Liquid drop model

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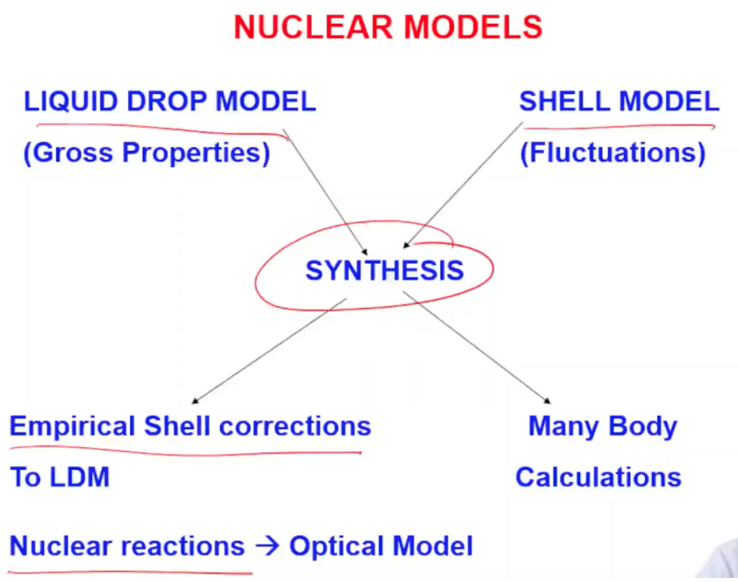
Lecture-4, Module-1

Hello everyone. Last lectures, we discussed about the radioactivity and the decay, radioactivity and also the nuclear structure and stability and the properties of nuclei. Now, whenever we have a system and it has got certain characteristics, it is required to make a model that will explain the properties of the system. So in the case of nucleus, there are many properties that nuclei possess and we need to have a model which will explain almost all properties of the nuclei. So today, we will start discussing the different models that are proposed and we discuss them how these models work and how they explain the properties of nuclei. So I will just give a concise slide depicting different models that have been proposed to explain the nuclei structure, nuclear properties and so on.



The first one is the liquid drop model, which in fact explains the gross property of the nuclei. And then there is another model called shell model, which explains the fluctuations in the gross properties of nuclei. So as we discussed, you will find that liquid drop model gives you the gross properties, the cell model explains the fluctuations in those properties. And there is a synthesis of these two types of models to give you empirical shell models, wherein you use the experimental data to generate the properties of nuclei.

These are like masses of nuclei that are predicted using the empirical models. And there are many body calculations from the first principles you can try to calculate the properties of nuclei. So this is with regard to the structure and properties. In nuclear reactions, when we are studying the nuclear reactions, there is another model called as the optical model. There are also some of the models called collective models, which explain the collective properties of nuclei, like rotation, vibration, etc.

We will be concentrating only on the liquid drop model and the shell model in this particular course. So let me first start with the liquid drop model.

Liquid drop model: Niels Bohr

1. Nucleus is like a droplet of incompressible matter
2. All nuclei have same density
3. Nuclear Force is short range and attractive and operates between nearest neighbours.
4. Nuclear Force is saturated – like Van der Waals force between molecules
5. Binding energy ~ heat of vaporisation of liquid drop

The liquid drop model was proposed by Niels Bohr in 1930s. And in fact, he was trying to explain the nuclear fission process, how to explain the splitting of a nucleus, heavy nucleus particularly into two fragments of nearly equal mass. And he thought of a concept like if you have a drop of liquid, if you try to deform it, how it can split into two droplets.

That is how he got the concept of the liquid drop model. So we have discussed the different properties of nuclear matter. And we make use of those properties to build up this liquid drop model. So in the nuclear structure, we found that nuclear matter has got very high density. If you recall the previous lecture, we discussed that the density of nuclear matter is 10^{14} grams per cc or 10^{38} nucleons per centimeter cube.

So nuclear matter is highly incompressible. So it is envisaged in the liquid drop model that the nucleus is like a drop of incompressible matter. We also saw that the density, the central density of all nuclei is same. And that is on the order of 10^{38} nucleons per cc. So all nuclei have the same density.

Then nuclear force, nuclear force is short range and attractive. What are the evidences for that? The short range property of nuclear force, the evidence comes from the saturation property of binding energy. So binding energy per nucleon is constant. And if it was not constant, if each nucleon is interacting with all nucleons in the nucleus, then the binding energy per nucleon would be proportional to the mass number. But actually the binding energy per nucleon is constant means the nucleons are interacting only with those nucleons which are in its immediate vicinity.

