Radioactive decay chain

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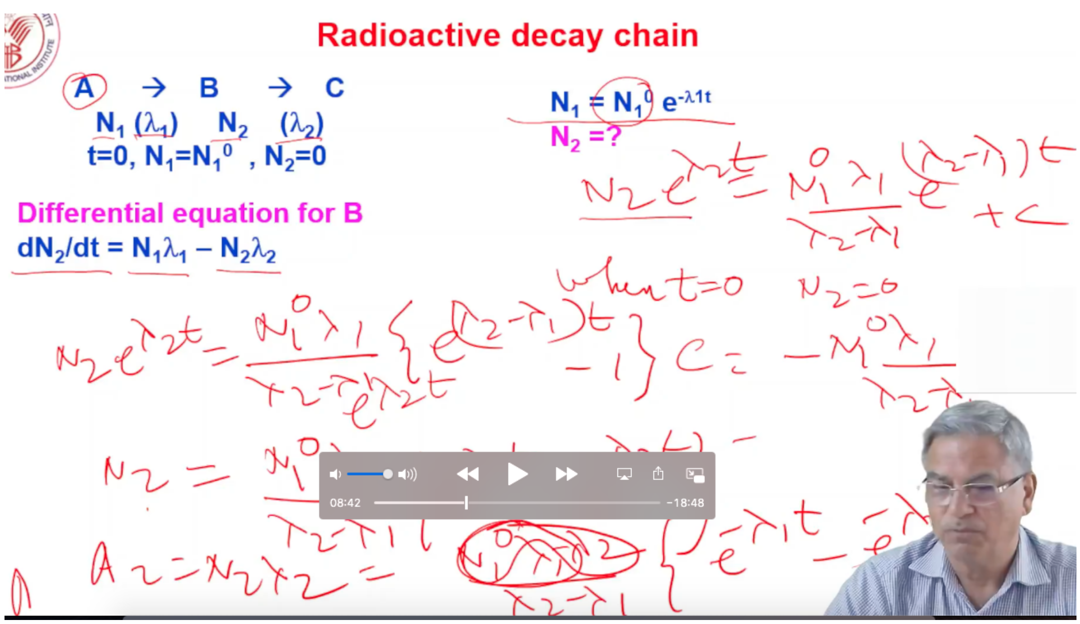
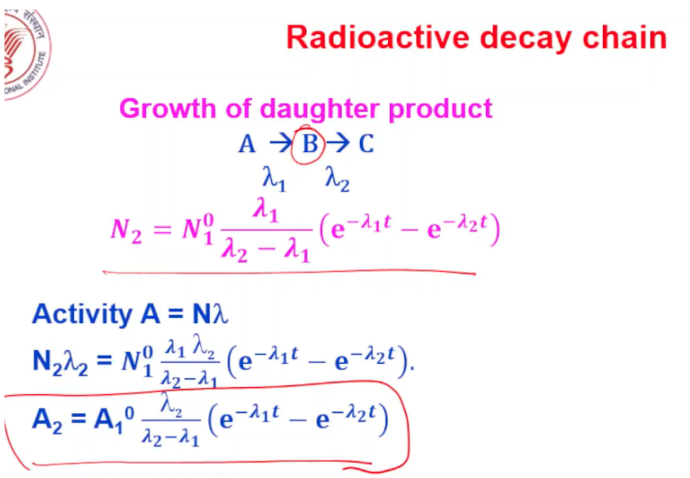
Lecture-2, Module-1

Hello everyone. So, we go to the second lecture of this course on Nuclear and Radio Chemistry. So far we have discussed the radioactive decay. We learnt that radioactive decay follows the first order rate law. And we also discussed the different types of conditions like, two independently decaying radioisotopes, how does the activity change with time, and also branching, a radioisotope decaying by two modes, like beta plus and beta minus, and how does the half-life depend upon the partial half-lives.

Today we will discuss the decay of an isotope to another isotope which is also radioactive. So there is a chain. The decay happens in a chain of radioisotopes. And so there will be different aspects of that and lot of applications also which I will be discussing in this lecture.



