Transport Processes I: Heat and Mass Transfer
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Lecture - 60
Diffusion equation: Equivalence of spherical harmonics and multipole expansion

We continue our discussion of the solution of the diffusion equation. We were looking at solutions in the spherical coordinate system. I had derived solutions for you by separation of variables and we got this rather complicated form for the temperature field.

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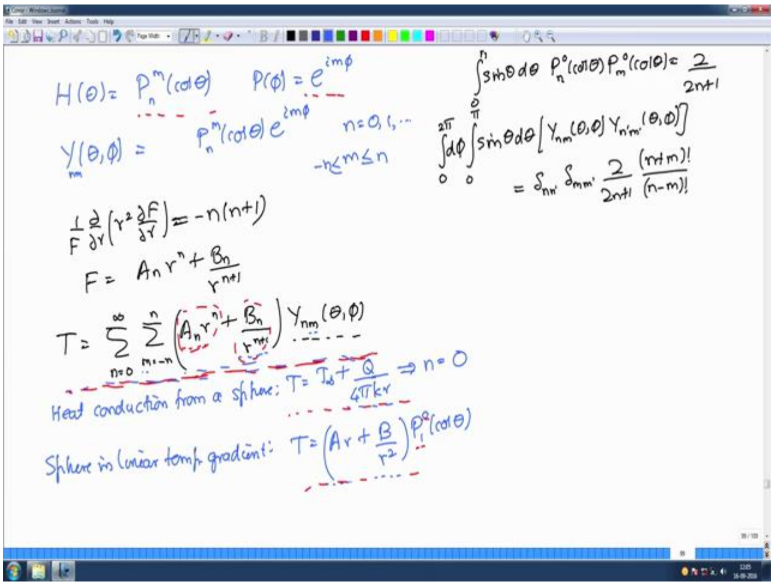

It was basically spherical harmonic expansion, where Y n m of theta phi were the spherical harmonics and that was multiplied by two terms: one of which was increasing as r increased the other was decreasing as r increased; one was increasing proportion to r bar m the other was decreasing proportional to one over r bar n plus 1. It also solves the same equation in a different way and that was to consider the delta function solution.

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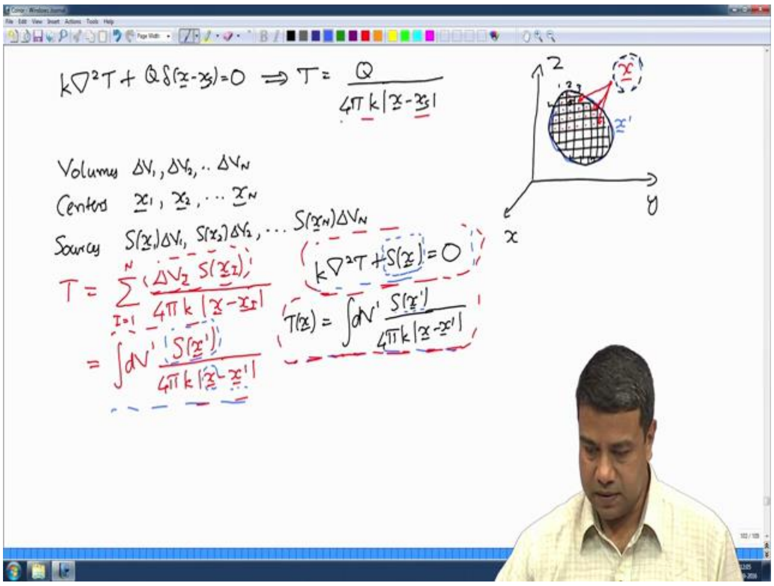

To consider inhomogeneous terms in the form of a delta function and for that we got the solution as Q by 4 phi k times the distance between the source and the observation point.

In the last lecture we were trying to see what it is that the relationship between these two kinds of solutions.

