Transport Processes I: Heat and Mass Transfer
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Lecture - 59
Diffusion equation: Method of images

Welcome to this our series of lectures on the fundamentals of transport processes, where we were looking at how to solve the diffusion equation in the limit of small Peclet number, where the transport is basically diffusion dominated because the convective transport is much smaller than that due to diffusion. We had looked at different problems, problems in a Cartesian coordinate system for example, problem in a spherical coordinate system, the effective conductivity of a composite. I had in the previous lecture given you the general solutions for a spherical coordinate system as a series of spherical harmonics.

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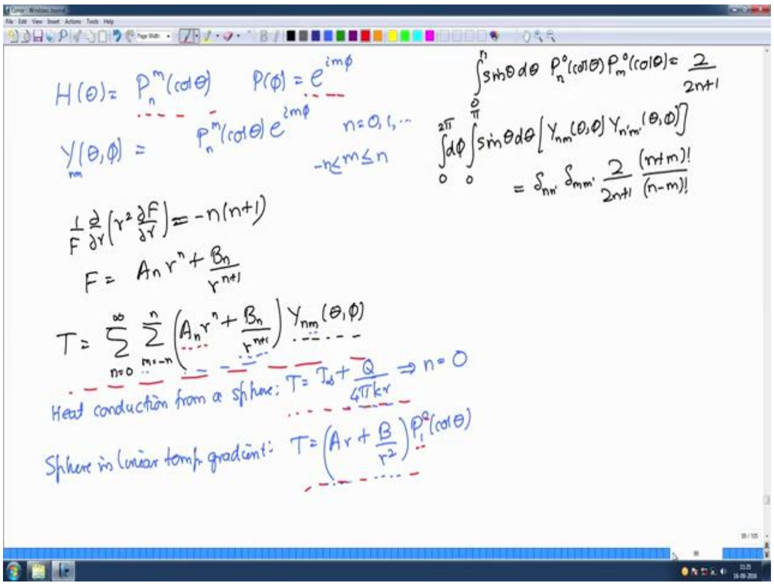

You can see there are two of these spherical harmonics: one is increasing proportional to r power n and the other is decreasing as 1 over r power n plus 1.

These are the general solutions and I had told you that the solution for heat conduction from a sphere was a special case where n was equal to 0, the sphere in a linear temperature gradient was the special case where n was equal to 1, but we still did not have a clear picture of what these spherical harmonics actually mean and I was trying to give you a physical interpretation based upon the heat conduction due to point sources.

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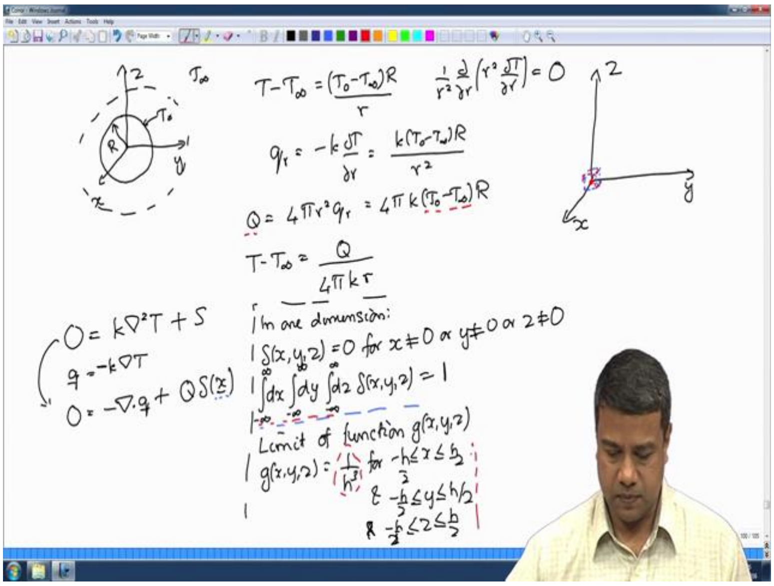

So, I had defined the heat conduction due to a point source, wherein the inhomogeneous term in the equation is equal to an energy generated per unit time times the delta function. Delta function is an idealization of a function which is nonzero only at 1 point, it is 0 everywhere else in space, the integral of that of that function over the volume is some finite number; in this case the delta function is defined so that it is normalized to 1 and I try to give you some physical understanding of the functions which can be idealized to a delta function.

