Applications of Liquid drop model

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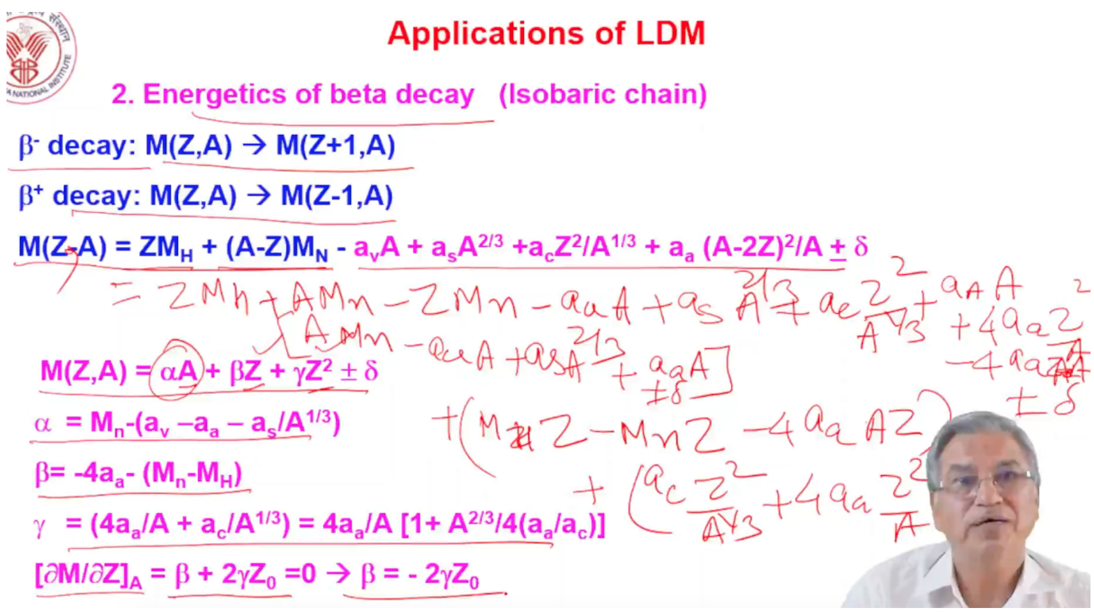
Lecture-4, Module-2

Hello everyone. Last lecture, we discussed the liquid drop model, where we derived an expression for the mass of a nucleus using the phenomenon of a liquid drop and considering the Coulomb energy and the asymmetry between neutrons and protons. Using this semi empirical mass formula, we also discussed that it can predict the masses of nuclei quite accurately. Today, I will discuss the other applications of liquid drop model.



One of the most important applications of liquid drop model is to predict the energetics of beta decay in an isobaric chain. We know that β- decay occurs in an isobaric chain, and in this process, the mass atomic number is increased by 1, while in β+ decay, the atomic number decreases by 1.

Now, let us try to calculate the Q for beta decay, and then we will see the systematics of the Q beta decay. So for beta decay, let us calculate, we can in fact write the mass of a nucleus in terms of the coefficients of Z and Z^2. So the mass of a nucleus or an atom M(Z, A) can be written as mass of Z protons, mass of (A – Z) neutrons minus the binding energy term. So, this we have already seen that there is the semi empirical mass formula.

And now let us try to rearrange these terms in terms of, we will try to get this formula in terms of Z independent term, Z dependent and Z^2 dependent terms from this above form. So you can write this as,

$$M(Z - A) = ZM_H + (A - Z)M_N - a_v A + a_s A^{\frac{2}{3}} + a_c \frac{Z^2}{A^{\frac{1}{3}}} + a_a \frac{(A-2Z)^2}{A} \pm \delta$$

