slide Time: 03:37)



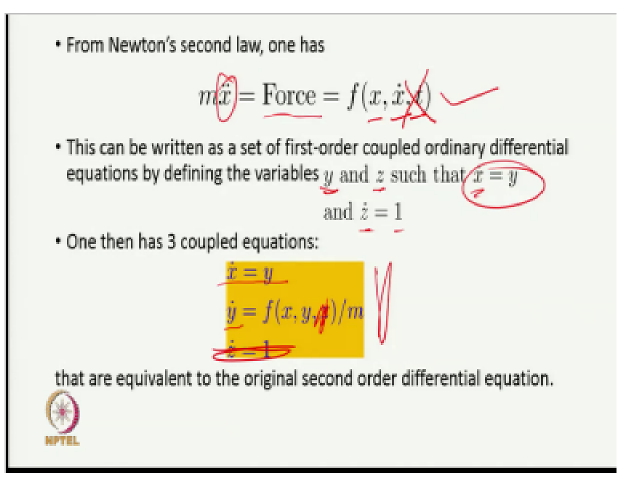
Now, in order to define dynamical systems, we typically have independent variables there could be several, but one is almost always present and that is the time the time is not dependent on anything else usually. So, we take that to be almost always the only independent variable. There could be a bunch of dependent variables and we will call them x of t and y of t and so, on and these dependent variables could represent you know for example, the position of a particle the position of a planet y of t could be its velocity or its angular momentum or whatever, it could be a variety of different properties talking about the atmosphere, it could be the temperature at a particular point, the velocity of the wind at a particular point and so on. Now there can be many different variables that the system would have and the number of different variables tells you the dimension of the problem. All these variables together they constitute the phase space of the system. This phase space is a mathematical space and sometimes it corresponds to the real configurational space as well, but most often it is just the mathematical space defined by these variables x y as many as there are and this is denoted by some symbol we will call it P. Now, at every instant of time the system is at some point in the phase space because all these variables tell you where the system could possibly be ah. So, it goes from a point in phase space to at a later time another point in phase space and at is still later time to yet another point in phase space and the path that it traces in the phase space is known as the orbit or the trajectory of the system. (Refer Slide Time: 05:53)



There is one more feature that is required in a dynamical system and that is rule an evolutionary rule which tells you how a given variable will evolve. So, the equation of motion could have a rule for each of the dynamical variables. So,

$$\dot{x} = \frac{dx}{dt} = f_x(x, y, ..., t)$$
$$\dot{y} = \frac{dy}{dt} = f_y(x, y, ..., t)$$

and there can be many such equations one for each of the variables that is there in your phase space and which describes your system. Now time has you know time has a slightly special role in in these in this discussion because in many systems it is not necessary to observe all the variables at each instant of time. (Refer Slide Time: 07:05)



Now, there are familiar laws of motion that we are that we are used to starting for example, from Newtons second law, which says that mass times the double derivative of the position which is the acceleration is equal to the force or usually it is written as f is equal to m a where a is the derivative of the velocity. Now Newtons second law can be actually rewritten as the following

$$m\ddot{x} = Force = f(x, \dot{x}, t)$$

and this force is a function of the position x the velocity $\dot{x}$ and perhaps the time. Frequently one finds equations of motion that come out of classical mechanics written in the so, called Newtonian form. Newtons second law just to recall it, says that force is equal to mass times the acceleration. If x is the position of a particle, its derivative dx by dt is the velocity and the derivative of the velocity dv by dt or d squared x by dt squared is the acceleration. So, one has Newtons second law written as

