Nuclear shell model

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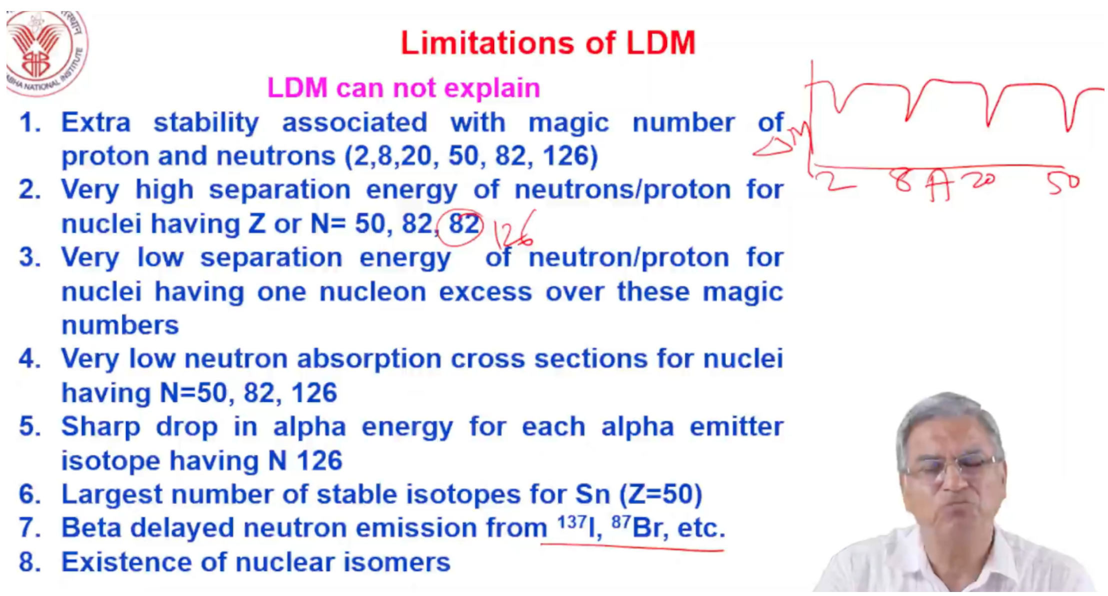
Lecture-5, module-1

Hello everyone. In the previous lecture, we discussed about the liquid drop model, which explained the masses of the nuclei, the energetics of the beta decay and fission. But then we found that there are certain observations, it could not explain and they are essentially the fluctuations in the gross properties of the nucleus. So, to explain those phenomena, which could not be explained by liquid drop model, another model had been proposed and that is the nuclear shell model. So, today we will discuss the nuclear shell model. Just to recapitulate, what are the limitations that the liquid drop model had, which it could not explain certain observations I have tried to list here.



So, the most important aspect that the liquid drop model could not explain was the extra stability associated with the magic number of protons and neutrons. In fact, what we discussed in the previous one that if you draw the difference in the liquid drop mass and the experimental masses, then we found that the experimental masses are lower than the liquid masses for certain number of neutrons and proton. And this corresponds to, let us say, 2, 8, 20, 50, 82 and so on. So, that means these nuclei have lower mass, that means they have higher binding energy and therefore there is extra stability associated with these magic numbers.

So, this part could not be explained using the liquid drop model. And associated with these magic numbers, there are other properties, other observations like the very high separation energy. Separation energy means the energy required to remove a neutron or proton for nuclei having this number of protons and neutrons, again the magic numbers, 50, 82, 126. So, this is 126. Similarly, now if you have a nucleus having one neutron more or one proton more than the magic numbers, then that nucleus, that neutron or nucleon is very easy to remove.

That means the separation energy is very low for these nuclei which have magic number plus one nucleon excess, again associated with the magic number. And also the nuclei having the magic number of neutrons and protons have, particularly magic number of neutrons have very low neutron absorption cross-sections. So, that the next level is very high. So, the probability of capturing a neutron become very small.

Then the, again, if the, in the case of alpha decay, which we will discuss later on in the probably subsequent lectures, if the radioisotope has got 126 neutrons, that is again a magic number, then the alpha energy for such a nucleus is very low. Also, if after the alpha decay, the daughter product is having 126 neutrons, then the alpha energy is very high. So, again associated with the masses. Again, the magic number of neutrons and protons, they have very large number of isotopes, particularly if you have 50 protons, such as tin isotopes, tin has got a large number of stable isotopes, again, extra stability.