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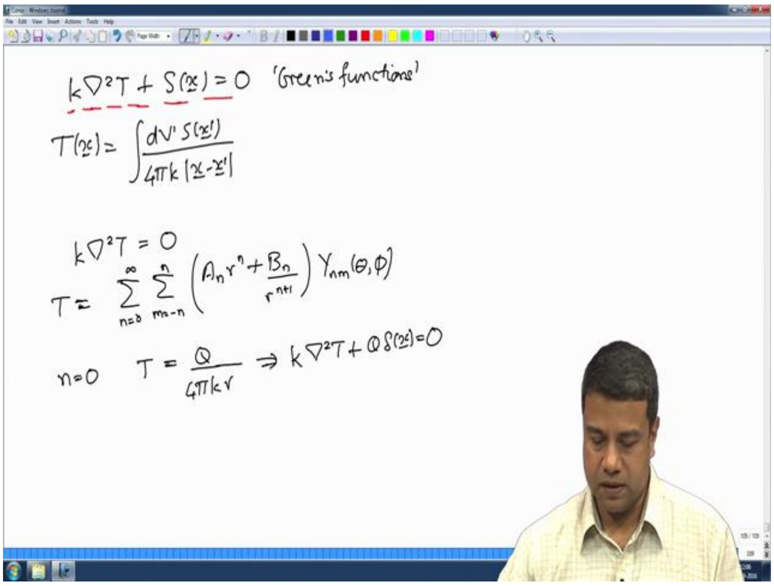

You have one solution where in you express the inhomogeneous term has a source term and then you get the temperature field, the other is this expansion A times r power n and B n by r power n plus 1.

So, for n is equal to 0, I told you it is a point source, it will be solved the equation for a point source is get Q by 4 pi k r that is the exact solution that you get by solving this as well for a point source a delta function source, what do the other terms in this expansion mean?

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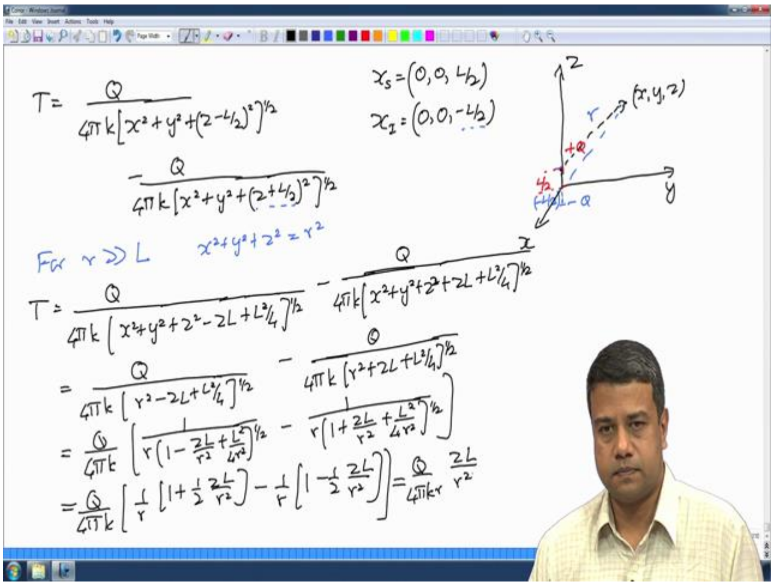

So, for this let us take a source plus Q at a location L by 2 along the z axis and then take a sink minus Q at the location minus L by 2. So, we have a combination of a source and a sink which are separated by a distance L along the z direction, what is the result for the temperature field that I would get for this configuration?

We actually solved this, what we need to do is to take a source and the sink and the super force the temperature due to each one of them individually. In this configuration since I have a combination of a source and a sink, the flux lines will be parallel perpendicular to the mid plane between those two. So, therefore, the temperature is equal to Q by 4 pi k into the distance between the any point observation point x, y, z and the source location. The source location x s is that 0, 0, L by 2 it is along the z axis, so the x and y coordinate have 0 and the z coordinate is at plus L by 2; the image of the source is at 0, 0 minus L by 2 that is the sink.

So, this is going to be equal to x square, plus y square, plus z minus L by 2 whole square whole square root off; there is going to be equal to the temperature due to the presence of the source. The sink has strength minus Q. So, the second term in this will be equal to minus Q by 4 pi k into x square, plus y square, plus z plus L by 2 the whole square whole to the half and that is because the sink location is that z is equal to minus L by 2. So, z minus the sink location is going to be equal to z plus L by 2 ok.

Now, what happens if the distance from the source of the sink is much larger than the separation between these two; in other words what happens for r much greater than L? The distance from the source of the sink is much larger than the separation between these two. So, what you can do is to expand this temperature field. So, I can write this as x square plus y square plus z square, I will get minus z L, plus L square by 4 whole to the half. Just expanding out this z minus L by 2 the whole square and I will get the second term as minus Q by 4 pi k into plus z L, plus L square by 4 whole to the half.

