Nuclear reactions: Energetics

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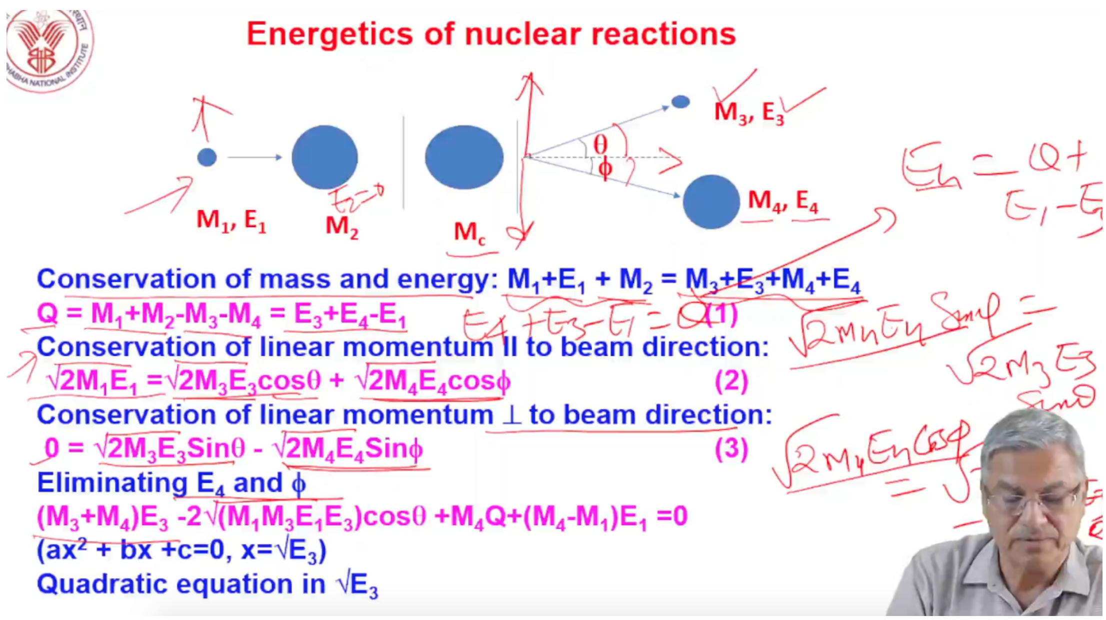
Lecture-11, module-2

Hello everyone. In the previous module, we discussed the conservation laws that govern a nuclear reaction and also the Q value, how to calculate the Q value from the masses of the nuclei. Q value could be positive or negative and accordingly they are called as exo-ergic and endo-ergic.



Now we will discuss the energetics of nuclear reactions, particularly what is that energy that is actually available for inducing the nuclear reactions that is called as the energy available in center of mass system and also the kinematics of nuclear reactions whereby you can in fact calculate the energy of a particular reaction product at a particular angle. So that will be the main focus of the particular lecture.

So let us discuss the kinematics of a nuclear reaction.

a + A □ b + B

M1 + E1 + M2 + E2 = M3 + E3 + M4 + E4

We set up the equation which we will solve to find out the energy of the ejectile as a function of angle and energy of the projectile. So, we will see here we have the projectile of mass M1 and kinetic energy E1 bombarding a target which is at rest, E2 equal to 0. So the target is stationary here and then it can form a compound nucleus, composite nucleus or a compound nucleus. We will discuss the compound nucleus in more details in subsequent lectures.

So, the composite M1+M2 will form a compound nucleus. It may not form a compound nucleus but for the time being we will say it is a composite nucleus and after these reactions then you have an ejectile formed of mass M3 and energy E3 which are at angle θ and heavy residue of mass M4 and energy E4 which are at angle φ. So now we use the conservation laws to determine relationship between E3 a function of E1 and θ. So that is the problem that if you know the projectile and its energy hitting a target then at a particular angle theta what is the energy of the ejectile for a particular energy of the projectile. Once you set up this equation then for any reaction in fact it is independent of the reaction mechanism you can calculate suppose you are putting a detector at a particular angle you know what is the energy of the ejectile.