So by radioactive decay chain, I mean that a radioisotope A undergoes some decay to B and B is also radioactive which also decaying further to C. We have already discussed the radioactive decay of A which follows the exponential decay. So N1 is the number of atoms of A at any time and which we discussed in previous lecture that it follows an exponential decay with N_1^0 the initial atoms of A and the decay follows an exponential path. Now let us discuss how the activity of B will change with time if it is growing from a freshly purified isotope A.



A □ B □ C

N_1 (λ_1) N_2 (λ_2)

$t=0$, $N_1=N_1^0$, $N_2=0$

So the lambda value for A is lambda 1 and for B is lambda 2 and the number of atoms of A and B at any time t are N1 and N2 respectively. Let us assume the initial conditions that when time is 0 then there is no daughter product. We have only the N_1^0 the initial number of atoms of A and N_2 is 0. So we set up a differential equation for the number of atoms of B how it is changing with time and solve that equation to find out at any point of time t how does the number of atoms and hence the activity of B change with time. So let us see here the rate of change of number of atoms of B will be given by,

$$dN_2/dt = N_1\lambda_1 - N_2\lambda_2 \qquad (1)$$

B is growing from A so that is N1λ1, that is the rate of decay of A is the rate of growth of B and the B is decaying by the rate dB_B by dt equal to N2λ2. So there is a growth part and there is a decay part. Now this is a differential equation and we have to solve it for N2. So N2 is the number of atoms of B at any time t. So let us see how do we solve.

First we separate the variables.

$$dN_2/dt + N_2\lambda_2 = N_1\lambda_1 \qquad (2)$$

N_1 we can write as

$$N_1 = N_1^0 e^{-\lambda 1t} \qquad (3)$$

Thus we have

$$dN_2/dt + N_2\lambda_2 = N_1^0\lambda_1 e^{-\lambda 1t} \qquad (4)$$

To solve such a differential equation we use the integration factor, that is, $e^{\lambda 2t}$. If I multiply both sides of this equation by this factor it becomes easy to integrate.

$$dN_2/dt\, e^{\lambda 2t} + N_2\lambda_2\, e^{\lambda 2t} = N_1^0\lambda_1 e^{(\lambda 2t-\lambda 1t)} \qquad (5)$$

Now let us integrate this equation. The above equation can be written as

$$\int d(N_2 e^{\lambda 2t}) = N_1^0\lambda_1 \int e^{(\lambda 2-\lambda 1)t} + C$$

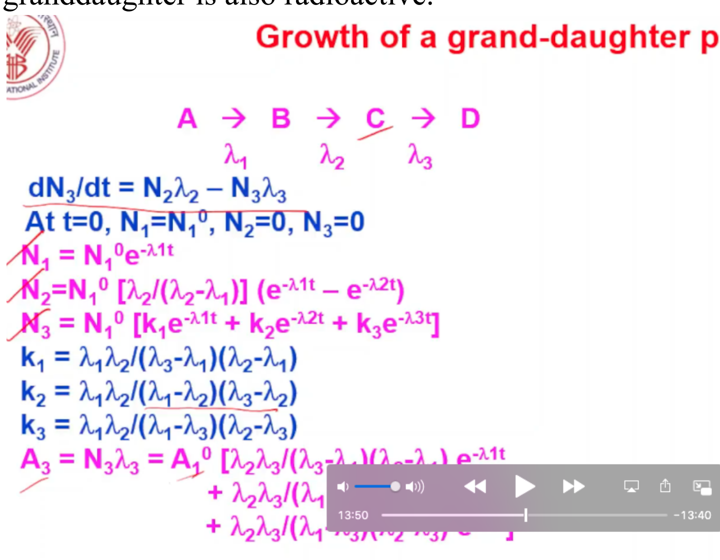
So if I integrate then we can see the solution of the integration of this will become,

$$N_2 e^{\lambda 2t} = N_1^0\lambda_1/(\lambda_2-\lambda_1)\, e^{(\lambda 2-\lambda 1)t} + C \qquad (6)$$

So now you can now find out the value of the integration constant c that is when t equal to 0 and N2 equal to 0. This gives, $C= - N_1^0\lambda_1/(\lambda_2-\lambda_1)$