In three dimensions it is a function which is equal to 1 over h cubed only in cubic differential volume its 0 everywhere else and the integral of that over volume is equal to 1, and I showed you that the temperature field due to this point source is just equal to Q by 4 pi k r; where Q is the energy that is generated per unit time from the source. So, even though it has 0 volume, the energy generated per unit time is finite which means the flux has to go to infinity at the surface.

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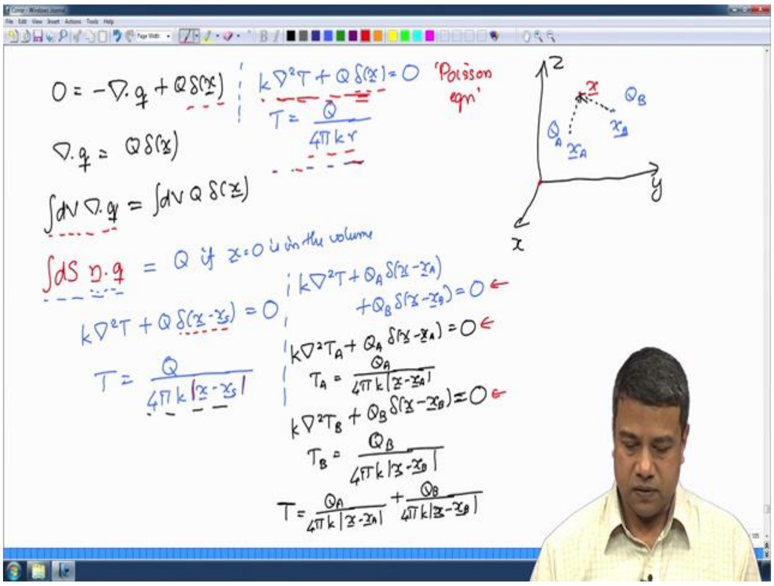

Now, for this source we had seen how to solve problems; if you this is basically an inhomogeneous equation with the delta function as the inhomogeneous term. So, it is a Poisson equation, which contains the laplacian of the temperature field plus a delta function as the inhomogeneous term. There showing you that the solution for that is Q by 4 pi k r. The temperature of course goes to infinity as r goes to 0, the flux which is the derivative the temperature goes as 1 over r square volume goes as r square; so the total heat generated is finite.

If you had the source at some other point you just have to take the distance from the source point. If you had two sources calculate the temperature due to each one of those sources and add them up, this works only for delta functions as I had shown you previously it does not work for finite objects because when you separate the problem into two different problems, the physical boundaries all have to be the same in both of those problems.

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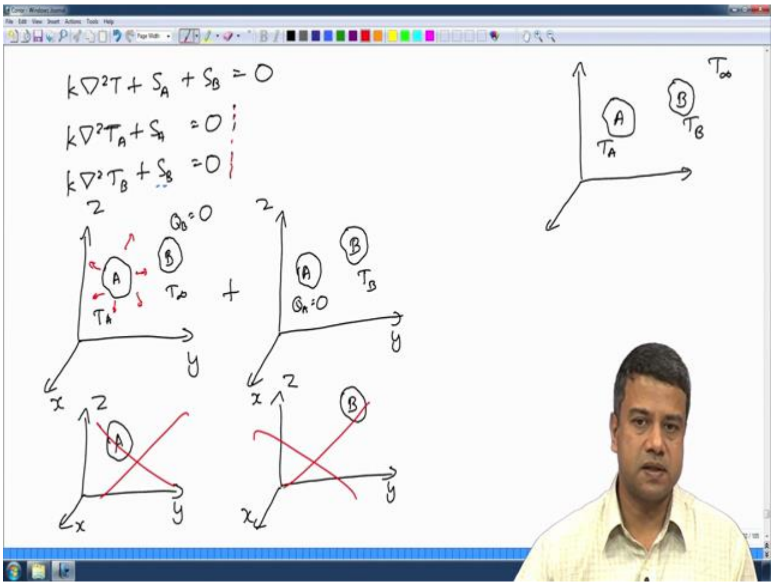