So that is the purpose of this exercise and there are many corollaries of this derivation which may become apparent subsequently. So let us set up the equation for the conservation of mass and energy which we derived in the previous lecture. So M1+E1 is the mass and energy of the projectile plus M2, target is stationary so E2 equal to 0. So M1+E1+ M2 = M3+ E3 + M4 + E4. So, this is the conservation of mass and energy before and after the reaction.

So now we can rearrange this equation in terms of the Q value. The Q value is nothing but

$$Q = M_1 + M_2 - M_3 - M_4 = E_3 + E_4 - E_1 \qquad (1)$$

i.e., (mass of the reactants - mass of products) or (the kinetic energy of products - kinetic energy of reactants). You can arrange it this way also. Second equation is the conservation of linear momentum parallel to the beam because linear momentum is a vector quantity so it has got components along the beam and perpendicular to the beam. So for parallel to the beam along this direction so you will have the cos θ component along the beam direction and the sin θ component perpendicular to the beam direction.

The momentum P = Mv or you can write

$P^2/2M = E$

$P^2 = 2ME$ or $P = (2ME)^{1/2}$.

So, you can write in terms of this for the incoming projectile it is coming at 0 degree so you can write

$$\sqrt{2M_1E_1} = \sqrt{2M_3E_3}cos\theta + \sqrt{2M_4E_4}cos\varphi \qquad (2)$$

Linear momentum is conserved in along the beam direction.

Similarly, the linear momentum is conserved perpendicular to the beam direction. Perpendicular to the beam direction there is no momentum for the projectile so it is

$$0 = \sqrt{2M_3E_3}sin\theta - \sqrt{2M_4E_4}sin\varphi \qquad (3)$$

So now these are the three equations and you can eliminate E4 and φ from these equations to get the relationship between E3, M1, M2, M3, M4, and E1 and θ of course. So how do you do? You eliminate E4 and ϕ.

$$(M_{3+}M_{4)}E_3 - \sqrt{\left(2M_1M_3E_1E_3\right)}cos\theta + M_4Q + \left(M_4 - M_1\right)E_1 = 0$$

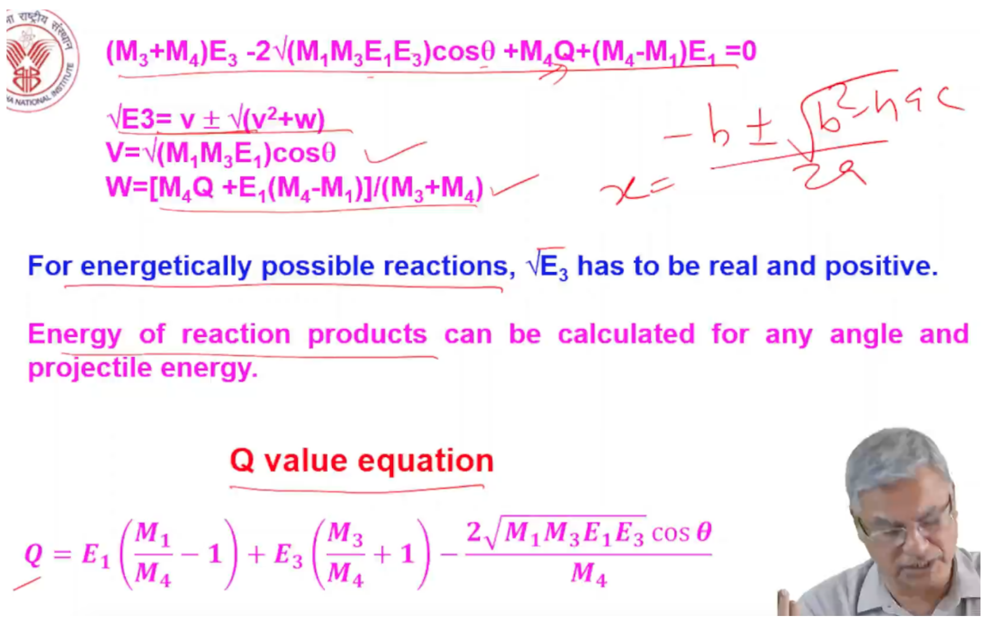
So essentially what you do you bring the $sin\varphi$ and $cos\varphi$ term on the left hand side then the remaining in the right hand side now you square and add so this M_4E_4 this square when you add the $sin^2\varphi + cos^{2\varphi}$ that will become 1 so you will have $2M_4 E_4$ and then the rest will be on the right hand side. So the result of that will be $(M_3+M_4) E_3$ you can rearrange the terms

$-2M_1 M_3 E_1 + M_4 E_3 \cos\theta + M_4 Q + M_4 - M_1 E_1 = 0$.