So if we substitute for C in above equation we get,
$N_2e^{\lambda 2t} = N_1^0\boldsymbol{\lambda}_1/(\boldsymbol{\lambda}_2\text{-}\boldsymbol{\lambda}_1)\ e^{(\lambda 2\text{-}\lambda 1)t} - N_1^0\boldsymbol{\lambda}_1/(\boldsymbol{\lambda}_2\text{-}\boldsymbol{\lambda}_1)$
Now if I take this $e^{\lambda 2t}$ to right hand side then,
$N_2 = N_1^0\boldsymbol{\lambda}_1/(\boldsymbol{\lambda}_2\text{-}\boldsymbol{\lambda}_1)\ \{e^{-\lambda 1t} - e\text{-}^{\lambda 2t}\}$ (7)

So this is the equation for the number of atoms of B at any time t. So it contains the exponential terms due to the decay of A and B. Now if we want to find out the activity then activity of daughter product B will be
$A_2 = N_2\boldsymbol{\lambda}_2 = N_1^0\boldsymbol{\lambda}_1\boldsymbol{\lambda}_2/(\boldsymbol{\lambda}_2\text{-}\boldsymbol{\lambda}_1)\ \{e^{-\lambda 1t} - e\text{-}^{\lambda 2t}\}$ (8)
Taking $N_1^0\boldsymbol{\lambda}_1 = A_1^0$, we get
$A_2 = A_1^0\ \boldsymbol{\lambda}_2/(\boldsymbol{\lambda}_2\text{-}\boldsymbol{\lambda}_1)\ \{e^{-\lambda 1t} - e\text{-}^{\lambda 2t}\}$ (9)
So this is an important relationship.
I hope you remember the derivation of this which will be used many times in subsequent lectures also. So the net result of the radioactive decay chain is that if you want to find out the activity of B at any time t we determine the number of atoms of B as given above (7). Similarly for the activity A_2 of the daughter isotope (9). So this is an important relationship and there are several corollaries of this equation which we will discuss in details subsequently.
Now there will be situations where not only the daughter product is radioactive but the granddaughter is also radioactive.



A □ B □ C □ D
λ_1 λ_2 λ_3
Granddaughter means C. So when C is also radioactive one can set up similar equation like we discussed for B the equation for the activity of C as a function of time. So to set up that equation you can write,

$dN_3/dt = N_2\lambda_2 - N_3\lambda_3$ (10)
Where,

$N_1 = N_1^0e^{-\lambda 1t}$
$N_2=N_1^0\ [\lambda_2/(\lambda_2\text{-}\lambda_1)]\ (e^{-\lambda 1t} - e^{-\lambda 2t})$
As derived above.

Again you can put the conditions the initial conditions,
At $t=0$, $N_1=N_1^0$, $N_2=0$, $N_3=0$

So only purely freshly purified parent isotope is present. So under those conditions how the activity of C will change with time that is the problem we have to solve here. So already you know the number of atoms of A how they will change as a function of time (3). Similarly for B we have just now solved the equation (7). And so if you substitute the value of N1 and N2 in equation (10) and then you solve the differential equation similar way like we solved the one for N_2 then the solution I am not going into the solution of the differential equation because the same way you can do for this also. So the solution of that equation will be.

$$N_3 = N_1^0 [k_1 e^{-\lambda 1t} + k_2 e^{-\lambda 2t} + k_3 e^{-\lambda 3t}] \qquad (11)$$

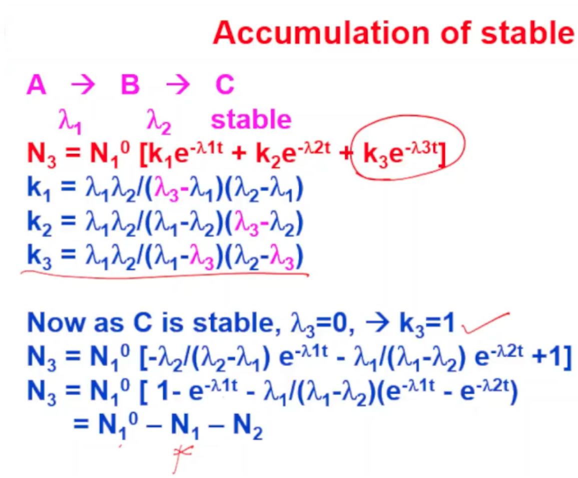
Where $k_1 = \lambda_1\lambda_2/(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)$, $k_2 = \lambda_1\lambda_2/(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)$ and
$k_3 = \lambda_1\lambda_2/(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)$