Now, use the substitution that x square plus y square plus z square is equal to r square that is the distance from the origin. So, this I will get Q by 4 pi k into r square minus z L plus L square by 4 whole to the half.

Now, I take divided take r out as a common factor in the denominator in both of these expressions. You take r out as a common factor in both of the denominator. So, I will get Q by 4 pi k into 1 over r into 1 minus z L by r square plus L square by 4 whole to the half minus 1 over and now since r is much larger than L I can use an expansion in l over r. I can use an binomial expansion in L over r and just retain the first term in the series; 1 over r into 1 plus z L just retain the first term in the expansion, 1 by 1 minus x to the half is 1 minus half x sorry 1 by 1 minus x to the half as x goes to 0 is 1 plus half x. So, now the second term here which is 1 over r to 1 minus 1 by 2 z L by r square. So, this is effectively equal to q by 4 pi k r into z L by r square. So, let us look at what this means. So, I had started off with a problem.

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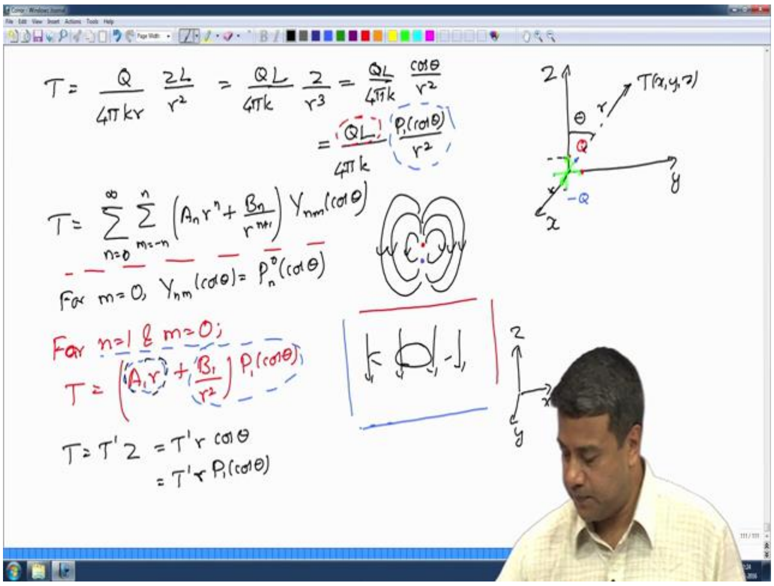

Use the source plus Q; let us sync minus Q separated by a distance L and I was trying to find out what was the temperature at some location T of x, y, z for the case where this distance r is much larger than L and retaining just the first term in the expansion, what I got was T is equal to Q by 4 pi k into r into z L by r square.

I can write this as Q by 4 pi k into L into z by r cubed; since z is equal to r cause theta should recall when we did a spherical coordinate z was equal to r cause theta is equal to Q L by 4 pi k into cause theta by r square and since when we do all the expansion the legendre polynomial series, we know that this is also equal to Q L by 4 pi k into P 1 of cos theta by r square; let us refer back to the expression for the temperature that we had derived earlier.

For m is equal to 0, Y n m of cos theta, just equal to y n 0 cos theta; so therefore, if you take this expression and take the value for n is equal to 1 for m is equal to 1 and m is equal to 0, the temperature becomes A 1 r plus B 1 by r square into P 1 of cause theta. I am sorry yeah this term here 1 over r square times P 1 of cause theta is exactly what I have got over here. So, therefore, the superposition of a source and a sink separated along the z axis, gives me the solution for n is equal to 0 and n is equal to 1 and m is equal to 0. This is called a dipole a point dipole; in the limit as L goes to 0, the separation of the source and the sink goes to 0; in the limit as L goes to 0, while Q times L is finite, as you make the source and sink closer and closer to each other, while retaining Q times L to be a finite number you get a point dipole.

Solution for the point dipole the temperature decreases as 1 over r square. I have a source and the sink close to each other; that means, that outside of these there is no next source because whatever is coming out of one source is going back into the sink, there is no net generation of energy, the net source is equal to 0. For a point source I have told you that the temperature decreases as 1 over r; in this case there is no net source, you have a superposition of a source and a sink, so I said the net energy from one goes back into the other, there is no net generation of energy, in that case the temperature decreases as 1 over r square. So, this is the dipole the point dipole with a dipole strength, which is given by Q times L and the combination of these if you plot if you plot a source and a sink close to each other, the combination of these is going to give you flux lines that start from one and go to the other.