It is attractive of course, it is imperative that you want to break the nucleus into constituent nucleons where you have to supply energy. So the nucleons are tightly held together and which works at a very short range. Nuclear force is saturated, we already discussed and so it is like the Van der Waals forces between the molecules, it is like a very short range force. So that is how we explain that the nuclear force is like a Van der Waals force between molecules in a drop of liquid. And then in that analogy, the binding energy is analogous to the heat of vaporization of liquid drops.

If you want to remove a nucleon, it is like evaporating a drop, a part of the liquid. So they are the analogies and we will build the nuclear liquid drop model based on these properties and these assumptions. So the important part of the liquid drop model lies in its success in predicting the masses of the nuclei. So that is a very important success of liquid drop model and our focus will be more on the calculating the binding energy of the nucleus and thereby the nuclear mass.

Semiempirical mass formula: Von Weizsacker (1935)

$M(Z, A) = ZM_H + (A-Z)M_N - B$

A= mass number, Z= atomic number, N= neutron number

B= Binding energy

$B = B_v + B_s + B_c + B_a + B_p$

B_v = Volume energy

B_s = Surface energy

B_c = Coulomb energy

B_a = Asymmetry energy

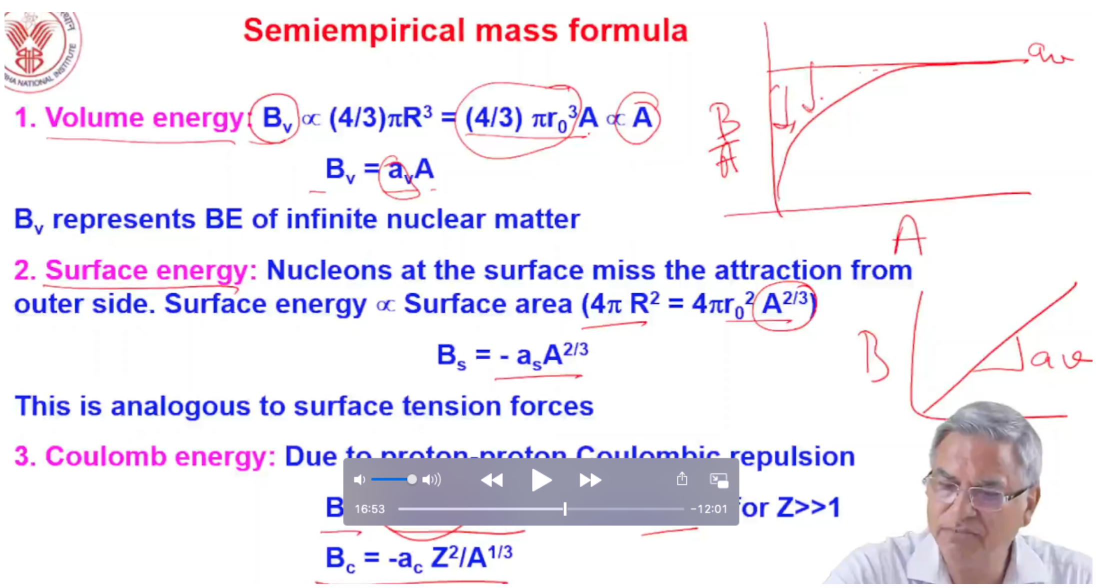
B_p = Pairing energy (δ)

So we have the semi empirical mass formula. By semi empirical essentially we mean that it is based on some phenomenology analogous to liquid drop model, but the constants of that liquid drop model mass formula are obtained empirically using the experimental masses of nuclei and it was proposed by Von Weizsacker in 1935.

So in the liquid drop model, the most important thing is to predict the mass of a nucleus or essentially which requires the predicting the binding energy of it. We are focusing more on binding energy of the nuclei. The binding energy is essentially given in terms of the mass of the nucleus equal to mass of the Z protons in the nucleus and the mass of the A minus Z neutrons in the nucleus. So when we combine Z protons and (A-Z) neutrons to form a nucleus of mass M, then energy equivalent to the binding energy is released. In other words, if you want to break this nucleus into its constituent nucleons, then we have to supply energy equivalent to the binding energy of nuclei.