So this is the result of adding so you can substitute E4 in terms of Q and this one so see here you can replace $Q = E_4+E_3 - E_1 = Q$ so you can write $E_4 = Q + E_1 - E_3$ so you can substitute for E_4 in terms of Q, E_1, E_3 so you will be left with the terms corresponding to E_1, E_3, M_1, M_2, M_3 and θ. So this equation if you see here it is a quadratic equation in the $\sqrt{E_3}$ energy of the ejectile and so

$$ax^2 + bx + c = 0\ where\ x = \sqrt{E_3}$$

and the solution of this will be the familiar equation you can write the equation the solution root of a quadratic equation in terms of a b c.



So that is what I have tried to explain here the equation which is quadratic in $\sqrt{E_3}$ so what you write

$$X = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

is the root of that quadratic equation but here I have written it in the following form.

$$\sqrt{E_3} = V \pm \sqrt{V^2 + w}$$

-b/2a is v and $-\ 4ac/2a$ is w so that is how you can write that and so

$$V = \sqrt{M_1 M_3 E_1 cos\theta}$$

and

$$w = \left[M_4 Q + E_1\left(M_4 - M_1\right)\right]/(M_3 + M_4)$$

So now this equation which depends upon the Q value the Q value is inbuilt in this equation and the E1 and cos θ so if you want to find out the energy of the ejectile E3 at a particular angle and for a particular energy of the projectile you can use this equation irrespective of the mechanism it is valid for elastic scattering, inelastic scattering or any type of nuclear reaction, direct reaction or compound reaction.

So for energetically possible reaction $\sqrt{E_3}$ has to be real and positive, sometimes there will be situations where some roots are imaginary or negative so those roots are ruled out. So based on this equation you can calculate the energy of the reaction products for any angle and projectile energy. In fact, you can also rearrange this equation in terms of the Q value if you could put Q equal to then this is called the Q value equation.

Q value is nothing but

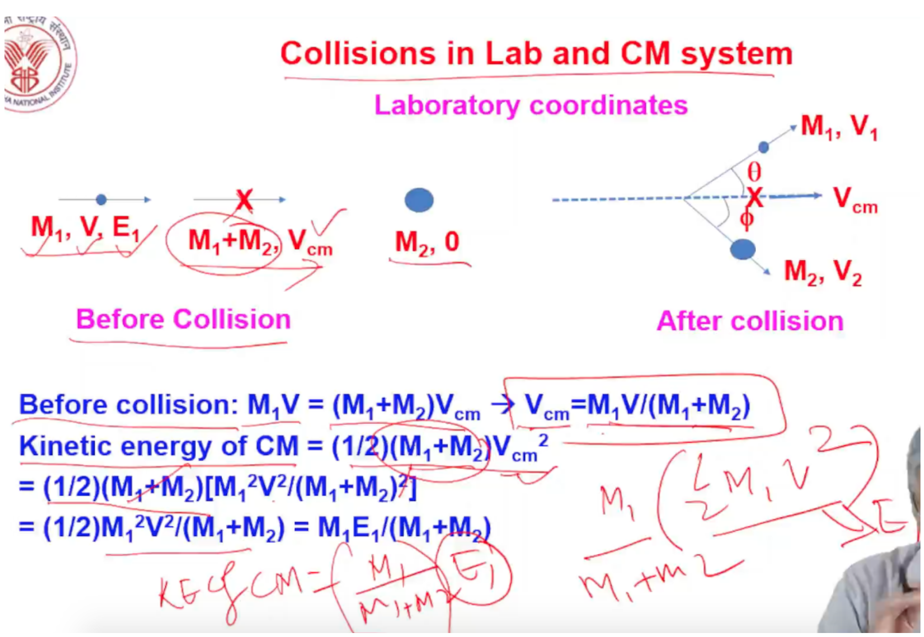
$$Q = E_1\left(\frac{M_1}{M_4} - 1\right) + E_3\left(\frac{M_3}{M_4} - 1\right) - \frac{2\sqrt{M_1 M_3 E_1 E_3}cos\theta}{M_4}$$

So, this Q value equation essentially it is the same equation in a different fashion it has been represented here and the solution of the Q value equation is,

$$\sqrt{E_3} = V \pm \sqrt{V^2 + w}$$

where V and w are given above. So this is a generalized solution of the Q value equation and for any type of reaction mechanism you can use the same formulations.