To obtain the activity of C, you multiply N_3 by $\boldsymbol{\lambda}_3$.

$$\begin{aligned} A_3 = N_3\lambda_3 = A_1^0 [&\lambda_2\lambda_3/(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)\, e^{-\lambda 1t} \\ &+ \lambda_2\lambda_3/(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\, e^{-\lambda 2t} \\ &+ \lambda_2\lambda_3/(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\, e^{-\lambda 3t}] \end{aligned} \qquad (12)$$

So, using this equation we can calculate even the activity of the granddaughter product and there are several cases you will see that we in fact we do produce the radio isotopes by two decay processes. So, that will come in the discussion subsequently.

Now, there can be situations where the granddaughter instead of being radioactive is a stable isotope.



In fact this is very common as we discussed A going to B going to C that is the same situation and the granddaughter is stable it is not undergoing further radioactive decay.

$$\begin{array}{ccccc} A & \rightarrow & B & \rightarrow & C \\ \lambda_1 & & \lambda_2 & & \text{stable} \end{array}$$

But you can use the formula for the number of atoms of C which we derived just now to calculate the number of atoms of C that is formed at the end of some particular time. So, what we saw just now that the number of atoms of C is given by equation (11).

Now in this case $\boldsymbol{\lambda}_3$ is zero as the granddaughter is not radioactive. If it is not decaying means the half-life is infinite so $\boldsymbol{\lambda}_3$ is 0.

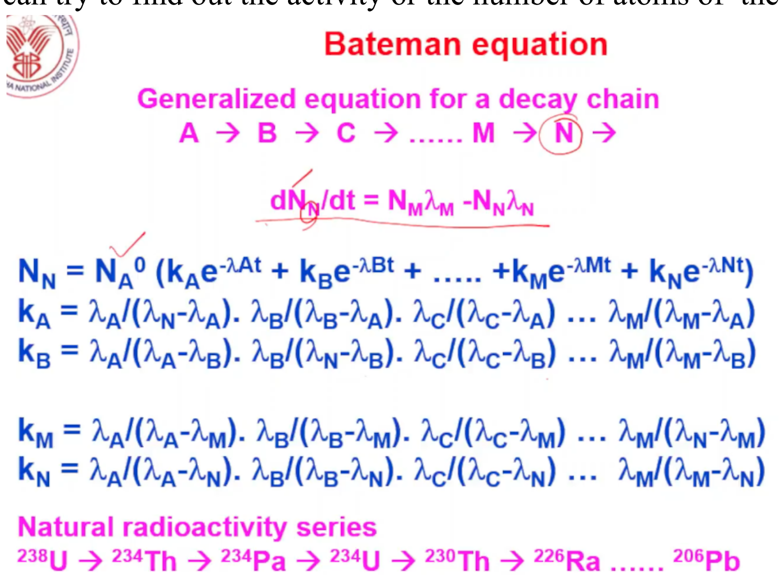
Now as C is stable, $\lambda_3=0$, $\rightarrow$ $k_3=1$

$N_3 = N_1^0 [-\lambda_2/(\lambda_2-\lambda_1)\, e^{-\lambda 1t} - \lambda_1/(\lambda_1-\lambda_2)\, e^{-\lambda 2t} + 1]$ (13)

and you can now rearrange above equation in such a way that these terms represent N1 and N2. N1 and N2 are given in equations 3 and 7 respectively.

$$N_3 = N_1^0 [\, 1- e^{-\lambda 1t} - \lambda_1/(\lambda_1-\lambda_2)(e^{-\lambda 1t} - e^{-\lambda 2t})$$
$$= N_1^0 - N_1 - N_2 \qquad (14)$$

So, the number of atoms of C at any time will be initial atoms of A minus the number of atoms of A and B at that particular time that is N1 and N2. So, this is how you can calculate like for example if you have a freshly purified parent and after a long time you want to know how many atoms of grand daughter will be formed which is stable we can use this equation provided you know the number of atoms of N1 and N2 at that point of time. If you know initial activity you can calculate N1 and N2 also. So, that is how we can try to find out the activity or the number of atoms of the grand daughter.