$$m\ddot{x} = Force = f(x, \dot{x}, t)$$

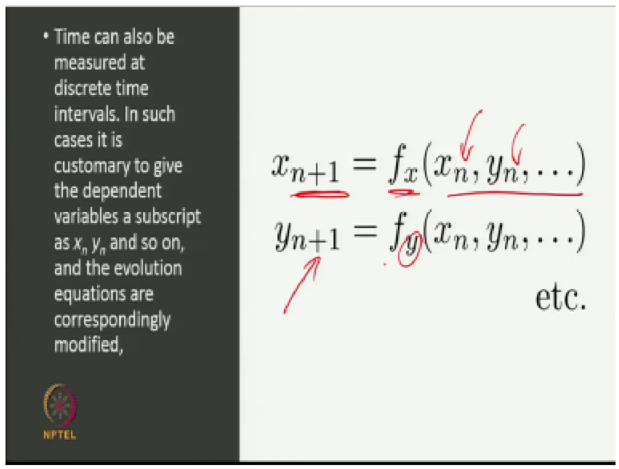
. Now the force is a function of the position and possibly the velocity and still more possibly the time. How does one consider this in the framework of dynamical systems in particular the set of coupled first order differential equations that I have just written down on the previous slide. Well, we can do this in the following sense by redefining some variables in particular if I define the variable y to be $\dot{x}$ and a new variable z to be such that $\dot{z} = 1$, then you note that z itself is just the time and I can rewrite these this particular requite this second order differential equation over here as 3 coupled equations

$$\dot{x} = y$$

$$\dot{y} = f(x, y, z)/m$$

$$\dot{z} = 1$$

just defines for me the time and these 3 equations are exactly equivalent to the first equation of second order that we wrote down. Now, if time does not appear explicitly in this equation if time is not there; there is no need for this particular equation and no need for this variable. So, a second order differential equation is equivalent to 2 first order differential equations which are coupled. Therefore, one can easily generalize this syndrome and you can work it out quite simply that if I have the nth derivative of time in the equation, then I can write rewrite this as n first order differential equations and henceforth and in this course we will only consider first order ordinary differential equations that are coupled. (Refer Slide Time: 11:13)



Time can also be measured at discrete intervals rather than continuously. This is natural in some systems and some and you can also consider this to be a particular choice that one makes, that instead of describing the system over the entire duration of time one just measures it at discrete intervals. In such cases we rewrite these equations not in terms of discrete arguments of time, but as subscripts and this subscript tells you what is the value of the variable x at the time step number n+1 and this is given as a rule.

$$x_{n+1} = f_x(x_n, y_n, ...)$$
$$y_{n+1} = f_y(x_n, y_n, ...)$$

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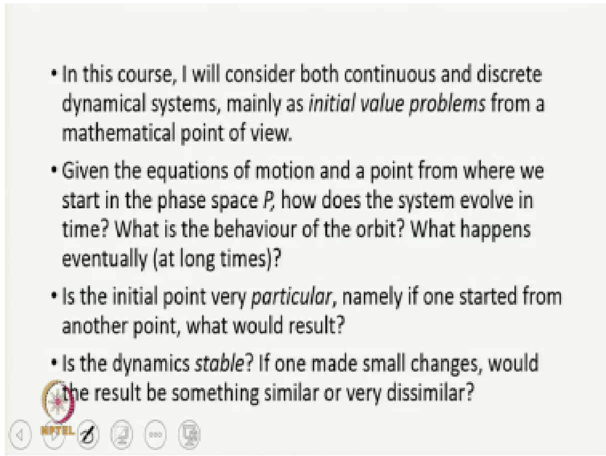
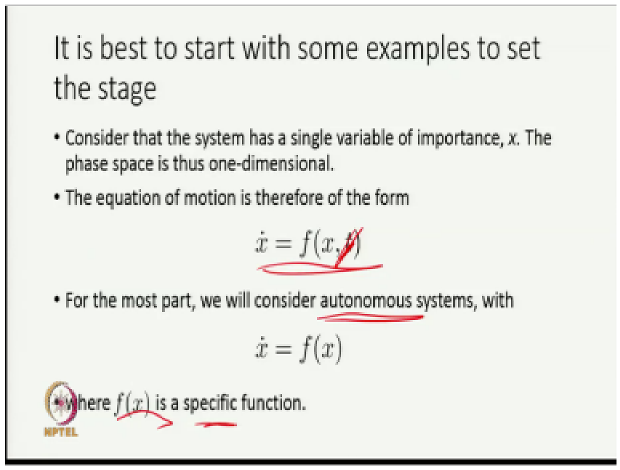