Now when we have a projectile bombarding a target as we will go along we will see that all the energy of projectile is not available for the nuclear reaction to take place. So certain energy is lost in moving the whole center of mass system and therefore the discussion of the collisions in the laboratory and center of mass system becomes important.



So we will discuss the two scenario in the laboratory system and in the center of mass system how does this collision take place. So in the laboratory system the projectile with mass M_1 energy E_1 and velocity V is moving towards the target and target is stationary. So the projectile is moving the center of mass is also moving in this direction center of mass will have the mass M_1+M_2 and velocity V cm. So this is the kinematics of the nuclear reaction in the laboratory we will see the projectile is moving with velocity V towards the target which is stationary. And after the reaction let us consider the simple elastic scattering M_1 will go at θ and M2 can go at φ.

So first let us focus on the incoming reaction channel. Let us set up the equation for the momentum and energy of the system before the collision the momentum is M_1V because the target is stationary and that is equal to the momentum of the center of mass system that is

$$M_1V = (M_1 + M_2)Vcm$$

because the momentum has to be conserved. So this momentum is same as the momentum of the center of mass system. So this you can now calculate the velocity of the center of mass

$$Vcm = \frac{M_1V}{(M_1+M_2)}$$

So this is an important relationship we will use this subsequently.

So in the laboratory with the center of mass system is moving with velocity $\frac{M_1V}{(M_1+M_2)}$ and the centre of mass system is moving with the kinetic energy

KE of cm = ½$(M_1+M_2)Vcm^2$

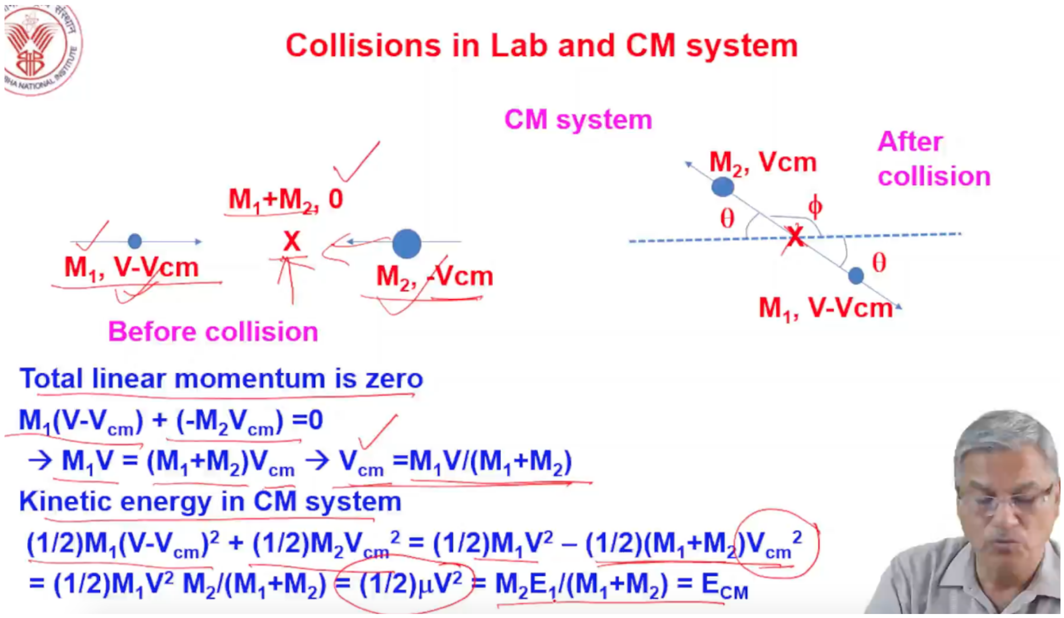
Velocity of center of mass $\frac{M_1V}{(M_1+M_2)}$ energy of the center of mass ½$(M_1+M_2)Vcm^2$ this is the mass into Vcm^2. Now you see you can substitute for the Vcm from this formula so