To extend this further into a larger series like natural radioactive series in fact this condition was there in the early 20th century when the scientists were trying to separate the daughter products from the natural radioactivity series and there are n number of decays as I mentioned uranium-238 undergoes 8 alpha and 6 beta decay to lead-206. So, suppose at any point of time you want to find out what will be the activity of thorium-230 which is the 4th product or radium-226 is the 5th daughter product. So, in such situation whatever we discussed just now may not be useful. So, you can go for a generalized equation and that is called as the Bateman equation.

A □ B □ C □ …… M □ N □

So, the generalized equation for a decay chain where there are the parent isotope A decays by subsequent decay to B, C so on to M and so on then you can write a generalized equation for nth daughter product like,

$dN_N/dt = N_M\lambda_M - N_N\lambda_N$

So, this is a similar equation which we just now solved. Now the solution of this equation will be similar to that we found for the granddaughter but in a generalized way now and so the number of atoms of this particular isotope at any time t will be given by,

$N_N = N_A^0 (k_A e^{-\lambda At} + k_B e^{-\lambda Bt} + \ldots\ldots + k_M e^{-\lambda Mt} + k_N e^{-\lambda Nt})$

$k_A = \lambda_A/(\lambda_N-\lambda_A).\ \lambda_B/(\lambda_B-\lambda_A).\ \lambda_C/(\lambda_C-\lambda_A) \ldots \lambda_M/(\lambda_M-\lambda_A)$

$k_B = \lambda_A/(\lambda_A-\lambda_B).\ \lambda_B/(\lambda_N-\lambda_B).\ \lambda_C/(\lambda_C-\lambda_B) \ldots \lambda_M/(\lambda_M-\lambda_B)$

$k_M = \lambda_A/(\lambda_A-\lambda_M).\ \lambda_B/(\lambda_B-\lambda_M).\ \lambda_C/(\lambda_C-\lambda_M) \ldots \lambda_M/(\lambda_N-\lambda_M)$

$k_N = \lambda_A/(\lambda_A-\lambda_N).\ \lambda_B/(\lambda_B-\lambda_N).\ \lambda_C/(\lambda_C-\lambda_N) \ldots \lambda_M/(\lambda_M-\lambda_N)$

So, you can set up a generalized equation, in fact you can make a computer program to calculate the number of atoms of any daughter product at any point of time using this generalized Bateman equation and this becomes very useful when you have to calculate the activity of a daughter product like for example you want to separate radium-226 from an old uranium sample but you want to know a priori what will be the number of atoms or what will be the activity of radium-226. So, there you can do this calculation and arrive at whether it is worth going for separation at this point of time or not. So, these Bateman equations are very useful in finding out the activity of daughter products in a natural radioactivity series.

Now we will come to another aspect of this radioactive decay chain.

Parent daughter decay growth relationship

$$\begin{array}{ccccc} A & \rightarrow & B & \rightarrow & C \\ N_1 & (\lambda_1) & N_2 & (\lambda_2) & \end{array}$$