So now let us try to rearrange the terms which are independent of Z, so that will be $AM_N - a_v A + A^{\frac{2}{3}} + a_a A$. These are the terms which are independent of Z. So then it becomes

$$M(Z - A) = \alpha A + \beta Z + \gamma Z^2 \pm \delta$$

Where $\alpha = M_N - (a_v - a_a - \frac{a_s}{A^{\frac{1}{3}}})$, $\beta = -4a_a - (M_N - M_H)$

and $\gamma = \left(\frac{4a_a}{A} + \frac{a_c}{A^{\frac{1}{3}}}\right) = \frac{4a_a}{A}[1 + \frac{A^{\frac{2}{3}}}{4\left(\frac{a_a}{a_c}\right)}]$

So we have actually tried to rearrange this formula in terms of coefficients of A, Z and Z^2. That is the purpose of rearranging this mass formula and then you can write in terms of $\alpha A + \beta Z + \gamma Z^2 \pm \delta$. These are the terms and you can rearrange them so that you know this equation looks familiar to you in terms of the coefficients of Z.

Now this α, β and γ actually are called the local constants because they are constant for that particular isobaric chain, they depend upon the mass term. And if you want to find out the most stable isobar in this isobaric chain, then you can differentiate this formula with respect to Z and so from here you get

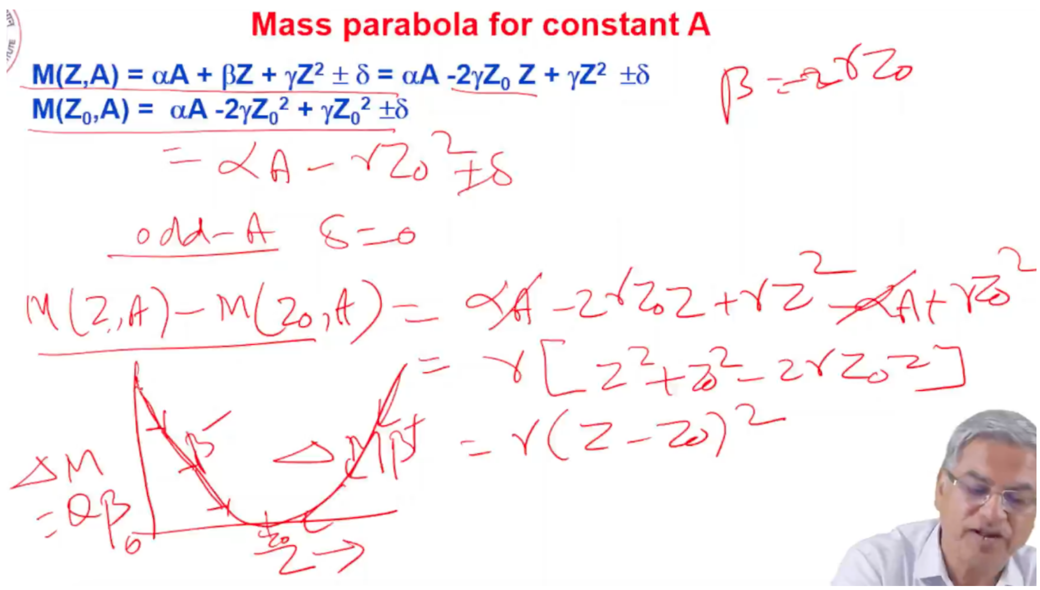
$$[\frac{\delta M}{\delta Z}]_A = \beta + 2\gamma Z_0$$

And if you equate this to 0 to get Z_0, then it becomes

$$\beta + 2\gamma Z_0 = 0 \; or \; \beta = -2\gamma Z_0$$

So essentially you have $\beta = -2\gamma Z_0$.

So you will have γ and Z_0. These are the two local constants. If you know the γ and Z_0, then you can find out the Q beta for the particular isobaric chain. So that we will discuss in the next slide.



Now let us discuss the isobaric chains for the particular mass number wherein the beta plus and beta minus decay is going to take place.