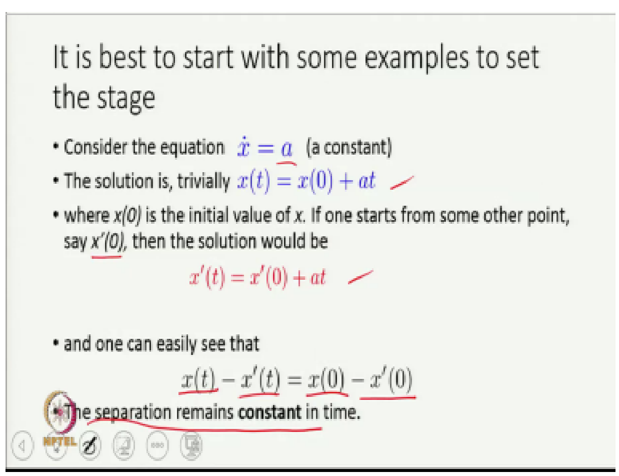
Now, we are going to be interested in both continuous and discrete dynamical systems in this course, but I will see them as initial value problems. Namely if I start at some particular point in the phase space use these evolution equations, how does the system progress in time, where does the orbit go from one point to the other to the other and in particular I am going to be interested in what happens at long times. We are going to be interested in this question from several points of view, but let me just enunciate a few for today. One thing that we would like to know is the initial point that you are considering is it somehow special. If I started from another point would I go would I get some other behaviour, would I would I get would I get something which is drastically different from what I am observing with this particular initial point or will I am am I going to get something which is fairly similar. A related question really is the following that is the dynamics stable? Namely if one made small changes would the results be something similar or would it be something very very different. These kinds of questions have actually dominated the field of dynamical systems over the last century also the stability of the solar system in particular is supposed to have sparked off a competition in Paris in the 1800s, late 1800 and it actually led to the birth of this field of non-linear dynamics and so, on. But that is for another forum. (Refer Slide Time: 14:19)



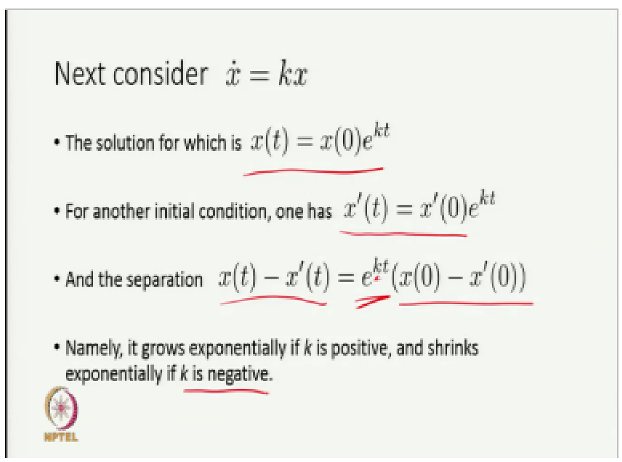
Let us get down to business by starting with some simple examples. Supposing the system is a very has a simple and it has only a single variable of importance let us call that x. The phase space of the system is thus one dimensional and the equation of motion has the form

$$\dot{x} = f(x, t)$$

and as I have said possibly the time for the most part in this course we are going to consider autonomous systems that is to say systems where the right hand side does not depend on t explicitly. It is of course, implicit there in $\dot{x} = f(x)$ and f(x) is given by some specific function. (Refer Slide Time: 15:15)