You can separate out into two problems in one of which the object, one object is generating heat at the other is not. In the second the second object is generating heat and the first is not and then add them up that is always valid, but when you do that separation the physical boundaries have to be exactly the same in both of those objects and because of that you do not gain much for finite objects, since the delta function has infinite decimal volume, the volume is 0, physical boundaries do not matter because the volume itself is 0 so therefore, in the case of delta functions the superposition is simple, it yields useful results and it can be used for cases where you have a distributed source.

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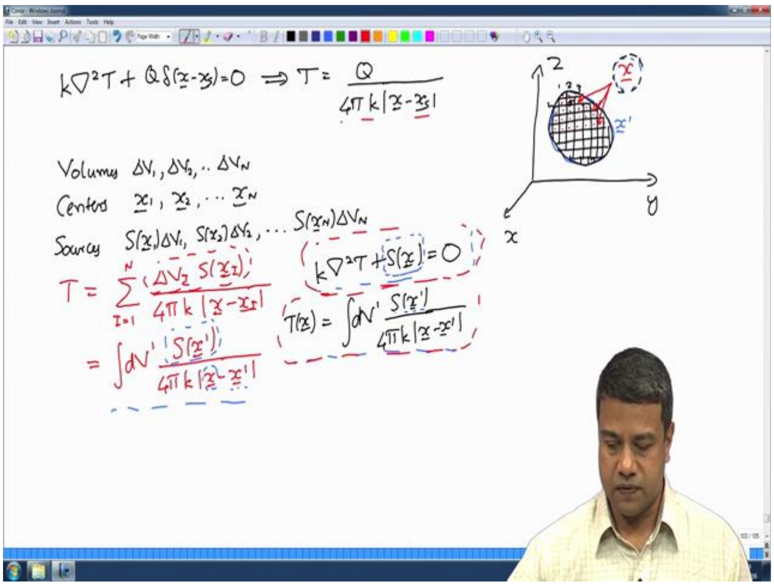

If you have a distributed source, you can separate that source into small infinitesimal volumes; and the entire heat generated within that volume can be placed at its center because the volume itself is going to 0 so the heat generated from that volume approximates the delta function as the volume goes to 0 and then you can add it all up, you can find out the value of the temperature at a location due to each one of those sources and in the limit as the volume goes to 0 this basically becomes an integral and therefore, this is the solution for the temperature field due to the distributed source and we had seen an example how to solve that.

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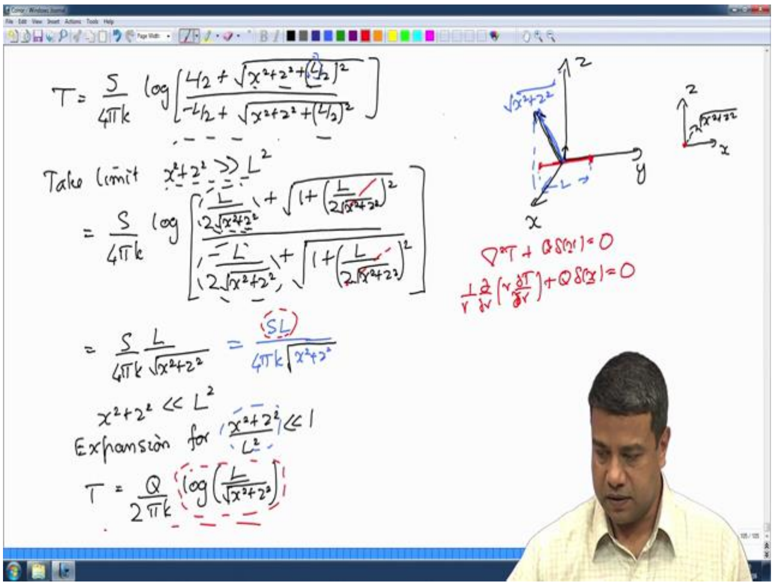