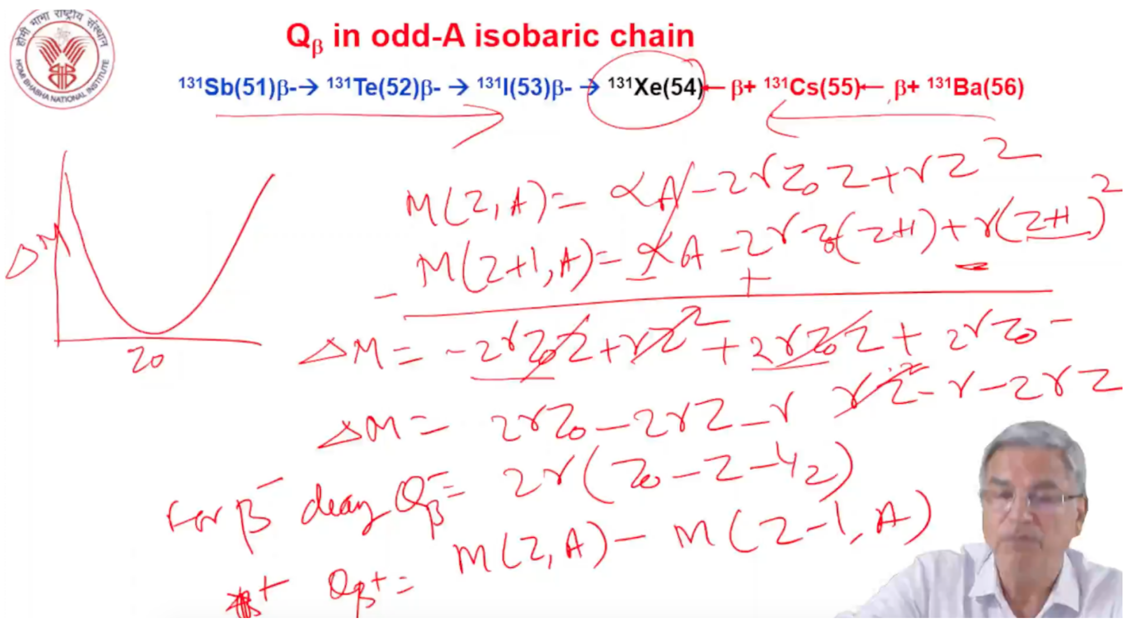
So we just discussed that the mass can be written in terms of $\alpha A + \beta Z + \gamma Z^2 \pm \delta$ and the β can be replaced by $= -2\gamma Z_0$. So I have put here

$$M(Z, A) = A - 2\gamma Z_0 Z + \gamma Z^2$$

Now as we discussed that for the minimum mass, the nucleus is most stable. So we can write the mass of the most stable isobar by replacing Z by Z_0.

So it becomes

$$M\left(Z_0, A\right) = \alpha A - 2\gamma Z_0^{\ 2} + \gamma Z_0^{\ 2} \pm \delta$$



So now let us see for odd A isobaric chain where $\delta = 0$. For the moment we will forget about δ term because for odd A δ is equal to 0. So let us try to find out the mass difference M(Z,A)-M(Z_0, A) equal to, so you subtract from here it will be

$$M(Z, A) - M\left(Z_0, A\right) = \alpha A - 2\gamma Z_0 Z + \gamma Z^2 - \alpha A + \gamma {Z_0}^2$$

So you see here αA will cancel out and it will become you see here γ is out what we have is

$$M(Z, A) - M\left(Z_0, A\right) = \gamma[Z^2 + {Z_0}^2 - 2\gamma Z_0 Z]$$

$$\Delta M = \gamma(Z - Z_0)^2$$

Now you see here the ΔM the mass difference between any isobar and the most stable isobar follows a parabolic relationship. So if you plot this then we get ΔM versus Z as a parabola where minimum corresponds to Z_0. Now this ΔM is nothing but Q_β for isobaric chains. But now you can see here that this is what we mean by the mass parabola. The masses of the isobars of a beta decay chain fall on a parabola.

Now you see here as the Z is increasing from left to right you will see that the up to Z_0 the lighter Z values will undergo β-. The heavier Z values will undergo β+. So irrespective of where you produce for example if you produce a fission fragment it will be highly neutron rich so it will undergo beta minus decay to ultimately lead to stability. If you produce a highly neutron deficient nucleus it will undergo beta plus decay or electron capture and come to the rest at Z stable. So this is what we mean by the mass parabola and using this formula we can find out the systematics of beta decay.