Other than that, there are other observations like beta delayed neutron emission. Certain isotopes like fission products, ^{137}I, ^{87}Br, these isotopes emit delayed neutrons, means they undergo beta minus decay. And after beta minus decay, the daughter product is left with an excited state that is more than the binding energy of neutron. So, instead of gamma emission, that excited state emits a neutron, they are called delayed neutrons because they follow the half-life of these precursors.

Also, the isomers, the nuclear isomers are those, certain states of nuclei, which have a long half-life and they decay by gamma ray, sometimes their decay by gamma ray hindered, they can also emit beta minus or beta plus. So, the nuclear isomers also could not be explained by the liquid drop model. And that is why there is urgent need to have another model that we call the shell model.

Nuclear Shell model
M.G.Mayer and J.H.D.Jensen 1949

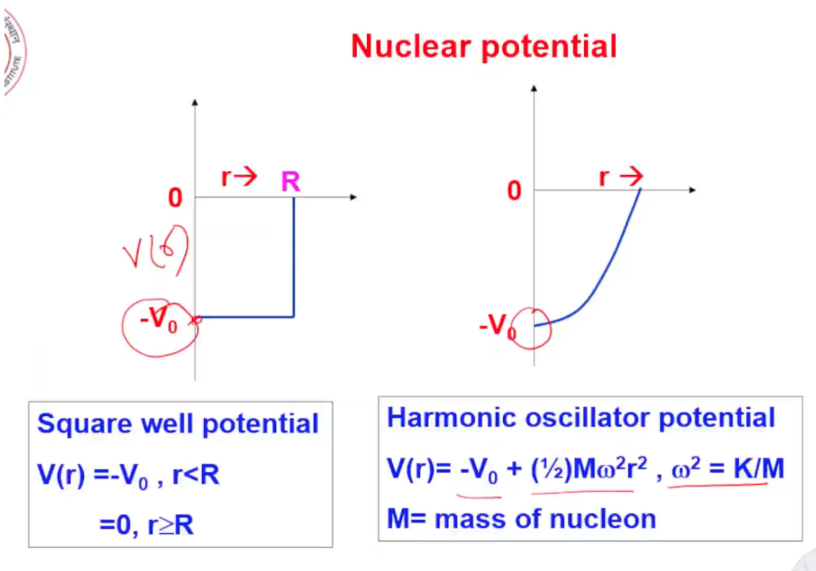
Assumptions in the shell model

1. Nucleons move inside the nucleus under the attractive potential generated by the remaining nucleons.
2. Neutrons and protons move in separate potential wells.
3. Nuclear force is assumed as weak force as the nucleons in different orbits do not interact.
4. Pauli Exclusion principle is followed

So, this is the shell model, which was proposed by Meyer and Jensen in 1949. And the shell model has certain assumptions, like, here now it is more like atomic structure, where the electrons are evolving around the nucleus in certain fixed orbits, stationary orbits. Here, the nucleons move, now, unlike in the atom, where we have a central potential offered by the nucleus, the nucleus is positively charged and the electrons are negatively charged and electrons move around the nucleus because of the central potential, effective potential of the nucleus. And so, you have the assumptions about these stationary states of atoms by Niels Bohr.

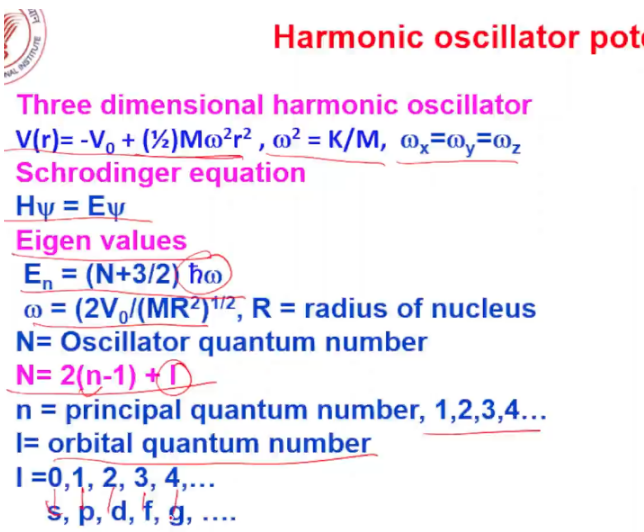
Here, in the case of nucleus, there is nothing like the central potentials because the nucleons themselves constitute of the potential. So, what is assumed that every nucleon moves inside the nucleus under the attractive potential generated by the remaining nucleons. So, any nucleon you take, it is in a potential well that is formed by the other nucleons. Now, in the case of protons and neutrons, they are in separate potential wells. So, you start seeing two potential wells, one for neutron, one for protons.