KE of cm = ½(M_1+M_2) $[\frac{M_1^2V^2}{(M_1+M_2)^2}]$

So, this M_1+M_2 will cancel with one M_1+M_2 we are left with

KE of cm = ½$\frac{M_1^2V^2}{M1+M2} = M_1E_1/(M1 + M2)$

So what we have here is, kinetic energy of center of mass equal to $M_1E_1/M1 + M2$. That means out of the kinetic energy of the projectile a fraction $M_1/(M_1+M_2)$ is involved with the motion of the center of mass that is called the kinetic energy of center of mass.



Now let us discuss in the center of mass system we will calculate the energy in center of mass system. So this kinetic energy of center of mass system at least is wasted in the motion of center of mass and what we are going to discuss is what is the energy available in the center of mass system that is useful to induce the reaction.

In the center of mass system the center of mass is stationary here with the mass $M_1 + M_2$ the energy of center of mass is 0. So Vcm is 0 here. So the velocity of center of mass in center of mass system is 0. The mass of projectile M1 velocity will be V - Vcm where Vcm is the velocity of the target. Now the target is moving towards the center of mass with the velocity minus Vcm because we are considering the center of mass to be stationary.

So target is moving towards this side where in the frame of reference of center of mass target is moving towards the Centre of mass with velocity minus Vcm and after the collision you will find the Centre of mass remains in the same place the projectile and target move in the opposite direction because the momentum has to be 0. So the bottom line is that in the Centre of mass system the total linear momentum is 0 because the Centre of mass is stationary. So let us write the equation

$M_1(V-V_{CM}) - M_2V_{CM} = 0$

Which can be rearranged to $M_1V = (M_1+M_2)V_{CM}$

So if you see the total momentum is 0 then you can write in this way and so Vcm will be equal to M_1 V /M_1+M_2.

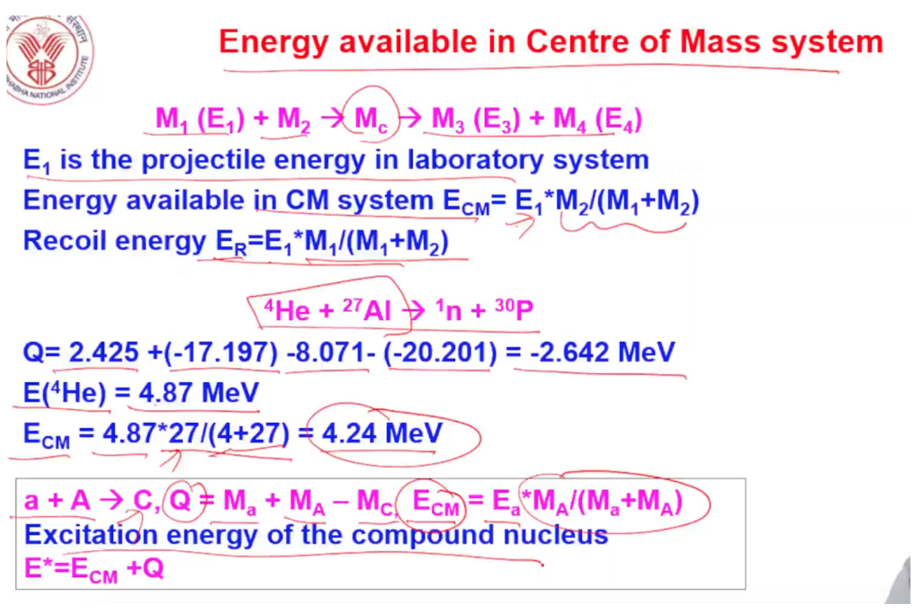
So this is the same formula which we got from the laboratory frame of reference the velocity of center of mass is given by M_1 V /M_1+M_2.

Now the kinetic energy in the center of mass system mind you earlier we discussed the kinetic energy involved in the motion of center of mass what we are discussing is now the kinetic energy available in center of mass system. So that is the kinetic energy of projectile in center of mass system plus kinetic energy of target nucleus in center of mass system.