Let us now see for an odd A isobaric chain how we can derive the Q_β. So this is a typical example of a isobaric chain for odd A. You have one stable isobar shown in black and the ones on the left hand side are undergoing beta minus decay and the ones on the right hand side undergoing beta plus decay to reach the stability. And so this can be shown as a mass parabola this is Z_0 this is ΔM or Q_β. Now let us try to derive expression for the Q_β from this graph.

So we write

$$M(Z, A) = \alpha A - 2\gamma Z_0 Z + \gamma Z^2$$

And now you have the next isobar.

$$M(Z + 1, A) = \alpha A - 2\gamma Z_0(Z + 1) + \gamma(Z + 1)^2$$

So this is what is the ΔM value for adjacent isobars that means you are now trying to find out the Q_β for a β- decay. So it will get cancelled out. So what we have is,

$\Delta M =- 2\gamma Z_0 Z + \gamma Z^2 + 2\gamma Z_0 Z + 2\gamma Z0 - Z2 - \gamma - 2\gamma Z$

For now we can find out ΔM equal to so it will be cancelling out here you see here

$$\Delta M = 2\gamma Z_0 - 2\gamma Z - \gamma = 2\gamma(Z_0 - Z - \tfrac{1}{2})$$

this is the relationship for the Q_{β}- so we showed $Q_{\beta-}$ because we are going from Z to Z +1.

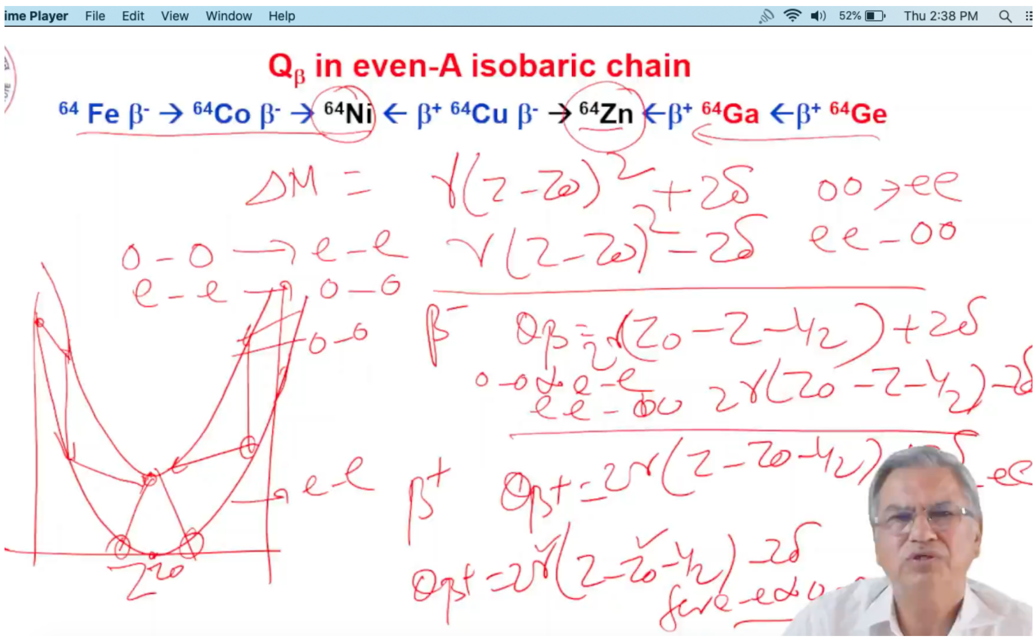
$$Q_{\beta-} = 2\gamma(Z_0 - Z - \tfrac{1}{2})$$

the same exercise for $Q_{\beta+}$ will give

$$Q_{\beta+} = M(Z, A) - M(Z - 1, A) = 2\gamma(Z - Z_0 - \tfrac{1}{2})$$

So using these expressions you can find out the Q_β using the local constant γ and Z_0. So what essentially you need is, two Q beta values along this decay in this parabola if you know the two Q beta values then you can find out the local constants Z_0 and γ and thereby the Q beta value of any other isobaric chains. So this is the application of liquid drop model.