And like, you know, for the atomic structure, you have to solve the Schrodinger equation for an electron, you find out the energy levels of electron, K, L, M and so on shells. Similarly, in this case of nucleus, you will have to solve the Schrodinger equation for the neutron and the proton in a potential well, you will get the energy states and that energy states are analogous to that for the electrons. Now, as you know, the neutrons and protons are fermions. So, they follow the Pauli exclusion principle. And so, you will find that in a particular state, you know, only two nucleons will be occupying, they have to be paired up. So, no two nucleons will have the same quantum numbers. So, this is followed. And in that way, when the neutrons and protons are orbiting in different orbitals, so essentially you assume this nuclear force is a weak force in that sense, it is not a weak force associated with the beta decay, but it is compared to the strong force that we envisaged in the liquid draw model, in the shell model that the neutrons and protons are in separate potential wells and they don't interact with the other nucleons in that potential. So, in that sense, it is a relatively weak force. Okay, so essentially the shell model constitutes solving a Schrodinger equation for the nucleon in a particular potential well.



So, what are the different types of potential wells that you can envisage? The simplest potential well is the square well potential. We have the attractive potential well having the depth of $-V_0$, this $-V_0$, so the potential well of the nucleus is of the order of 30, 40 MeV deep. And so when we are solving the Schrodinger equation for a neutron or a proton, you can assume the potential as a square well. So, essentially you have Vr is equal to $-V_0$ for r less than the nuclear radius (R). And then for more than R it is 0.

So, that is the typical attractive square well potential. You can in fact solve the Schrodinger equation assuming the square well potential or more realistically you can use as a harmonic oscillator. In the harmonic oscillator again you have the negative $-V_0$, the depth of the potential and now it is a parabolic expression with r. So, you have the potential $-V_0$ plus $\frac{1}{2}M\omega^2 r^2$ that is the energy of the oscillator. And so the frequency of this oscillator is nothing but K/M the force constant upon the mass of the nucleon.



So, essentially in the shell model we solve the Schrodinger equation for this harmonic oscillator potential and then we get the energy eigenstates and the eigenvalues. Okay, so let us try to now generate the shell model scheme by solving the Schrodinger equation for a three-dimensional harmonic oscillator. So, the potential for the harmonic oscillator can be given in terms of V(r)

$$V(r) = -V_0 + \frac{1}{2} M\omega^2 r^2$$

where ω is the oscillator frequency that is given by the force constant upon the mass of the nucleon

$$\omega^2 = K/M$$

and assuming that this oscillator is isotropic that means the oscillation frequency in x, y and z direction is same,

$$\omega_x = \omega_y = \omega_z$$

So, the Schrodinger equation for this will be

$$H\psi = E\psi$$

We will not go into details of the solving Schrodinger equation.

Our interest is to get the eigenvalues for this equation and the eigenvalue of this Hamiltonian will be given in terms of the oscillator quantum number $(N+3/2)\hbar\omega$, 3/2 is coming from the three-dimensional harmonic oscillator and this is the unit of the oscillator energy. So, the frequency of oscillation can be given by

$$\omega = \left(\frac{2V_0}{MR^2}\right)^{1/2}$$

where R is the radius of the nucleus and N is the oscillator quantum number and so the oscillator quantum number can start from 0 to higher values 0, 1, 2, 3, 4 and so as the N value increases the energy of the shell increases.

Shell model states for harmonic oscillator potential

$N = 2(n-1) + l$

N	n	l	Orbital	Occupancy	Cumulative
5	3	1	p	6	
	2	3	f	14	
	1	5	h	22	
4	3	0	s	2	70
	2	2	d	10	
	1	4	g	18	
3	2	1	p	6	40
	1	3	f	14	
2	2	0	s	2	20
	1	2	d	10	
1	1	1	p	6	8
0	1	0	s	2	2

Does not reproduce the magic numbers 2,8,20, 50, 82, 126

Now, let us try to set up the eigenstates, the different shell model states. For that, like I told that the electrons are occupying the K, L, M shells in the atoms. Similarly, the nucleons, neutrons and protons are separately occupying different orbitals like S, P, D, F and so on and so there is a formula for the oscillator quantum number in terms of the other quantum numbers.