E_{CM} = ½M_1 (V – Vcm)^2 + 1/2M_2Vcm^2. This is the kinetic energy of projectile plus kinetic energy of target in the center of mass system.

So now you can write 1/2 M_1(V^2 + Vcm^2 - 2 V. Vcm). You open it up and you substitute for the Vcm from this equation Vcm = M_1 V/(M_1+M_2) so you can you can write out so this will be equal to ½ (M_1+M_2)Vcm^2. So it is equal to now (1/2M_1V^2)(M_2/M_1+M_2). See the mass of the target upon mass of projectile plus target.

This can also be written as in terms of reduced mass of the system M_1 M_2/(M_1+M_2) is μ, reduced mass but let's not bother about it we can write it in terms of M_2 /(M_1 +M_2) E_1 where E_1 is 1/2 M_1V^2.



So here if you recall the kinetic energy of center of mass as discussed in previous one was M_1/(M_1 +M_2)E_1 and kinetic energy in center of mass system equal to M2/ (M_1+ M_2) E_1. In the previous slide we derived the expression for kinetic energy of center of mass that means this much energy tied up in the motion of center of mass whereas here in the center of mass system the kinetic energy available in center of mass system is $M_2/(M_1+M_2)E_1$. So this much fraction of the projectile kinetic energy is available for the reaction to take place. This much fraction of projectile energy is not available for reaction to take place and that's why it is also called as the recoil energy.

Recoil energy means when the projectile hits the target, some fraction of energy is tied up in moving the whole system in the forward direction or it is the target gets give the recoil from the projectile that much energy is not available for the reaction to take place. So out of the projectile energy E1 only this much fraction is available in the center of mass system which will be useful to induce the reaction. So let us see now the energy available in the center of mass system. Just now we discussed mass of projectile and having kinetic energy E_1 + M_2 going to a composite nucleus M_3 + M_4 where E_1 is the projectile energy in the laboratory system and the energy available in the center of mass system is now Ecm equal to initial energy of projectile into M_2 /(M_1 + M_2). This is the fraction which is available in the center of mass system and the remaining fraction $M_1/(M_1+ M_2)\ E_1$ that's called as the recoil energy.

So this much energy is not available for the reaction to take place. In fact, like you know you call free energy in chemical reactions. The free energy only is available useful energy to induce the reaction. Similarly, here in the nuclear reaction is the energy available in center of mass system E_{CM} that is the useful energy to induce the reaction. Let us try to explain this in more details.

So again for this reaction, $^{4}He+^{27}Al \rightarrow ^{1}n+^{30}P$, the Q value we have calculated earlier is

Q =2.425+(-17.197)-8.071-(-20.201) =-2.642 MeV, which is an endo-ergic reaction.

And now the projectile energy 4.87 MeV that is the energy of alpha particle available from polonium-210 decay. So Ecm will be for this reaction.

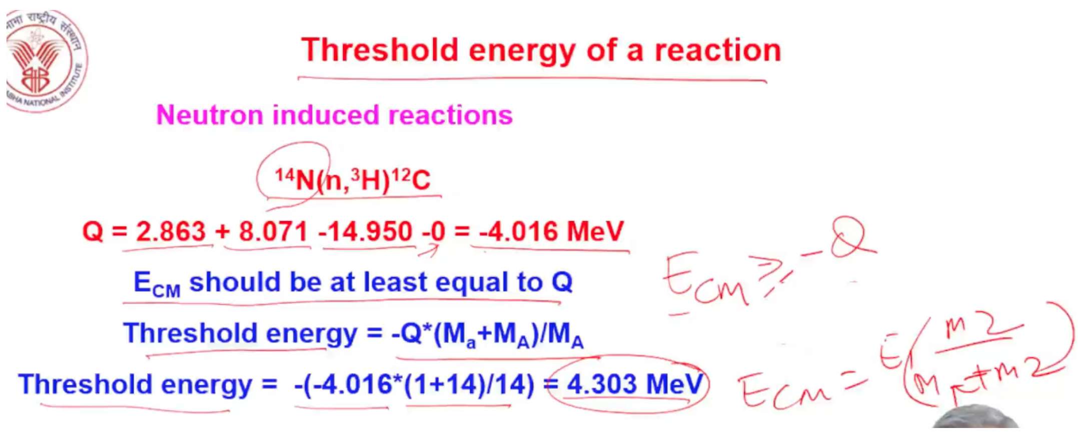
Ecm = 4.87*27/(4+27) = 4.24 MeV

So when the alpha is bombarding the ^{27}Al, Ecm will be (projectile energy) *$M_2/(M_1+M_2)$. Target mass upon projectile plus target mass and that comes to be 4.24 MeV. So out of the initial energy of alpha particle of 4.87 MeV only 4.24 is available to induce the nuclear reaction.