So this binding energy is the term which we will focus in the present lecture and the binding energy B is constituted of several terms which we call as volume energy B_v, the surface energy Bs, the Coulomb energy B_c, the asymmetry energy B_a and the pairing energy B_p. So the volume, the surface are the properties of the liquid drop. We derive the expressions of the volume energy and surface energy from the liquid drop properties. And then to that there are some additional terms which depending upon, like the nucleus instead of the liquid drop, the nucleus contains charge and the like charges repel each other. So that is how the Coulomb term came into picture.

And the neutron and proton numbers are not same, particularly for the heavy nuclei and so we have the asymmetry energy term and the pairing energy term that means the nucleons like to pair up inside the nucleus. So there is a pairing energy term. So let us see how to construct the semi-empirical mass formula. So we know that the mass of the nucleus is proportional to the mass number, because the mass number is nothing but proton number plus neutron number. So mass is very close to the mass number. There is a slight difference between actual mass and the mass number. And the binding energy is the energy when the nucleons combine to form the nucleus then the energy is released.



So the volume and mass are related to the density and the density of nuclear matter is constant. So if you take an infinite nuclear matter, we know that the nucleons interact with the other nucleons. Then for the infinite matter, if you take the A nucleons, then the binding energy is proportional to A because we know that the average binding energy is constant, B proportional to A or B by A is constant.

So that is what we have tried to generate here that the B, the binding energy of nucleus is proportional to mass number and this proportionality constant is nothing but this term. What is that? The volume energy constant. So the volume is $4/3\pi R^3$ which is nothing but proportional to mass number because density is constant. So binding energy you can write as proportional to the volume. So this should have been

$$B_v \propto \frac{4}{3}\pi R^3 = \frac{4}{3}\pi r_0^3 A \propto A$$

So this $\frac{4}{3}\pi r_0^3$ is termed as the a_v, volume energy constant. So this volume energy essentially constitutes the binding energy of an infinite nuclear matter. If there were no boundaries of the nuclear matter, then the binding energy per nucleon is constant and we can say that like we have this binding energy per nucleon, it should be constant. So that is what is the per nucleon. But if you take the binding energy, then it should have been like this because the slope is nothing but a_v.

So binding energy per nucleon is constant because the binding energy is equal to $a_v \times A$. Now this is a very hypothetical situation. The nucleus is not infinitely large in size. It has got certain size because there are finite number of nucleons. That is where if it is not infinite, then it has got a surface.

So the surface energy term in fact is like a correction to the volume energy term. So those nucleons which are at the surface of the nucleus, they do not have the neighbors on the outside. So they are pulled up by the nucleons inside the nucleus. So this is like the surface tension force. That is how a drop of liquid tries to retain a spherical shape because the surface tension is minimum for a spherical shape.

So the surface area tries to minimize because the surface tension is proportional to the surface area. So if you see the surface energy is proportional to surface area of the nucleus and surface area is $4\pi R^2$. You can again write $R=r_0 A^{1/3}$. So it becomes $\frac{4}{3}\pi r_0^2 A^{2/3}$. So now you see here the nucleons at the surface, their contribution has already been taken into consideration in the volume energy term.

The volume energy term takes care of all nucleons in the nucleus, assuming that there is no surface. So out of those A nucleons, there are a few nucleons which are at the surface. So there is a negative correction to the volume energy because of nucleons at the surface.

$$B_s = -\, a_s A^{2/3}$$

Now you see $A^{2/3}$, if you see this was the a_v term, then the surface energy term will become like this. The surface energy term will be dominating at low mass; all the nucleons will be at the surface. As you go to higher and higher mass, there will be less and less fraction of nucleons at the surface. So the surface energy term is dominating at lower masses and becomes less and less as you go to higher and higher masses.

Third term is the Coulomb energy term. And this arises because protons repel each other and so they will try to destabilize the nucleus. So they will try to be away. And so the Coulomb energy term, is again a subtraction to the volume energy term. Now how to calculate the Coulomb energy? The Coulomb energy of a sphere, suppose there are Z protons, each proton will interact with the remaining Z minus one protons. So the Coulomb energy of a sphere containing Z charges is