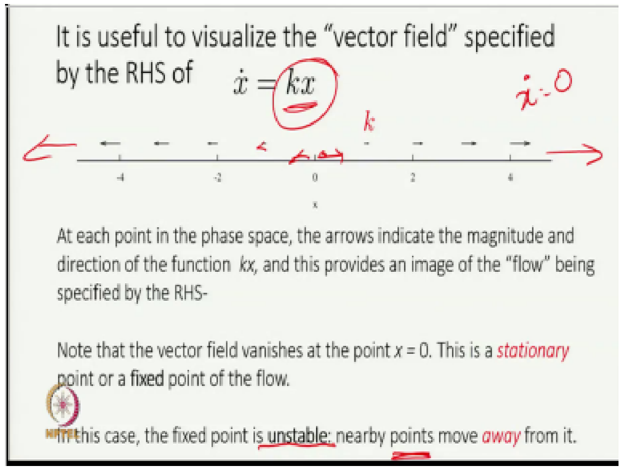
So, if we want to consider the simplest possible example, this function f(x) is a constant. So, $\dot{x} = a$ and you can trivially integrate this equation to say that $x(t) = x(0) + at$is equal to 0 and if you start from some other point let us say $x'(0)$, then the solution just putting in primes in this equation gives you $x'(t) = x'(0) + at$. To come back to the question that I asked earlier is a solution somehow special well depending on what your value of x(0) is and x'(0), both the solutions look absolutely identical. So, there is nothing particularly special about the solution. You can also see that if I look at the initial separation between these two solutions namely if I look at $x(t) - x'(t) = x(0) - x'(0)$ has exactly the same value. The separation remains constant in time it neither grows nor shrinks this is going to be in sharp contrast to some other examples that we will see. (Refer Slide Time: 16:49)



In particular if I were to consider this equation $\dot{x} = kx$, the solution for which is again rather simple and can be written down almost by inspection that $x(t) = x(0)e^{kt}$. Notice that for another initial condition you have $x'(t) = x'(0)e^{kt}$ and the separation

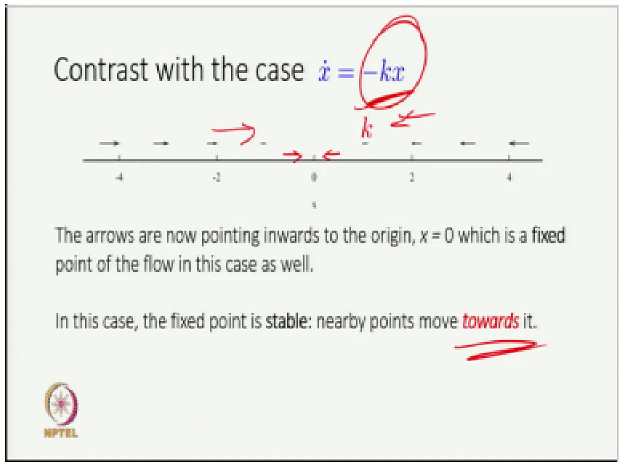
$$x(t) - x'(t) = e^{kt}(x(0) - x'(0))$$

So, this separation in time it is going to grow exponentially if k is positive and it is going to shrink if k is negative and in fact, go to 0. So, this rather simple linear equation already has something different compared to this rather trivial equation namely, the separation here remains constant in time and here the separation either increases exponentially or decreases exponentially depending on the sign of k. (Refer Slide Time: 18:13)