That is the meaning of Ecm. The remaining 4.87- 4.24 is not useful for the reaction that will go as a recoil energy moving the whole system forward. We will also discuss in the case of compound nucleus: suppose projectile and target form a compound nucleus. For example, here they can combine to form phosphorus-31 and then phosphorus-31 emits a neutron. So the phosphorus-31 will be called as a compound nucleus.

So in such cases what is the excitation energy of the compound nucleus is also calculated using the center of mass energy. So a+A projectile plus target going to form a compound nucleus. The Q value for formation of compound nucleus mass of projectile plus mass of target minus mass of compound nucleus and so the energy available in center of mass is energy of projectile into the ratio of target mass upon projectile plus target mass. So this is the energy available in the center of mass system and this is the Q value.

The compound nucleus let us say phosphorus-31 in this particular reaction will be formed with an excitation energy of Ecm + Q. Whether Q value is positive or negative that is immaterial the compound nucleus will have excitation energy E* = Ecm + Q. So that is the significance of the compound nucleus excitation energy which is governed by the energy available in the center of mass system and the Q value of the reaction.



Now if there is a positive Q value reaction, such as for neutron induced reaction there is no threshold even the thermal neutrons can undergo reaction. But if the Q value of the reaction is negative then for neutron induced reaction also we require a threshold energy neutron should have a minimum energy that is called as the threshold energy of the reaction.

So let us consider a nuclear reaction induced by neutrons which is having a negative Q value.

$^{14}N(n,^{3}H)^{12}C$

So let us calculate the Q value of this reaction. Q value will be

Q =2.863+8.071-14.950-0 = -4.016 MeV

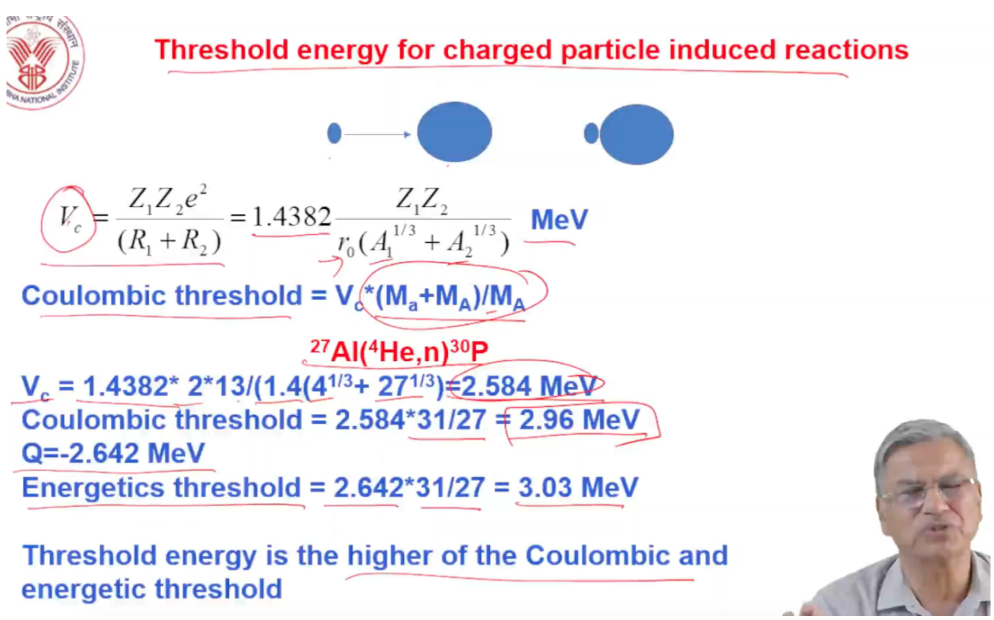
So this is a negative Q value reaction means it is endo-energic. So if a neutron has to induce this reaction then the energy in the center of mass system should be at least equal to Q value. You have to supply energy as it is a negative Q value reaction. So Ecm should be at least equal to Q value. So when we say Ecm should be at least equal to Q value that means -Q but this is nothing but Ecm $=E_1 * M_2 / (M_1 + M_2)$ and so you can write,