So let us now discuss the Q beta systematics for even A isobaric chain and an example of that is given here that is this iron-64 decaying into cobalt-64 decaying into nickel-64 by beta minus and then we have from germanium-64 to gallium-4 beta plus to zinc-64 and as you can see here there are two stable isobars in this particular isobaric chain that is very common for the even A isobaric chains. So we have already discussed for the odd A isobaric chains, that is, the ΔM for beta decay will be γ (Z-Z_0)2 that we have seen for the case of odd A isobaric chain.



Now in the case of even A isobaric chain there can be two situations that is we can have beta decay from odd-odd nucleus to even-even nucleus and if this happens then there will be a term 2δ for odd-odd to even-even beta decay and it can have even-even to odd-odd then δ the term will be - 2δ so it will be

$$\Delta M = \gamma(Z - Z_0)^2 + 2\delta$$

for odd-odd to even-even. So what essentially you have is that you have if you see the mass parabola for even A isobars then you will have two parabola one for the even-even and one for odd-odd nuclei and so the beta decay will take place from even-even to odd-odd to even -even to odd-odd and similarly here you will have this way this way this way and so finally you may end up with two stable isobars and this can undergo probably both beta plus and beta minus like in the case of copper-64. So in case of even A isobaric chain you have two parabola because of the pairing energy difference so odd-odd isobars lie on the upper parabola and the even-even isobars lie on the lower parabola.

So let us now calculate the Q beta for the isobaric chain so for beta minus decay we have already seen the Q beta equal to for odd A isobaric chain it was

$$Q_{\beta-} = 2\gamma(Z_0 - Z - \tfrac{1}{2})$$

So if it is odd-odd to even-even then it will be

$$Q_{\beta} -= 2\gamma\left(Z_0 - Z - \frac{1}{2}\right) + 2\delta$$

and if it is even even-even to odd-odd then it will be

$$Q_{\beta} -= 2\gamma\left(Z_0 - Z - \frac{1}{2}\right) - 2\delta$$

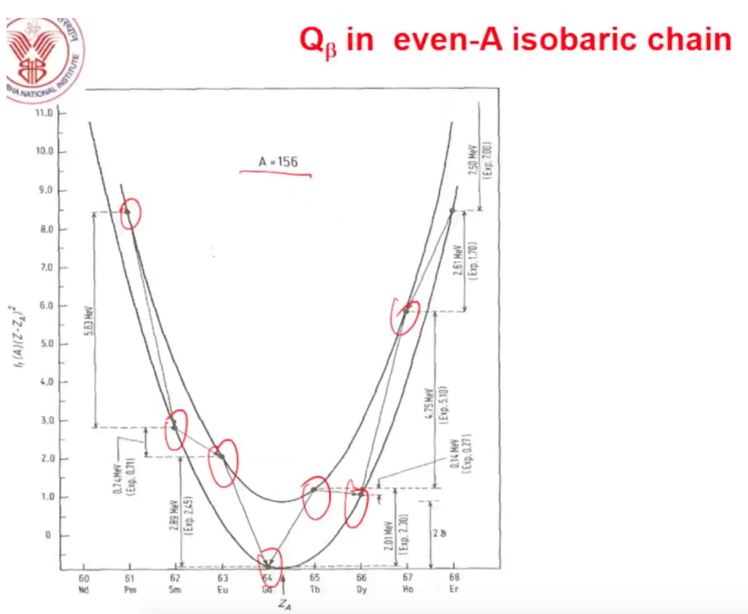
So that is the only difference in terms of for the odd-odd A to even-even A isobaric chain. Similarly, for beta plus decay Q beta