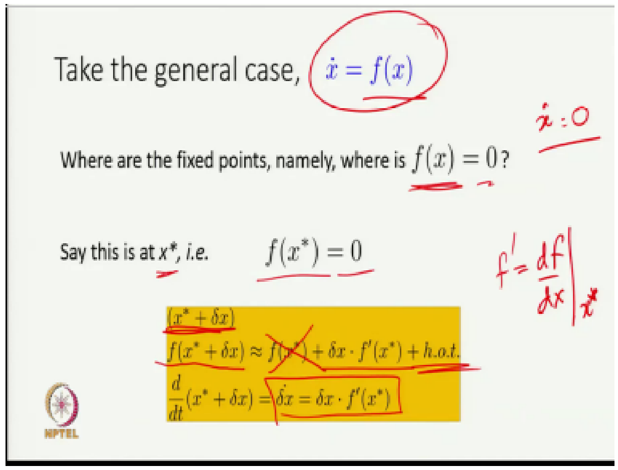


Now, to make things simple we will not just take k to be a positive number so, that I can write it in 2 different ways and let us see now how to bring another geometric point of view. The right hand side of this equation $\dot{x} = kx$ this function is specifying a velocity alright because if x if I think of x as a position then $\dot{x}$ is a velocity and kx tells me what is the functional form of the velocity depending on those position. So, this specifies a certain vector field alright and this vector field and I am just going to visualize it in the following way along the x axis, at each point x let me draw a little vector which is got the magnitude depending on where it is. So, at the position x is equal to 1 the size of this vector is k, at position 2 it is 2 k at position x is equal to 3 it is 3 k and so, on and similarly at x is equal to minus 1, x dot is equal to minus k and that is now a vector of pointing in the opposite direction, at -2 it is -2k,-3k and -4k.

So, at each point in the phase space the arrows are indicating the magnitude and the direction of the flow this in this particular case kx and this provides an image of this flow which is specified for us by the right hand side. In this simple case of course, is just the arrows are going get bigger and bigger as you keep going out outwards, but in other cases there is going to be something interesting that happens. Now, I would like to draw your attention to one feature and that is that the vector field vanishes at the point $x = 0$, because that x=0,$\dot{x} = 0$ this is called a fixed point of the flow and this is a stationary point. In this case this fixed point is such that any motion close by is going to be flowing outward away from this fixed point, such behaviour is called unstable. Namely we will turn this fixed point unstable because nearby points move away from it. (Refer Slide Time: 21:03)



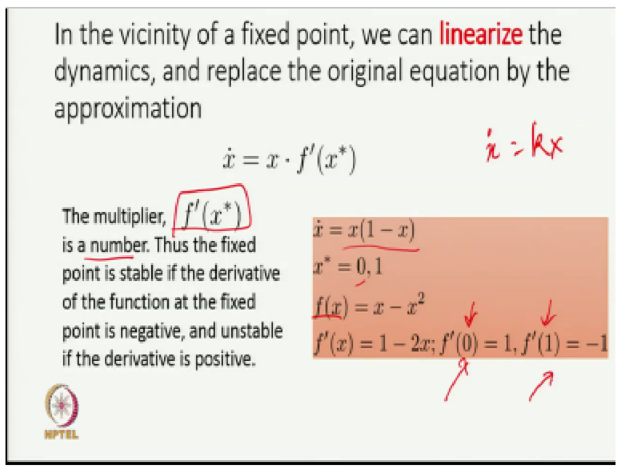
Take the contrasting case $\dot{x} = -kx$ as I pointed out I am going to keep k positive and just put the negative sign over here to show you that I want to consider this case. Now if I redo this exercise of drawing arrows, you notice that the arrows over here let me see if I can rub out let me erase alright. So, in this case you see how the arrows are pointing away from the fixed point $x = 0$. In this particular case because k is negative the arrows are pointing inwards from here at the point x is equal to 2 for example, the value of the vector field is 2 times minus k. So, it is minus 2 and that is a vector pointing in this direction, it is a vector pointing in this direction. Similarly, at the point x=-2,, the vector field is pointing in the plus direction because the size of the vector is plus 2 k. Nearby points over here therefore, are going to be flowing inwards into this fixed point and in this case we call this fixed point stable because nearby points move towards it. So, this is we have now looked at the case where the velocity vector the term on the right hand side was either a constant or it was a simple linear function either minus kx or plus kx. (Refer Slide Time: 23:05)



What about the general case? In the general case we understand and this is where non-linearity starts showing its effects, in the most general case we there is very little that one can say immediately, but there is one set of points around which we can say a few things. In particular we these are the fixed points of the system, at the fixed point note the fixed points are those points where f(x)=0.