For in this case for a wire which is extended along the y axis we had seen this is non; the source, is now a source per unit length therefore, we had idealized it as 2 delta functions in the x and z direction, because it is confined to x is equal to 0 and z is equal to 0; this is a wire that is extended along the y axis and we had seen that when the distance from the wire becomes much larger than the extent, it basically looks like a point source, the details do not matter it looks like a point because the distance from the source is much larger than the extent of the source.

In the opposite limit you get a solution that is basically the solution for the diffusion equation the poison equation in two dimensions, because the distance from the wire is much smaller than the extent of the wire. So, the wire effectively looks like an infinite object, because the distance from the wire is much smaller than the extent of the wire.

Infinite object in three dimensions is effectively a point at the origin in two dimensions in this two dimensional case and to cover the solution in that case as well. So, we can do this for distributed sources for point sources and so on, real problems invariably have physical boundaries. So, how do we extend this for the case where we have a physical boundary?

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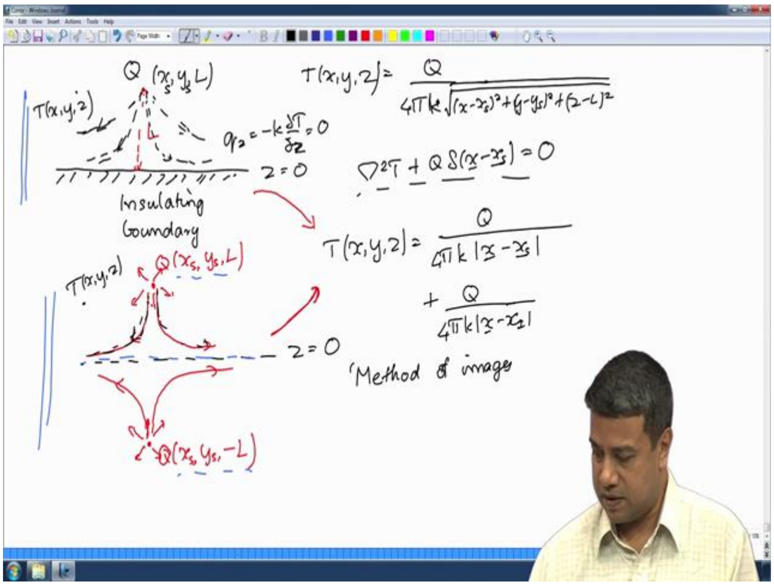

So, let us say that I had a boundary here, I will call this as z is equal to 0 and I had a point source at this location and the distance of the point source from the boundary was L and let us assume that this was an insulating boundary, there is no heat conduction across this boundary. So, this is an insulating boundary when z is equal to 0 and the source is at some location x, y, L set the height L from the insulating boundary.

If the boundary were not there then the temperature if Q was the energy that was generated per unit length, the temperature Q would just be equal to I am sorry the temperature T would just be equal to Q by 4 pi k into let me call this as the source point. So, this is x s, y s and L it that could be the temperature field at the location x y z that would be the temperature field at the location x y z at any point T at; however, this is not going to be the temperature field in this case, because I do have an insulating boundary. In the insulating boundary what should happen? If there were no boundary then the heat flux lines would all be isotropically outward this is generating some heat, so the heat flux lines would all be isotropically outward spherically symmetric.

In this case since the boundary is insulating, the flux lines cannot go through the boundary. So, therefore, there is going to be a distortion to the temperature field due to the presence of this insulating boundary and the question is how do we calculate that how do you calculate the distortion to the temperature field due to this insulating boundary? Basically at this boundary you require that q z is equal to partial T by partial z; minus k partial T by partial z this has to be equal to 0 at this insulating boundary there should be no temperature variation perpendicular to the surface, if its insulating because the flux has to be 0.