$$B_c \propto Z(Z - 1)e^2/R$$

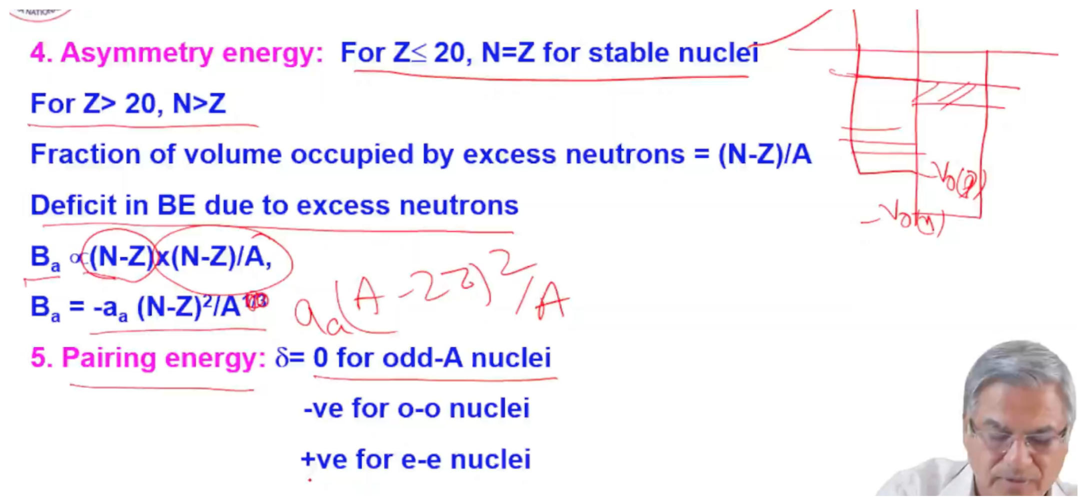
where R is the radius of the sphere. B_c is the Coulomb energy term. So you can now substitute for $R=r_0A^{1/3}$. it becomes and particularly when Z is very large compared to one, and Z(Z-1) is Z^2.

So it becomes

$$B_c \propto \frac{Z^2e^2}{r_0A^{1/3}}$$

So the Coulomb energy term becomes $-a_c$, where a_c is nothing but $\frac{e^2}{r_0}$. And it becomes $\frac{Z^2}{A^{1/3}}$. So this Coulomb energy term, again, it is a negative contribution to binding energy because this is a repulsive term. The volume energy term is attractive term, but the Coulomb energy term is a repulsive term.

So it is trying to decrease the binding energy of the nucleus. so the the Coulombic term you see, if you see $\frac{Z^2}{A^{1/3}}$, so for the lower mass region, the Z is low, the contribution of Coulombic term will be much less. So if you add to this, so actually by surface energy, and Coulomb energy, the binding energy has decreased. Now the Coulomb energy will be dominating for the heavy nuclei. So Coulomb energy will be more for the higher Z nuclei and for low mass number, it is less.



The origin of asymmetry energy term lies in the excess neutrons that we need to add to stabilize a nucleus, particularly for those nuclei which have atomic number more than 20. So you know, for calcium, for up to calcium 40, Z = 20, N = Z, means the number of neutrons, if it is equal to number of protons, then it is sufficient to stabilize the nuclei. But for nuclei having atomic number more than 20, then we need to have more neutrons to stabilize the nucleus because we need to compensate for the Coulombic repulsion between the protons. Now the origin of this term lies in, if you see the potential energy term, the potential for the protons and neutrons, so suppose protons are lying in a square well potential, then proton will have a Coulomb barrier. And so these are the proton levels. So and this is the Fermi level, this is the binding energy of, so this is a $-V_0$ for protons.

Now if you have more number of neutrons, then the Fermi level for proton and neutron has to be same, otherwise, it will start undergoing beta minus decay. So to accommodate the more number of neutrons than protons and have the same Fermi level, the neutron potential will start from here. So it is $-V_0$. So there are some neutrons which are excess over protons and they do not have the benefit of exchanging with the protons.