If f(x)=0 then at that point $\dot{x} = 0$ and; that means, the velocity vanishes nothing moves there is no further dynamic. There can be several fixed points in a given a particular form of this function f(x) and let us say that there is a fixed point and we will call this fixed point x star. So, at the fixed point $f(x^*) = 0$. In order to analyse the behaviour in the closed neighbourhood of the fixed point let me consider this variable x star plus delta x. So, a small perturbation away from the fix point x star and from Taylors theorem we know that we can rewrite f in the neighbourhood of this particular fixed point as f at x star plus delta x is approximately it is exactly equal to an infinite series, but I am just going to write the first two terms and that is it is f of x star plus delta x times f prime of x star where $f' = \frac{df}{dx}$ evaluated at x^*.

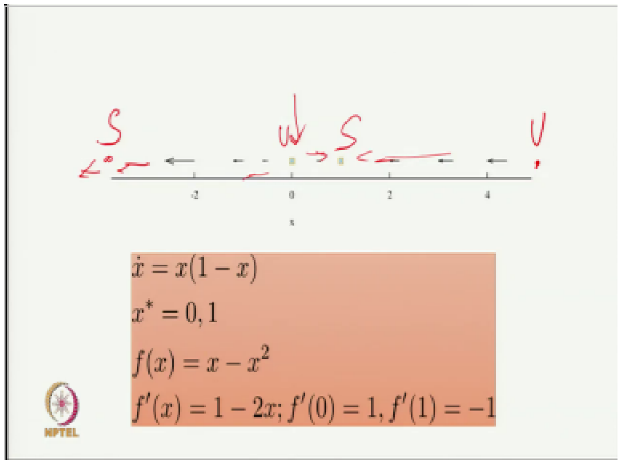
Now, how does the dynamical system look around this fixed point? Taking the derivative at I should also point out there are higher order terms, but I am not going to consider it because I would like to consider really small delta x. So, I can now look at the dynamical system in the neighbourhood of this fixed point and rewrite this equation as d by dt of x star plus delta x, which I can rewrite now as delta x dot and the reason I drop all this is that x star is some constant it is some number it is the fixed point. So, it does not have a derivative and delta x dot is just going to be equal to it is going be equal to f of x star plus delta x which is f of x star which is 0 and the only term that is remaining is delta x times f prime of x star. Now I can clear up the notation and rewrite this equation, I mean you can see the deltas are superfluous in this, but the linear approximation in the neighbourhood of the fixed point can just be rewritten as x dot is equal to x times f prime that is df by dx at x star. (Refer Slide Time: 26:33)



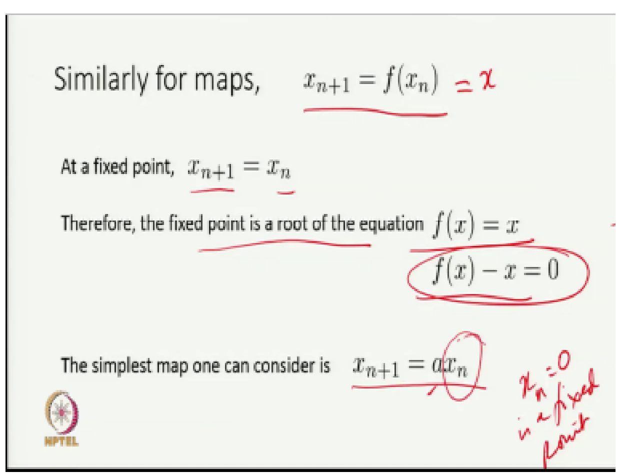
In the vicinity of this fixed point, this procedure is termed linearization of the dynamics and we can replace the original equation by the this approximation. Now, is this fixed point stable or unstable? We know that this equation $\dot{x} = kx$ is stable if k is negative and unstable if k is positive. So, the multiplier f prime of x star which is a number that tells me what is the value of the derivative of f at that fixed point and if it is positive that fixed point is unstable and if it is negative if fixed point is stable. To give another very simple example if I take