Threshold energy = $-Q(M_1+M_2)/M_1$

So this much additional energy you need to have in the threshold energy. So when we say threshold energy it is the projectile energy in the laboratory because the accelerators giving you or the neutron that is coming out has to have that much energy in the laboratory system. So when we say threshold energy that is the energy of projectile in the laboratory. So threshold energy of neutron will be -Q value

Threshold energy = -(-4.016*(1+14)/14 = 4.303 MeV

So actually you require 4.016 which is the Q value but since the neutron will be giving a slight recoil to the nitrogen-14 neutron energy should be actually more than the Q value by this much factor. And so this is the threshold energy in the laboratory of neutron which will be sufficient to cause the nuclear reaction. So that is how you can calculate the Q value of nuclear reaction or those which have got negative Q value.



Now we will come to charged particle induced reaction. So the charged particle induced reaction if the reaction is a negative Q value not only that you have to cross the threshold of the energetics but you have to also cross the coulombic barrier and so that is why for a projectile hitting a target charge particle projectile then you have to see what is the coulomb barrier. And the coulomb barrier for charged particle induced reaction can be calculated

$V_c = Z_1 Z_2 e^2/(R_1+R_2)$

Where Z_1 and Z_2 are the atomic numbers and the R_1, R_2 are the radii of the projectile and target. So if you recall the previous lectures we have dumped this some of certain units into 1.44 factor so that the energy becomes in MeV and r_0 is the radius constant 1.4 and this mass number of projectile and target to the power one third. So now the projectile has to have this much energy in center of mass system then only the coulomb barrier will be crossed. So for a charge particle induced reaction projectile should have energy equivalent to at least equivalent to Vc and so accordingly the coulombic threshold will be Vc into a factor for center of mass.

Coulombic threshold = $V_c(M_a+M_A)/M_a$

So by this much factor the laboratory energy should be more than the coulomb barrier. So for this reaction $^4He+^{27}Al \rightarrow ^1n+^{30}P$

$V_c=1.4382*2.13/(1.4(4^{1/3}+27^{1/3}))=2.584$ MeV

and so the energy in center of mass should be equal to 2.584 and accordingly the projectile energy in the laboratory will be higher by this much factor

Coulombic threshold = 2.584*31/27 = 2.96 MeV

So the coulomb barrier is 2.584 but the projectile should have 2.96 MeV in the laboratory so that we have Ecm energy available in center of mass equal to 2.504 MeV. Now the Q value for this reaction is -2.642 so energetics point of view threshold should be

Energetics threshold = 2.642*31/27 = 3.03 MeV

So now you have a threshold based on the negative Q value and we have a threshold based on the coulomb barrier whichever is the higher threshold value that will be the actual threshold. So when it comes to a charged particle induced reaction if the Q value is negative then you have to calculate the threshold not only for the coulombic barrier but also for the energetics point of view and whichever is the higher value that will be the threshold. So threshold energy is the higher of the coulombic and the energetics threshold.

So you have to compare the threshold for the coulomb barrier as well as the negative Q value and whichever is the higher one will be the threshold for the reaction. So today we have discussed the energetics of nuclear reaction what are the Q values it could be endo-ergic or exo-ergic and if it is an endo-ergic then there will be a threshold. Projectile will have some certain minimum energy to cross the threshold and if it is a charged particle not only that negative Q value threshold has to be crossed but the coulombic barrier also has to be crossed. Accordingly, the energy available in the center of mass has to be calculated. Accordingly the laboratory threshold value is higher than the energy in the center of mass by the factor that is $(M_1+ M_2)/M_2$ the factor takes care of the energy conversion from laboratory to center of mass. So I will stop here in the next lecture we will discuss the cross sections of the nuclear reaction. Thank you.