The n, principal quantum number and l is the orbital quantum number. The principal quantum number n can take values 1, 2, 3, 4 and so on, whereas the orbital quantum number can take the value 0, 1, 2, 3, 4 and so corresponding to the value of l we have S, P, D, F, G and so on. So the different combinations of n and l can give you the same value of oscillator quantum number and so you will find for the same oscillator quantum number there will be multiple orbitals contributing to that particular oscillator quantum number that we can see in the next one.

So let us try to now generate these shell model states for a harmonic oscillator, we will try to fill the values of n and l which will satisfy this condition starting from N equal to 0. So for the ground state of the oscillator we can have, n =1, 1 minus 1, 0 into 2, 0 and plus l equal to 0 also. So l = 0 means S orbital and S orbital can occupy two nucleons. So the occupancy of this orbital is 2 and the cumulative is also 2.

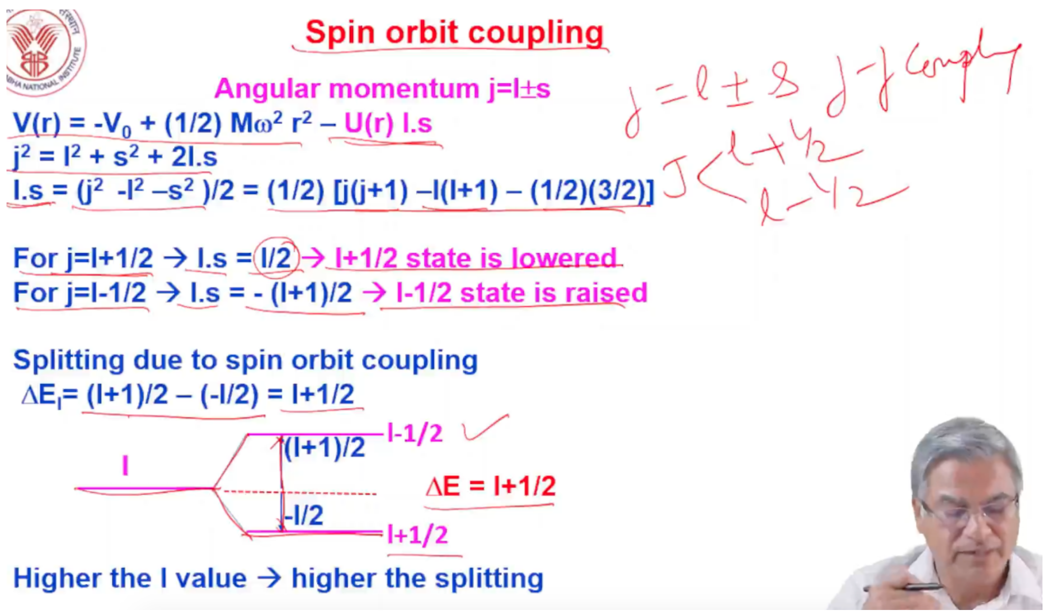
For the next oscillator quantum number N= l, again we can have, how can you get N equal to 1? So you start from small n equal to 1, 1 minus 1 equal to 0. So this term is 0, so we have l equal to 1, that is the P orbital. So capital N equal to 1 will have again one orbital P and the occupancy of P is 6 nucleons and the total occupancy becomes 8.

Now let us go to the next higher oscillator quantum number N equal to 2. Here we can have now two possible combinations of small n and small l. You can see here when you take n equal to 1, then 1 minus 1 is 0 and so you will have when we have n equal to 1, so you can see here, so here we are going to n equal to 2. So you will have n equal to 1 and 1 minus 1 0, so into 0, so l equal to 2. l equal to 2 is the D state and it can occupy 10 nucleons and in combination of that, so n equal to, now you can have n equal to 2, so 2 minus 1, 1 into 2, 2 plus 0, so you can have a 2S state. So basically you will have 2S, 1D, 1P, 1S. So you will have a 2S orbital close to the 1D orbital, which can take 2 nucleons and the total occupancy becomes, you can see here 6 plus 2 plus 10 plus 2 will have 20 nucleons.