This seems like a rather complicated problem to solve, it turns out it is not that complicated. The reason is that rather than solving this problem, I can replace it by another problem where I do not have a physical boundary, I do not have a physical boundary, but rather I put a point source at Q at x s, y s and L and another point source symmetrically about this plane at the mirror image of this point source, another source Q at x s, y s and L. So, I do not have a physical boundary, but rather I construct another problem where instead of the physical boundary, I put another point source at the mirror image of this point source reflected in the boundary.

Now what should happen? You know that the flux lines all go outwards on this, they go outward on this as well it has the same source strength and just from the symmetry of the problem, it is easy to see that the flux lines to be symmetric; based on the symmetry of the problem it is easy to see that if I have two such sources radiating symmetrically, the flux lines along the central plane have to go tangential to the plane; that means, there in this case just from the symmetry there will be no heat flux through the central plane, I should write this as minus L my apologies.

The second source is located at minus L, which is the reflection of this first source on the boundary. So, as far as the physical problem is concerned, the physical problem is in the upper half space, for that physical problem this 0 flux boundary condition it is identically satisfied by this second problem, where I do not have a physical boundary, but I have another source at the point where this boundary is reflected. So, since the solution of this the symmetric the boundary conditions for this problem are the same as the boundary conditions for this problem; the equation is the same in the physical space the equation in the physical space is del square T, plus Q delta of x minus x s is equal to 0, there is a physical problem in this upper half space this is the equation.

The boundary conditions in both cases are the 0 flux boundary conditions therefore, the solution has to be the same, the solution at an observation point for this is going to be equal to T at is equal to Q divided by 4 pi k into x minus, minus x s. So, for this problem the solution is going to be equal to Q by 4 pi k into x minus x s, where x s is the source, x s is the source location, but I also have a second source the image source, the image of that original source.

So, I will get a second term which is due to this image, that image has the heat generated per unit time by the image is exactly the heat generated per unit time by the real source, only then will the problem be symmetric and the flux through the surface be 0 therefore, I have a second image which is 4 pi k into x minus the image point, that is the solution for this problem and since this problem has the same equation in the physical space as well as the same boundary conditions as this problem that is the solution for this problem as well. So, I have a no flux boundary condition in the physical problem and the temperature field has to be calculated in the upper half space, I replace it by another problem which has a source in the upper half space and an image source in the lower half space such that the boundary conditions with the surface are identical to the original physical problem. The heat conduction equation is identical because you have a source only in the upper half space.

Therefore the solution has to be identical. So, basically this problem of a boundary I have replaced it by a problem where I have two sources in such a way, that the conditions at the mid plane are identical to the conditions that I would have had at the boundary. So, this is called the method of images; this works for a point source and it is not too difficult to see that it will also work for more complicated volumetric sources.

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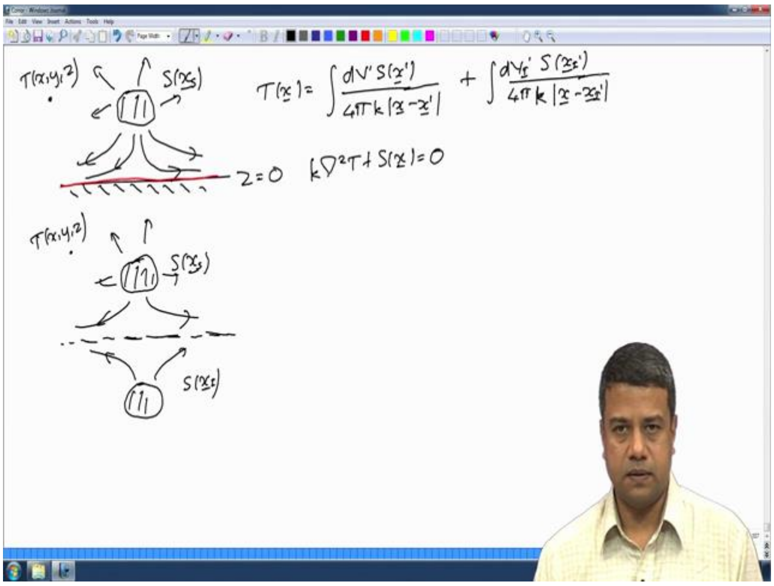