$$Q_{\beta} += 2\gamma\left(Z - Z_0 - \frac{1}{2}\right) + 2\delta$$

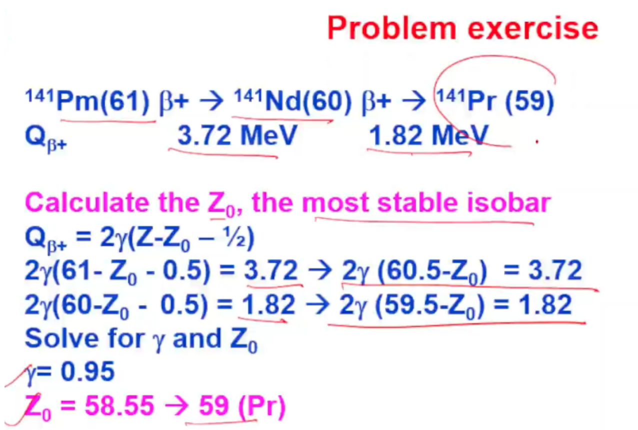
for odd-odd to even-even and Q beta plus equal to

$$Q_{\beta} += 2\gamma\left(Z - Z_0 - \frac{1}{2}\right) - 2\delta$$

for even-even to odd-odd. So this is how you can write the Q beta in terms of the local constants γ and Z_0 and if you know the Q beta values for two decays in isobaric chain then you can calculate γ and Z_0 and thereby you can calculate the Q beta for any other decays in the isobaric chain.



So this I have tried to show here in this graph you can see here this is a isobaric chain for mass number 156 and you can see here from neodymium, promethium, samarium, europium, gadolinium. So you can see the two parabola and there is beta decay from promethium to samarium, samarium to europium, europium to gadolinium and here again you have from erbium to holmium, holmium to dysprosium and erbium to dysprosium. So you see here that even A isobaric chain can have more than one stable isobar. This is a corollary of the two parabolas. So you can see this there can be more than one stable isobars for even-even isobaric chain. So this is the application of liquid drop model.



Just to see how we can use this model. We solve a problem here for mass number 141. The Q beta value for the promethium-141 to neodymium-141 is given here and for neodymium-141 to praseodymium the beta values are given. We have to calculate the Z_0, the most stable isobar from the information that is given here. So we can write the expression for Q beta plus

$Q_{\beta}+= 2\gamma\left(Z - Z_0 - \frac{1}{2}\right).$

$$Q_{\beta}+= 2\gamma\left(61 - Z_0 - \frac{1}{2}\right) = 3.72$$

$$2\gamma\left(60.5 - Z_0\right) = 3.72$$

So Q beta is 3.72.

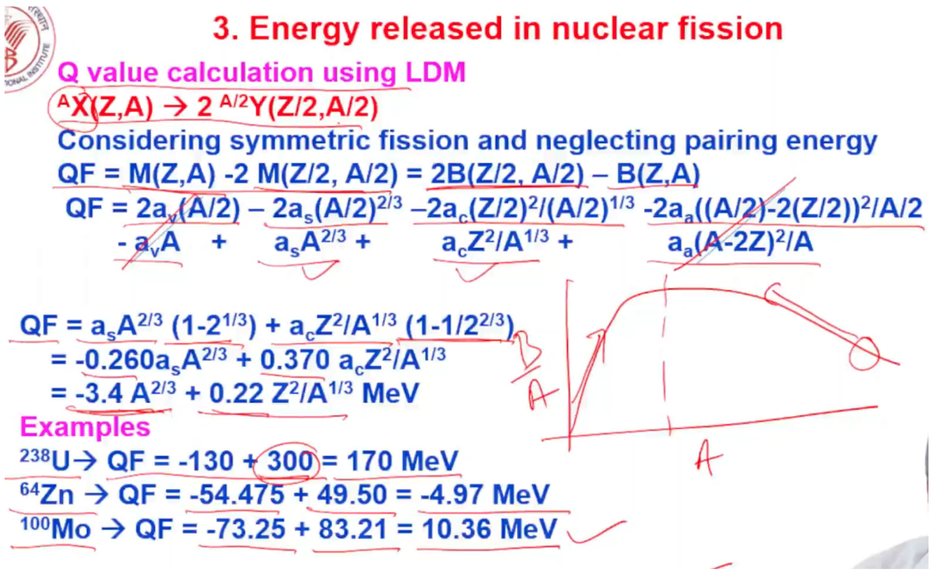
You can write this expression for the other beta decay.

$$Q_{\beta}+= 2\gamma\left(60 - Z_0 - \frac{1}{2}\right) = 1.82$$

$$2\gamma\left(59.5 - Z_0\right) = 1.82$$

So these two equations are there and you can solve them for Z_0 and γ and by solving you will find γ is 0.95 and Z_0 is 58.55. So that is close to 59.