Let's come to the higher oscillator quantum number 3. So you can again, you can have small n equal to 1, 1 minus 1. So small n will take 1, 2, 3, 4 values. So n equal to 1, 1 minus 1, 0. This term is 0. l equal to 3, so you will have, 1F orbital and F orbital can take 14 nucleons and in combination with that, you can have now n equal to 2, so you will have 2 minus 1 is 1, 1 into 2, 2 plus 1 is 3, so you have 2P orbital. And so these two, actually they are corresponding to same capital N, so their energies are very close and so P can take 6 nucleons, so total is 40.

Let's come to the capital N equal to 4 and here you can have now three combinations, n equal to 1, so l equal to 4, we have 1G, n equal to 2 and this l equal to 2, you have 2D and n equal to 3, you will have 3S. And so the occupancy of G orbital 18, D orbital 10 and you have 70.

So you can see here that by filling the orbitals using this formula and their occupancy, you can see that we are able to reproduce the magic numbers of 2, 8, up to 20 you can reproduce, but beyond that the important magic number 50, 82, 126 we are not able to reproduce. So that means there is something missing in the shell model which we still we have not taken into account.



And so that is where the spin-orbit coupling becomes important. You know in the atoms, in chemistry we have heard about the LS coupling, that means the small l orbital angular momentum of all electrons combined to give the capital L, that is total orbital angular momentum and the spins of all electrons combined to form the capital S that is spin angular momentum and the capital S and capital L combined that is called the LS coupling or Russell-Sanders coupling and that nicely explains the spins of the atoms. In the case of nuclei, because the spin-orbit coupling is very, very strong, the l and s of the individual nucleons combine to form what is called the j and so this is called the j-j coupling.

That means the j angular momentum of each nucleon, they combine to form the total spin of the nucleus. So it is not the individual l and individual s. l and s coupled together to form a j. And so if in the potential for this harmonic oscillator, you take a spin-orbit coupling term that is -U(r) l.s, then let us see how the level scheme will change. So you can see the vector sum of the l and s will be j and so you can write

$$j^2 = l^2 + s^2 + 2l.s$$

And we want to get the magnitude of $l.s$

$$l.s = (j^2 - l^2 - s^2)/2$$

and you can now write the magnitude of j^2, l^2, and s^2 as j(j+1) , l(l+1) and s(s+1),

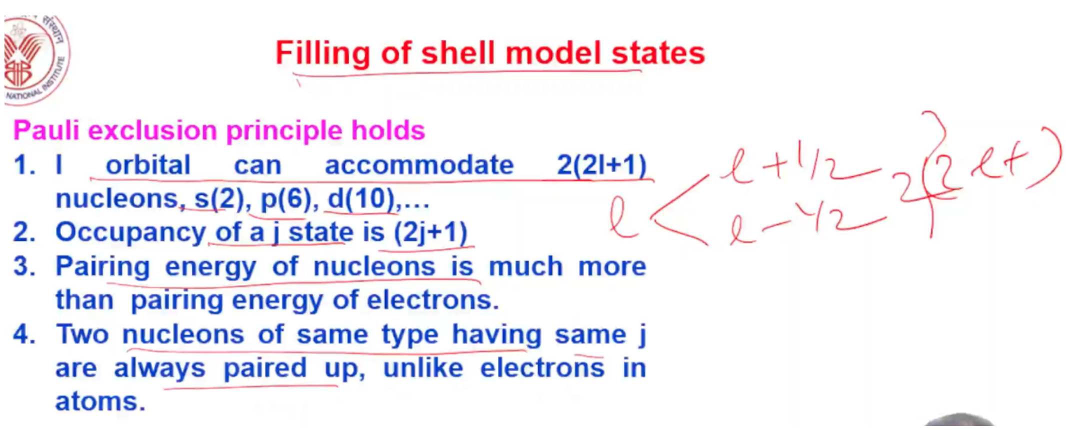
$$l.s = \left(\frac{1}{2}\right)[j(j+1) - l(l+1) - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)]$$

And so let us just see what happens to the spin-orbit coupling, what does it do to the particular state. So we have now instead of ls state, we have a j state and so for j, j can be now l +1/2 or j can be l -1/2 and so you will have different energy state. So each l state splits into two states, l+1/2 and l-1/2.

$$So\, when\, j \;=\; l \;+ \frac{1}{2},\;\; l.s \;=\; l/2$$

And so this $l\,\frac{1}{2}$ is, that means if you put l /2 here, then this is a negative term minus U(r) l.s. So this is the lower energy state and for $j \;=\; l \;+ \frac{1}{2},$ l.s, you put in this formula, you get –l+1/2. So essentially meaning that l-1/2 is raised because if you add, you put it here to minus of minus becomes plus and so l+1/2, state is lowered.