This is the dipole solution for the Laplace equation; these are the decreasing harmonics the decreasing dipole solution that decreases as 1 over r square. There is a second term that increases proportional to r that instead of having two points close to each other, if you have a source and a sink far from each other, the flux lines will look something like this between these two.

Therefore the temperature will be of the form T is equal to T prime time z there will be a linear variation of the temperature is equal to T prime times r times cos theta, which is equal to T prime times r P 1 of cos theta. So, this term that increases proportional to r basically corresponds to a source and a sink that are separated by a very large distance and you are looking at the temperature somewhere in between that is in a sense the inverse of this other problem, where I have a source in the sink which are separated by a small distance and you are looking for fun. So, those are the physical interpretations of the growing in the decaying harmonics. So, this is for n is equal to 0 and for m is I am sorry; for n is equal to 1 and for m is equal to 0, a dipole solution with the dipole moment that is this distance L aligned along the z axis.

What about for I just told you right in this expansion m can go from minus n to plus m. So, m can be minus 1, 0 plus 1; therefore there are three solutions, what do the other two solutions mean? I would not go into the details, but the other two solutions are basically dipoles that are aligned along the x axis with plus Q and minus Q along the x axis, that is for m is equal to 1 and m is equal to minus 1 corresponds to a source along the plus y axis and the sink along the minus y axis. In each of these cases the dipole moment is aligned differently, the three solutions for m is equal to minus 1, 0 and plus 1. For 0 it corresponds to an alignment of the dipoles, the source and the sink the separation between those two alignments of that separation, along the z axis. The other two are for the separation along the x and y axis; any dipole with separation in any direction can be expressed as a linear combination of these three because any vector can be expressed as a linear combination of the basis vector in this case there in the x, y and z direction.

So, any dipole can be expressed as a combination of these three; that is why you have 3 solutions: m is equal to 0, m is equal to minus 1 and m is equal to plus 1 with the dipole moments aligned along the three coordinate directions. I am sorry I should write with the 3 dipole moments along aligned along the three coordinate directions. So, those are the 3 solutions for n is equal to 1, m is equal to minus 1, 0 and plus 1 these are all dipoles combination of a source and a sink of equal magnitude, in such a way that the net source is 0, what is left behind when the net source is 0 is a dipole for which the temperature field decreases proportional to 1 over r square.

Since the next source is 0 there is no contribution which decreases the first 1 or r, the next higher term is actually a contribution that decreases to proposal to 1 over r square, those are the decaying harmonics. The growing harmonics on the other hand are constant temperature gradient solutions. So, for n is equal to 0 it is aligned along the z direction, for n is equal to 1 I will have an alignment along the x direction sorry. So, if I take this as the z, this is the x and this is the y, separation between these two surfaces much larger than the volume that I considering that is the inverse of the problem of point sources and n is equal to minus 1 corresponds to separation along the y axis and n is equal to plus 1 along the x axis. So, this is a 3 linearly independent solutions, for n is equal to 1 all of these decays as 1 over r square for the decaying harmonics, increase proportional to r for the growing harmonics along the three different coordinate directions.

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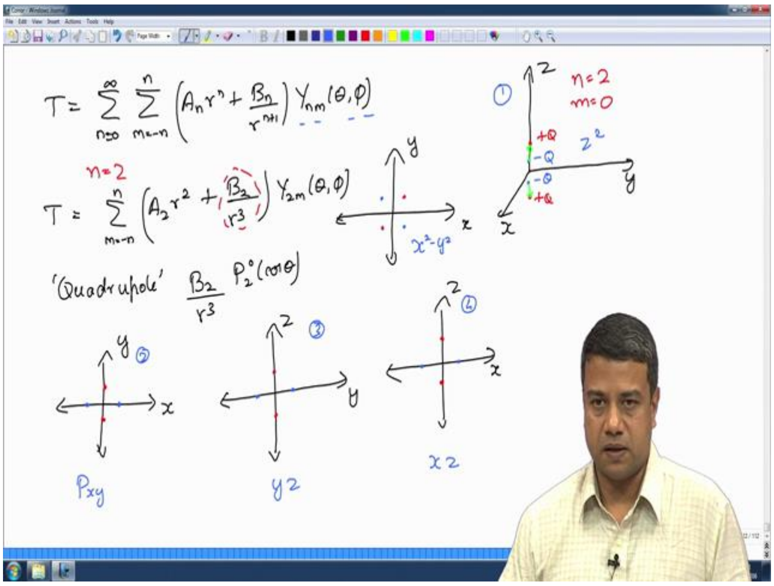

The next solution for n is equal to 2, the temperature is given by A 2 r square plus B 2 by r cubed into Y 2 m naught theta and phi. In this case n goes from minus 2, minus 1, 0 plus 1, plus 2. I would not be able to go through the details of how to derive these, the procedure is exactly the same as I have done it for the dipole what do these correspond to.