So 59 is the praseodymium. So praseodymium is the most stable isobar of this isobaric chain. So this is how we can study the systematic of beta decay and the liquid drop mass formula derived to find out the Q beta value for beta decay.



Now let us see how the liquid drop model can explain the energetics of nuclear fission process. We know from the binding energy data, binding energy curve that is binding energy as a function of mass number. The binding energy per nucleon reaches a maximum at about mass number 60 and so heavy nuclei when they split into two equal fragments, there is a gain in binding energy and so Q value should be positive like the light nuclei when they fuse together to form heavy nuclei there is a gain in binding energy and so these are in the exoergic reactions.

So grossly you can say that heavy nuclei can undergo fission and because of the change in binding energy from low binding energy to high binding energy. Let us now calculate the Q value for fission from the liquid drop model. So assuming that liquid drop model is considered the highest energy release for a symmetric splitting of a heavy nucleus, we will consider the splitting of the heavy nucleus A into two equal halves. So we will say Z going to Z/2, A going to A/2. So there will be two segments of mass and charge A/2 and Z/2.

So let us see what is the energy released in fission. We had the parent isotope, the M(Z,A) fissioning nucleus splitting into two fragments of mass Z/2 and A/2. And now the same thing can be written in terms of the binding energy because the mass number remains constant. So the difference in the masses can be written in terms of the difference in binding energy in other way around. Difference in masses of reactant minus product is equal to difference in binding energy of product minus reactant.

So binding energy of the products minus binding energy of the reactant that is the fissioning nucleus. So now we can see here, we can write this in terms of twice the volume energy term of the half the nucleus.

$$QF = M(Z, A) - 2M\left(\frac{Z}{2}, \frac{A}{2}\right) = 2B\left(\frac{Z}{2}, \frac{A}{2}\right) - B(Z, A)$$

$$QF = 2a_v(\frac{A}{2}) - 2a_s(\frac{A}{2})^{\frac{2}{3}} - 2a_c(\frac{Z}{2})^2/(\frac{A}{2})^{\frac{1}{3}} - 2a_a((\frac{A}{2} - \frac{2(\frac{Z}{2}))^2}{\frac{A}{2}} - a_v A + a_s A^{\frac{2}{3}} + \frac{a_c Z^2}{A^{\frac{1}{3}}} + a_a(A - 2$$

Now you can see here this volume energy term will cancel out because it is nothing but 2 into av into A by 2 means 2 av. Similarly, you can see here the asymmetric energy term will cancel out whereas the surface energy and the Coulomb energy terms will not cancel out because they contain exponential power to the A/2 and A. So you can write now the Q value for the fission will be