So the net result of spin-orbit coupling is a particular l state, you have splitting of the l state, l+1/2 being lowered and l-1/2 being raised. The splitting between these two states is, you can calculate from here, (l +1)/2 – (-l/2) so that is l+1/2. So this is the gap between the two states. As you can see here, that as you go to higher and higher angular momentum values, orbital angular momentum values, the gap between the l+1/2 and l-1/2 is increasing. And this is what actually explains later on, you can see that because of spin-orbit coupling, the higher shells are much more split and that leads to rebunching of the orbitals in different shells. So that we will see in the next slide.



So higher the l value, higher the splitting and that the repercussion of that we will see in this particular and then now we will fill the shell model states in the view of spin-orbit coupling. So let us see how do we start filling the states. First thing is that we follow the Pauli exclusion principle that means no two nucleons will have same quantum numbers. The main point is the filing of electrons in atomic orbitals and nucleons is quite different. In the case of atoms maximum multiplicity prevails, the state having maximum unpaired electrons having lower energy. But in the case of nucleus, the moment you have two unpaired nucleons, they get paired up because their pairing energy is very high. So the l orbital as you have seen previously can accommodate 2(2l+1) nucleons like s(2), p(6), d(10), f(14) and so on. Accordingly, the occupancy of j state because l has been split into l+1/2 and l-1/2. So each l has split into two and the total occupancy is equal to 2(2l+1) 2.

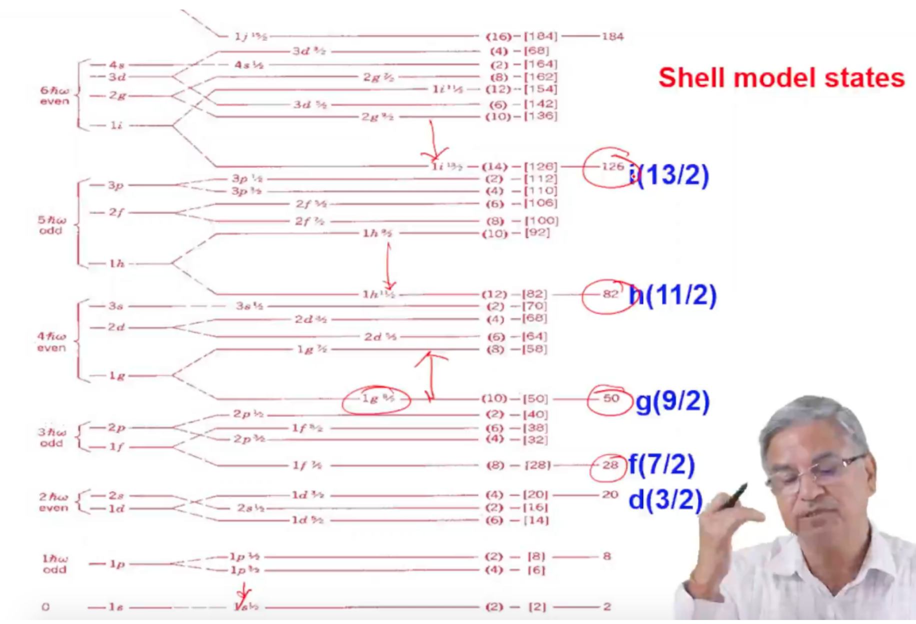
So the J states will have occupancy (2J+1). So (2J+1), every j state you will see (2J+1) occupancy. Then the pairing energy of nucleons is much more than the pairing energy of electrons. The electron pairing energy of the order of few electron volt, whereas in the case of nucleons, we have seen that delta value for pairing energy is 1 to 2 MeV. So it is a very high pairing energy. And the two nucleons of same type having same j are always paired up because of this high pairing energy, which is unlike the electrons in the atoms. So we will consider these facts by filling the orbitals by nucleons in the next slide.