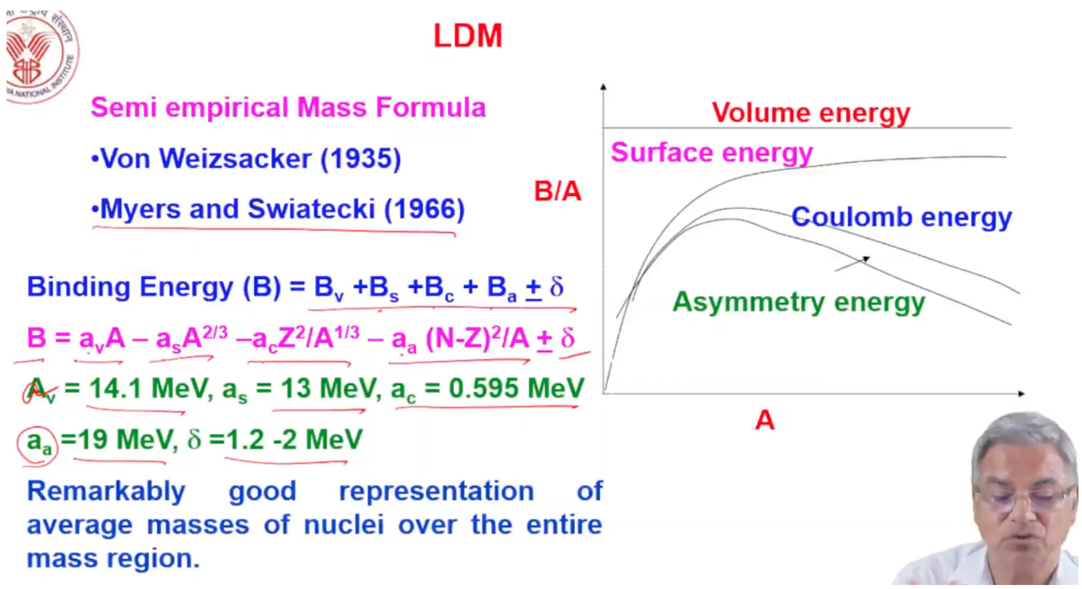
And so this excess neutrons, these excess neutrons in fact have higher kinetic energy compared to the protons. So there will be some fraction of neutrons which will be having higher kinetic energy than protons and hence the lesser binding energy than the other nucleons. So that is the deficit in binding energy due to asymmetry of neutrons over the protons. Now how to calculate this term? The asymmetry energy is proportional to the excess neutrons (N-Z) into the fraction of the volume occupied by the excess neutrons that is (N-Z)/A that is the fraction of the volume occupied by these (N–Z) neutrons. So excess neutron into the fractional volume is equal to the asymmetry energy.

And if you take the proportionality constant, it becomes a_a $(N-Z)^2$ /A. You can also write this as

$$B_a = \frac{-a_a(N-Z)^2}{A} = -\frac{a_a(A-2Z)^2}{A}$$

So this is the binding energy term corresponding to asymmetry energy and its origin lies into the excess neutrons that are in the nucleus to stabilize that. So their contribution has already been taken care in the average volume energy term but because they are excess neutrons and they that fraction is actually not having the benefit of binding having exchange with the other protons. So that is the origin of asymmetry energy.

The last one is the pairing energy. As you know the nucleons tend to pair up and so as a matter of convention for odd A nuclei the pairing energy has been taken as 0. If it is an odd-odd nucleus, then their binding energy is low. So the pairing energy is -δ and if it is even-even nucleus then the binding energy is high. So for them the binding energy is +δ. So that is the convention we use for pairing energy term.



So let us recapitulate whatever we have discussed in this slide. So the liquid drop model using the semi empirical mass formula gives the calculation of the binding energy in terms of different terms like volume term, surface term, Coulomb term, asymmetry term and pairing energy. So you can write if you recall the previous expressions that we have been deriving is the binding energy of a nucleus

$$B = a_v A - a_s A^{\frac{2}{3}} - a_c \frac{Z^2}{A^{\frac{1}{3}}} - a_a \frac{(N-Z)^2}{A} \pm \delta$$

Now there are five constants a_v, a_s, a_c, a_a, and δ these five constants have been determined empirically from experimentally known masses. Masses have been obtained using mass spectrometers and different other formulations. So if you have the experimental masses put them into this equation and do the fitting to get these five constants. That is how we get it we call it as a empirical relation. Empirically we have obtained these masses hence the name semi empirical mass formula. Formula is based on some phenomenology but the constants are obtained by empirical means.

So these are the terms av, the volume energy constant is 14.1 MeV surface energy constant 13 MeV Coulomb energy constant 0.595 MeV. You can see here it is all binding energy that constant the other things are constant. So that means other things are not having any dimensions. So the dimensions of energy coming from these constants only. Asymmetry energy constant 19 MeV and δ the pairing energy 1.22 to 2 MeV. For now these five parameters in fact they have been undergoing lot of refinement over the last many many years and Myers and Swiatecki in 1966 gave a very advanced prediction of the nuclear masses. So Myers and Swiatecki are known for development of the liquid drop model to a greater extent because the constant that had been obtained now they owe them to Myers and Swiatecki.