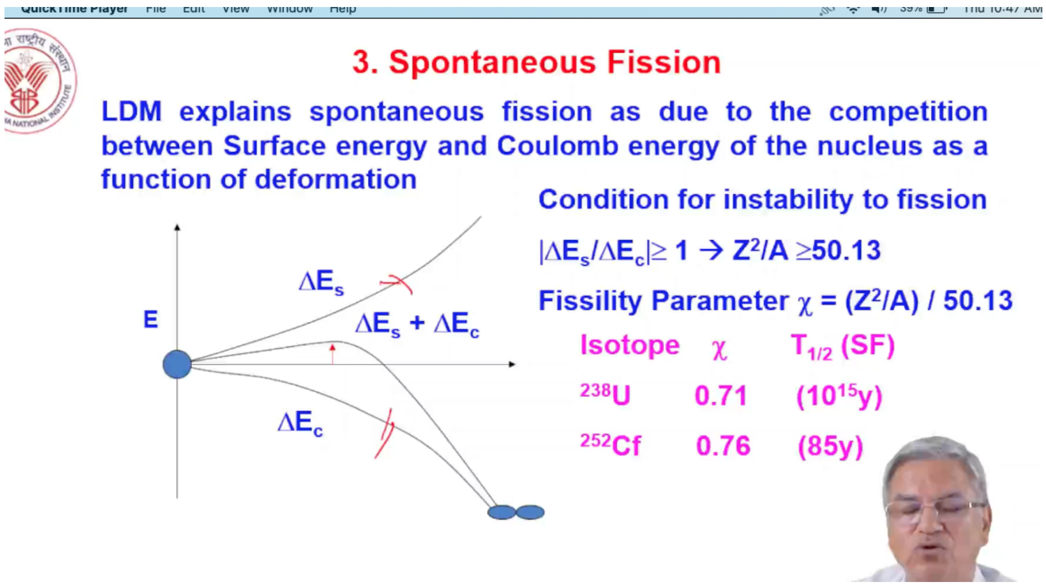
$$QF = a_s A^{\frac{2}{3}}\left(1 - 2^{\frac{1}{3}}\right) + \frac{a_c Z^2}{A^{\frac{1}{3}}}(1 - \frac{1}{2}^{\frac{2}{3}})$$

$$QF = 0.260 a_s A^{\frac{2}{3}} + 0.370 a_c \frac{Z^2}{A^{\frac{1}{3}}} =- 3.4A^{\frac{2}{3}} + 0.22\frac{Z^2}{A^{\frac{1}{3}}}\ MeV$$

So let us see the magnitude of this surface energy term and the coulomb energy term and then we can discuss which term dominates for which nuclei. For ^{238}U substituting in this value the mass number and Z the Q fission becomes this surface energy term will become -130 and the coulomb energy term become +300. So you have 170 MeV is the predicted energy released in the nuclear fission of ^{238}U. So you can see here that the Coulomb energy term is dominating in the fission of ^{238}U.

For Zinc-64 Q value for fission surface energy term -54.475, Coulomb energy term +49.5 so that is -4.9. So you can see here the Q value for fission of Zinc-64 is negative and so that explains why the light mass nuclei do not undergo fission.

For the intermediate nucleus the like Molybdenum-100 Q fission will be surface energy term -73.25 plus Coulomb energy +83.21 that is equal to 10 MeV. So this molybdenum-100 fission though it is you can see Q value wise it is positive but you will find that for the fission to take place there is a barrier. If you recall, we explained for the spontaneous fission it has to cross a fission barrier which is a resultant of the change in the surface energy and the Coulomb energy and the fission barrier is much higher than this Q value and therefore the half-life for this will be very high so you do not see fission taking place. So energetically it may be possible but practically you do not see it. That explains how liquid drop model can explain the energetics of the nuclear fission process.



The same thing I have tried to explain we will discuss this more when we discuss nuclear fission but the point, I wanted to bring home is that the liquid drop model explains the spontaneous fission of heavy nuclei wherein the competition between the surface energy and the Coulomb energy of a deforming nucleus is responsible for its fate towards the nuclear fission. So, this we will discuss more in the subsequent lectures on nuclear fission but right now we will try to summarize that the liquid drop model can explain the masses of nuclei, it can explain the beta decay energetics, it can explain the energetics of nuclear fission process but the picture is not that good always.

So there are some limitations of liquid drop model. Some of them are the unusual stability of Z and N having certain configurations and what I have shown here is the difference in the experimental and liquid drop predicted masses over a range of proton number and neutron number and you will find that there are dips that means the experimental masses are lower than the liquid drop masses at certain configurations and they correspond to the magic numbers. So that means the magic number nuclei have lower mass than that predicted based on liquid drop. In addition to that the ground state spin and parity of the nuclei also cannot be explained by the liquid drop model and the existence of nuclear isomers also cannot be explained by that. There are in fact many other properties which we will discuss later on they are difficult to explain in terms of liquid drop model. So, we will discuss this all in shell model next lecture how shell model can take care of many of the limitations of liquid drop model.

I will stop here and discuss this shell model in the next lecture. Thank you very much.