N	n	l	Orbital	j	Occupancy	Cumulative
5				3p1/2	2	
	3	1	p	3p3/2	4	
				2f5/2	6	
	2	3	f	2f7/2	8	
				1h9/2	10	
	1	5	1 h	1h11/2	12	82
4	3	0	3 s	3s1/2	2	
				2d3/5	4	
	2	2	2 d	2d5/2	6	
				1g7/9	8	
	1	4	1 g	1g9/2	10	50
3				2p1/2	2	
	2	1	2 p	2p3/2	4	
				1f5/2	6	
	1	3	1 f	1f7/2	8	28
2	2	0	2 s	2s1/2	2	20
				1d3/2	4	
	1	2	1 d	1d5/2	6	
1	1	1		1p1/2	2	8
			1 p	1p3/2	4	
0	1	0	1 s	$1s_{1/2}$	2	2

Now you can see here, so we have a j state, which is arising from l+1/2 or l-1/2 and occupancy of this is (2j+1). So this spin states, whatever I am showing, they are the j states of that particular orbit. So up to this we have already seen, so the S orbital is not split because l=0, and so it occupies the two nucleons. The p orbital is split into $1p_{3/2}$ and $1p_{1/2}$, and occupancy again (2j+1). So 2(3/2) +1 is 4, 2(1/2) +1 is 2. The d orbital will split into $1d_{5/2}$, $1d_{3/2}$, s orbital is not splitting. And 5/2 will have 6, 3/2 will have 4, 1/2 will have 2. f orbital again will split into 2, $1f_{7/2}$ and $1f_{5/2}$, and p orbital, $2p_{1/2}$ and $2p_{3/2}$. So this is 1s1/2, 1p3/2, 1p1/2, 1d5/2, 1d3/2, 2s1/2, 1f7/2, 1f5/2, 2p3/2. 2p1/2, and so on.

So these are the value of small n. So now you can see here, g orbital will split into $g_{9/2}$ and $g_{7/2}$, d orbital split into $d_{5/2}$ and $dd_{3/2}$, and s will not split, and so on, $h_{11/2}$, $h_{9/2}$. Now you can see the occupancy. So what happens, whenever the $f_{7/2}$ is lowered, this s, up to 20 we have explained in the previous scheme. But now, in fact, this s will give you 28, and at 28 also, there is a gap here. So this is called a semi-magic number, 28 neutron, 28 protons also have extra stability, but this is not a complete magic number, but 28, you know, the nickel, 28 protons, around 28 protons there is extra stability in the nucleus. So this is called a semi-magic number. But the major magic number is 50, again, because of the lowering of $g_{9/2}$ level. So this $g_{9/2}$ is a part of the lower level, lower shell, and so this bunching of these levels leads to what you call as the shell. So $g_{9/2}$ lowered gives you 50, magic number 50, that gives the stability. Similarly, $h_{11/2}$ when it is lowered compared to $h_{9/2}$, it explains the magic number 82. So this bunching of this $h_{11/2}$ in the lower shell explains the magic number.

So that is how, similarly, you will find $i_{13/2}$ will give you 126. The major factor responsible for reproducing the magic number is the spin orbit coupling, wherein the splitting of the l state into j, l+1/2 and l-1/2, with the l+1/2 being lowered significantly for high l value, that it is overlapping, it is enclosed to the lower shell, and so the magic numbers can be reproduced.



So this is now up to scale. You can see here the shell model states without spin orbit coupling, 1s, 1p, 1d, 2s, 1f, 2p, 1g and so on. And then you can now apply spin orbit coupling here, so $s_{1/2}$ $1p_{3/2}$, $1p_{1/2}$.

Now, as you go to, so here until 20, there was no problem. You have now $f_{7/2}$ being lowered. In fact, there is something called a semi magic number of 28. So there is extra slight higher stability for 28 protons or neutrons. And now you see here, because of the lowering of $1g_{9/2}$, there are now additional 10 nucleons here.

So there is a 50 nucleon shell magic number. And again, lowering of the $h_{11/2}$, you get 82, lowering of the $i_{13/2}$ you get 126 and so on. So you can see the gap, because of this gap between the l plus half and l minus half states for higher lvalues, the differences are rising. And because of that, we are able to explain the magic numbers of the nucleons.

So the orbitals that are responsible for this magic number are the l plus half state of d, f, g, h and i. So this is how we can see that l plus half state gets lowered and then it bunches with the previous shell. And so you get the magic number. So this is what the shell model explains. And the other aspects, how we can apply the shell model in predicting the different properties that we could not explain using the liquid drop model, I will discuss in the next part of the lecture. So I will stop here and take up the application of shell model in the next lecture. Thank you.