The growing harmonics correspond to 2 sources and 2 sinks of equal strength. So, the first one corresponds to a dipole which is aligned in this direction, the next one corresponds to a dipole that is a line in the opposite direction. The net source is 0 because the source and sink trends are equal. The net dipole is 0 because the dipole moments for the top one are exactly the opposite of the dipole moments of the bottom one, what your left with is what is called the quadrupole field.

I told you that the dipole decays 1 over r cubed, if the net dipole is 0 the next higher order term has to decay is one over r cubed; dipole decay is one over r square (Refer Time: 22:17) stick is 1 over r, dipole decay is 1 over r square the next higher term has to decay is 1 over r cubed. So, you get this by combination of 2 sources and 2 sinks in such a way that the net source is 0, the net dipole is 0, you noted left with is a quadrupole field which basically goes as, this is for n is equal to 2 and m is equal to 0; in that case we get something that goes as B 2 by r cubed, P 2 0 of cos theta.

So, that is for m is equal to 0, where you have axis symmetry along the z axis, you can get it for other values of m: minus 2, minus 1, plus 1, plus 2. I will try to explain to you what these physically represent. These are k of the form, the first one is along the x y plane, you have 2 sources and 2 sinks along the x y plane, the net source is 0 the net dipole is 0. So, this is called P x y ok then you have one along y z plane, I am taking projections of this. So, y z in this once again you have 2 sources and 2 sinks then you have one along x z, I am sorry I should made a mistake here.

So, this is one solution for m is equal to 0; then we have 2 3 solutions in which these sources and sinks are separated along the coordinate axis x and y axis first, y and z axis next and x and z axis next and finally, you have one final solution in which these are off center along the x y. So, this will be off axis, these are called x y, y z, x z this goes by the name of x square minus y square and this is z square. These are the fundamental mode in the spherical harmonic expansion, quadrupole all of them combinations of 2 sources and 2 sinks in such a way that the net source is 0 and the net dipole is 0, there are 5 fundamental solutions of these: one of which is axis symmetric about the z axis, the others are all in the x y, x z, y z plane.

These decreases all as 1 over r cubed and the next higher term will decreases 1 over r to the fourth that will have 4 sources and 4 sinks in such a way that the net source net dipole in a quadruple around all 0. So, by a combination of these point sources, just single point source gives us n is equal to 0, one source and one sink the dipole gives us n is equal to 1; two sources and two sinks gives us a quadrupole and so on. So, these are identical to what we got by looking at the heat conduction equation with a delta function source, the 2 are identical you can have a one to one correlation between the solutions for the delta function the sources and sinks and then Legendre polynomial expansion.

Since all of these are all orthogonal to each other, any temperature field due to any source can be expressed as a linear combination of these that is precisely what we did when we solve the problem for the heat conduction in a composite material, we expressed it in legendre polynomial expansion since each of these terms is orthogonal to every other term, any solution can be expressed in terms of these and just from the symmetries you can construct a solution for the legendre a polynomial expansion, give for the heat conduction equation with inhomogeneous terms I showed you how to superpose delta functions and how to incorporate boundary conditions.

With this we will complete our discussion of the diffusion equation, I shown you how to solve it in multiple ways one by separation of variables in Cartesian coordinate system spherical coordinate system; second by thinking of an inhomogeneous term which is in the form of a delta function and then generalizing it to a distributed source of energy along with boundary conditions, which can be satisfied using symmetry arguments.

Next lecture we will start the opposite limit convection dominated floor, where the peclet a number is large, the opposite of the present case. Here the peclet number was small, so we neglected all the convection terms, in that case the peclet number is large and you would think that you can just neglect the diffusion terms; however, transport two surfaces has to take place ultimately due to diffusion because convection is only parallel to the surface there is no velocity perpendicular to the surface, how do we solve those problems, they are all solved using boundary layer theory using similarity solutions that we had seen earlier. I will start convection dominates transport in the next lecture and I will see you then.