$$A_2 = A_1^0 \frac{\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

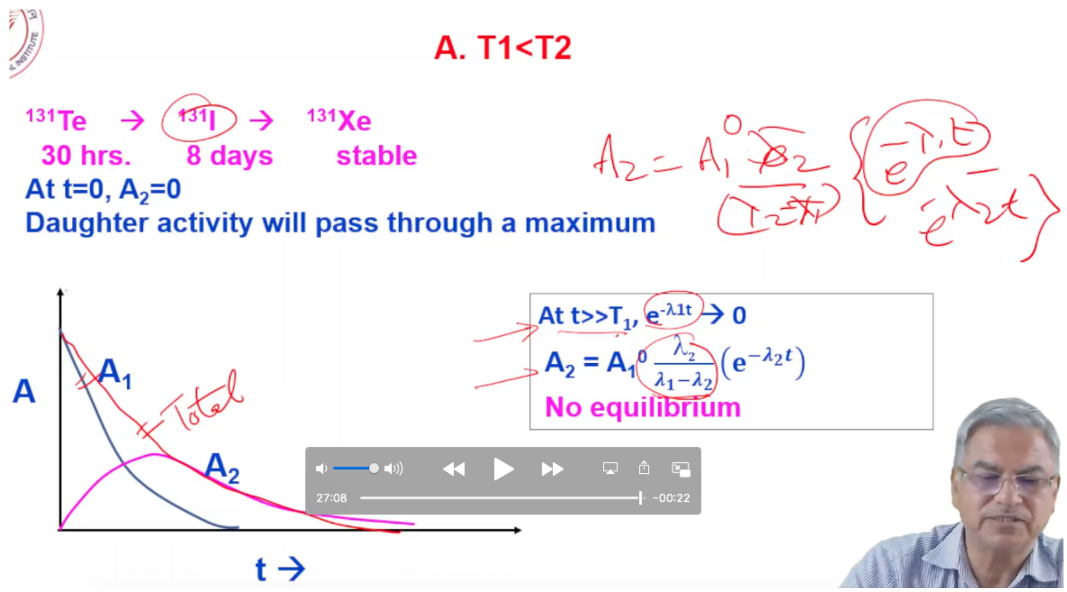
A. Parent shorter lived than daughter, $T_1 < T_2 \rightarrow \lambda_1 > \lambda_2$

B. Parent longer lived than daughter, $T_1 > T_2$, $\rightarrow \lambda_1 < \lambda_2$

Many a times we will come across situations where, depending upon the half-lives of parent and daughter you may have the growth and decay of the daughter product in a different fashion. Now I give this typical case of A going to B going to C with the number of atoms of A and B being N1 and N2 and their decay constant being lambda 1 and lambda 2 respectively and we derived the expression for the activity of B as a function of time in terms of the initial activity of parent as given in equation (7). So, now there are two situations.

(a) The parent is shorter lived than daughter, that means the half-life of parent is less than that of daughter and in terms of the decay constants the decay constant of parent is more than that of the daughter.

(b) The second situation is when the parent isotope is long-lived than daughter isotope. So, that means t1 is more than t2 or conversely **λ**1 is less than **λ**2. So, depending upon the magnitude of **λ**1 and **λ**2 or t1 and t2 we will discuss different situations.



Now here the first case where the half-life of the parent is short-lived than that of daughter and I give you a typical example where you have tellurium-131 having the half-life of 30 hours, that is, 1.25 day or so. It decays to iodine-131 having half-life 8 days and which is decaying to xenon-131 which is stable.

^{131}Te □ ^{131}I □ ^{131}Xe

30 hrs. 8 days stable

So, assume that we have initially a freshly purified tellurium-131. So, at t equal to 0 the activity of daughter that is iodine-131is 0. In such situations if you see the activity profile of a daughter product. So the general equation was,

$A_2 = A_1^0\ \boldsymbol{\lambda}_2/(\boldsymbol{\lambda}_2-\boldsymbol{\lambda}_1)\ \{e^{-\lambda 1t} - e^{-\lambda 2t}\}$

This is the generalized equation for the activity of iodine-131. Now here tellurium-131 is short-lived than iodine-131. So, the activity of tellurium-131 will decay faster than the growth and decay of Iodine-130. So, what happens let us put this condition here when t is very high compared to half-life of parent isotope then $e^{-\lambda 1t}$ tends to 0.

At t>>T_1, $e^{-\lambda 1t}$ □ 0

$$A_2 = A_1^0 \frac{2}{\lambda_1-\lambda_2}\left(e^{-\lambda_2 t}\right)$$

So, you can see here that the daughter activity, after some time after in fact a significant number of half-lives of the parent will decay with its own half-life. So, this I have tried to illustrate in the graph. This is in the linear scale. So, the activity, of A1 the parent tellurium-131 will decay exponentially and from this activity of parent the daughter activity is growing and then it will reach a maximum and then it will start decaying with its own half-life.

So, there is a case of no equilibrium. We will discuss the equilibrium case in the second part of this lecture. Essentially what is happening that it is difficult to resolve the parent and daughter activity from this data. So, this is a case of no equilibrium that means after the decay of the parent isotope the daughter isotope will decay with its own half-life. There are several cases like these cases but there are more useful cases when the parent is longer lived than daughter and then we have a case of an equilibrium. So, that I will discuss in the next part of this lecture. Thank you.