$$\dot{x} = x(1-x)$$

, you can easily see that there are 2 fixed points $x^* = 0, 1$ both of these are fixed points. This function f(x). I can write it explicitly as $f(x) = x = x^2$ and $f'(x) = 1 - 2x$. At the fixed point 0 this derivative has the value 1 at the fixed point one this derivative has the value minus 1. And so, without any further analysis one can declare that 0 is a fixed point which is unstable and 1 is a fixed point which is stable. (Refer Slide Time: 28:31)



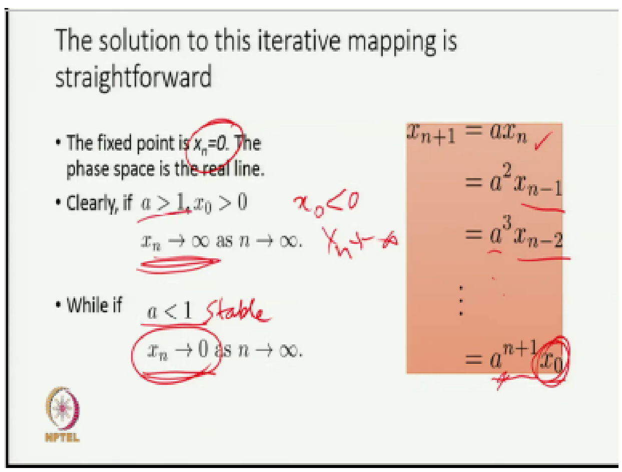
Again going back and looking at the vector field and looking at the flow, one can draw these points out here oops you have one fixed point over here 0 and you can see that the arrows are moving outwards, another fixed point 1 and you see the arrow from here moves outwards, but from here all the arrows are moving inwards. So, this point is stable and this point is unstable. You can easily convince yourself that if there was another fixed point over here not for this equation, but in general I am saying, if you had another fixed point over here just by continuity this on fixed point necessarily must be unstable and if there was a fixed point over here it is very clear that this fixed point over here because this one is unstable this fixed point must be stable namely in one dimension the fixed points must alternate in stability along the line. (Refer Slide Time: 29:51)



What about for maps? If you consider maps

$$x_{n+1} = f(x_n)$$

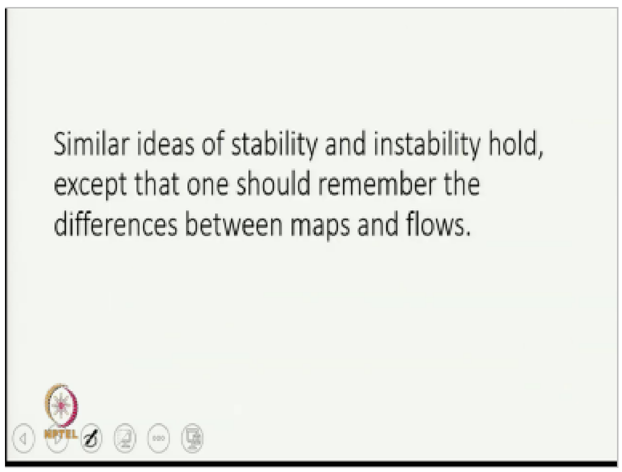
a fixed point is always the fixed point. So, the iteration at time n+1 is the iteration at time n this is the condition for the fixed point and therefore, you can easily determine that a fixed point must be a root of this equation $f(x_n) = x_n$ namely this equation $f(x) = x$ or if you would like to write it in this way, $f(x) - x = 0$ and I would like to know what are the roots of this equation. The simplest maps that one can consider our map linear maps like this $x_{n+1} = x_n$ and you can easily see that $x_n = 0$ is a fixed point because regardless of the value of a if $x_n = 0$ then $x_{n+1} = 0$. (Refer Slide Time: 30:57)