So in terms of the five constants can calculate the masses of the nuclei and they give a remarkably good representation of the average masses of nuclei over the entire mass region. What I try to explain here in graph that the what you plot the average binding energy the volume energy term av is constant about 14.1 MeV. So you can put here 14.1 and the surface energy term is dominating at lower lower mass at higher mass becomes less. The coulomb energy term is very small at low mass but higher mass becomes dominating. So it is decreasing the binding energy and asymmetry again becomes important above the mass number 40. So as for higher nuclei it becomes important. So you can see here by the liquid drop model we are able to generate the nature of the binding energy per nucleon.

We recall the binding energy curve in the previous lectures. This is the average binding energy which is rising in the beginning and falling in the end becoming maximum at mass number around 60. So liquid drop model successfully explains the trend of the average binding energy and these constants are used to find out the masses of the nucleus.

Applications of LDM

1. Prediction of atomic masses and binding energies

$B = ZM_H + (A-Z)M_n - M(Z,A) \rightarrow \quad M(Z,A) = ZM_H + (A-Z)M_n - B$

$= AM_n + Z(M_H-M_n) - [M(Z,A)-A] - A$

$(B/A)\ \text{exp.} = (M_n-1) - (M_n-M_H)Z/A - (M-A)/A$

$B(\text{cal.}) = a_vA - a_sA^{2/3} - a_cZ^2/A^{1/3} - a_a(A-2Z)^2/A \pm \delta$

$(B/A)\ \text{cal.} = a_v - a_s(1/A^{1/3}) - a_c Z^2/A^{4/3} - a_a(1-2Z/A)^2$

$N - Z$
$= A - Z - Z$

Nucl.	Vol. E	Surf E	Coul. E	Asym. E	(B/A) cal	(B/A) exp
^{17}O	14.1	5.05	0.87	0.07	8.71	7.75
^{65}Zn	14.1	3.22	1.92	0.22	8.74	8.75
^{195}Pt	14.1	2.25	3.20	0.76	7.90	7.92
^{245}Bk	14.1	2.08	3.66	0.82	7.54	7.52

All values are in MeV

So before I go into the complete applications of liquid drop in different areas just one example of the success of the liquid drop model is in predicting the atomic masses and their binding energies. So you see here the binding energy now you can substitute in the mass, the mass of the nucleus.

$$B = ZM_H + (A - Z)M_n - M(Z, A)$$

or you want to calculate the mass of the nucleus you can put into this formula.

$$M(Z, A) = ZM_H + (A - Z)M_n - B$$

So you calculate the binding energy by this formula and substitute in this equation we know the mass of proton and neutron can calculate nuclear mass. So you can now try to explain to put this try to make it do some calculations here. So binding energy equal to,

$$B = AM_n + Z\left(M_H - M_n\right) - [M(Z, A) - A] - A$$

so you can now take binding energy per nucleon divide by A experimental equal to

$$\left(\frac{B}{A}\right)exp = \left(M_n - 1\right) - \frac{\left(M_n - M_H\right)Z}{A} - \frac{M-A}{A}$$

So now you can calculate the binding energy by this formula

$$B(cal) = a_vA - a_sA^{\frac{2}{3}} - a_c\frac{Z^2}{A^{\frac{1}{3}}} - a_a\frac{(A-2Z)^2}{A} \pm \delta$$

So this is the calculated binding energy using these constants and so you can write now these constants.

And now what I am showing in the table here is the calculated and experimental masses. So you can get experimental binding energy from here and calculated masses from here which is a table here now for nuclei of different mass numbers. So oxygen 17 see the volume energy term is constant 14.1, that is, av, surface energy term is changing now it is decreasing with the mass number Coulomb energy term is increasing with the mass number and similarly asymmetric energy . Then the binding energy per nucleon calculated and experimental. So what we can see here it is very accurate quite accurately predicting the masses or the binding energy.

So once you know binding energy you can calculate the masses. So the calculated and the experimental binding energies are very close to each other. So this explains the success of the liquid drop form model in explaining the masses of the nuclei. So this in fact is the success of the liquid drop model. I will stop here and then subsequently I will discuss the applications of liquid drop model. Thank you very much. Thank you.