The solution to this kind of iterative map is actually very straight forward the fixed point is I have already pointed out is 0 and the phase space is the entire real line.

$$x_{n+1} = ax_n$$

xn plus 1 is a times xn and xn itself must be a times xn minus 1. So, xn plus 1 is a square times xn minus 1 and therefore, by iteration this is a cube time xn minus 2 and so, on and so, on until you reach it is a to the power n plus 1 times x of 0. Clearly if a is greater than 1 then as n goes to infinity xn xn plus 1 or xn must also go to infinity because if I am starting with some small some value some positive value of x naught. If this factor is if this multiplier is some number larger than 1 as n goes to infinity, xn will go to infinity. If x naught was is less sorry if x naught is less than 0 that is if I start with a negative value of x naught then as xn goes to infinity sorry as n goes to infinity xn will go to minus infinity. On the other hand if a is less than 1 then note that this becomes some number less than 1 to the power n plus 1 and as n goes to infinity xn will go to 0 namely it will go to the fixed point. So, if a is less than 1 in regardless of the value of x naught whether it is positive or negative eventually it will go it will lead you to the fixed point xn is equal to 0 all right. So, this situation is clearly what we have we can identify as stable and this situation is unstable. (Refer Slide Time: 33:31)



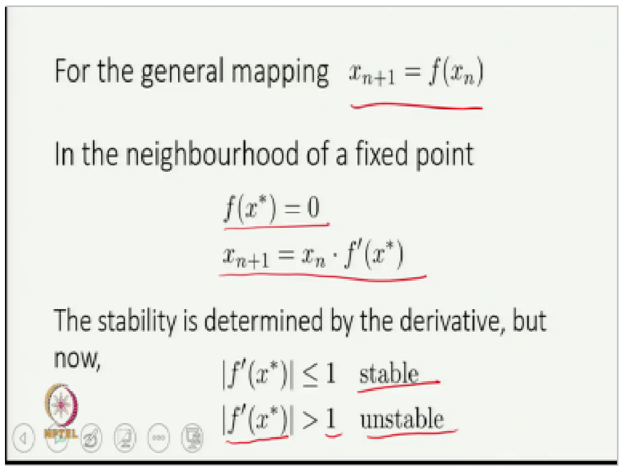
Namely similar ideas of stability and instability hold in both the cases except that one should remember the differences between maps and flows. If you have got a flow then the derivative of the function of the fixed point whether it is positive or negative tells you whether the flow is stable or unstable and when you have a map, the derivative of the fixed point the magnitude being bigger than 1 or less than 1 the absolute magnitude of it being bigger than or less than one that tells you whether the system is stable or unstable. Again for a general map if I write down

$$x_{n+1} = f(x_n)$$

, In the neighbourhood of this fixed point, I can do the same process of linearization and rewrite the system as

$$x_{n+1} = x_n.f'(x^*)$$

where f' is the derivative of this function and the stability is determined by the derivative, but now we note that if $| f'(x^*) | \leq 1$ it is stable this multiplier is less than 1. (Refer Slide Time: 34:51)



So, either as n goes to infinity it will shrink and if the modulus is bigger than 1 the system is unstable. Now, this is a good point to stop because tomorrow when we return I will take up small examples of this and discuss how one looks at these systems in general and then also go to one to extra dimensions let us look at 2 dimensions and see what are the differences between 1 and 2 dimensions see you tomorrow.