If I had for example, an insulating surface and I had some volumetric source over here with some distribution of energy per unit volume per unit time. The energy from this is going to be radiated, but; however, since I have a 0 flux condition, at the surface the flux lines have to go parallel to the surface and therefore, I can write the temperature at any observation point you cannot write it as.

This worked only when I had a source in an infinite medium, in this case I have a finite boundary at this location z is equal to 0; however, I can replace this problem by another problem, where I have a source identical to this one which is generating heat and I have an image of that same source, at the image location if it were reflected in the surface S of x I with exactly the source same source or strength. If I put those in, I am only interested in the temperature in the physical half space, but the temperature in the physical half space in this configuration is identical to the temperature in the physical half space in this configuration because both of these in the physical half space the equation is the same, k del square T plus S of x equation is the same the boundary conditions are the same both of these problems satisfied the 0 flux condition.

Therefore all I need to do is to add a contribution due to the image. V I image is an imaginary construct it is constructed so that I get back the correct boundary conditions at the surface is constructed so that the correct boundary conditions the 0 flux conditions are recovered at the surface, times the image divided by 4 pi k. So, this is not the correct expression because I have replaced the boundary conditions at this boundary by an alternate problem, which satisfies the exact same boundary conditions at that exact same boundary. This can be done for 0 flux conditions; it can also be done for example, for constant temperature boundary conditions at this boundary.

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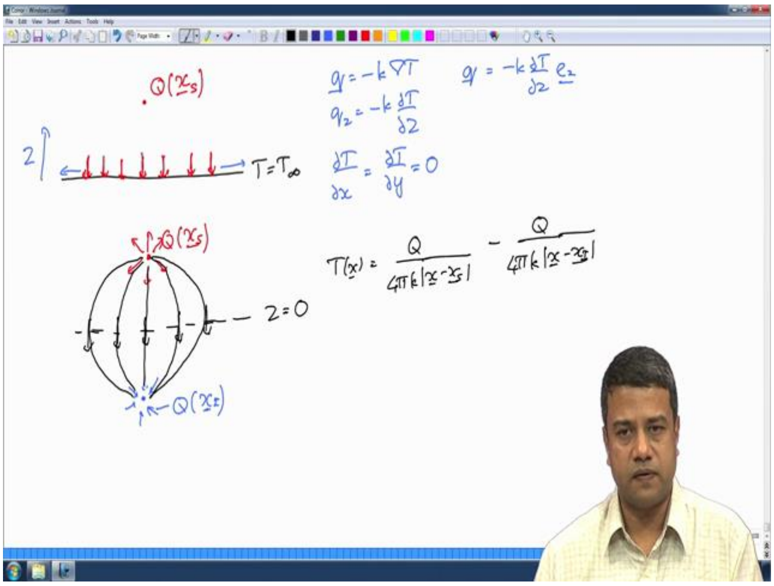

So, let us say that I had a boundary, in which I had specified the temperature at the boundary to be a constant value and I had a source here. Now how would the problem look like? If the temperature is right to be a constant at this boundary; that means, that the variation in temperature along this direction is 0, the variation temperature along the boundary is 0 because the temperature is a constant at this boundary.

What that means is that the variation in temperature can be only perpendicular to the boundary, we know that q is equal to minus k grad T, the variation temperature q z is equal to minus k partial T by partial z. If z is this axis here, the variation in the temperature in the x and y directions are 0, is equal to 0 the variations in temperature along the x and y directions are zero; that means, that this is equal to minus k partial T by partial z times e z. So, the flux is perpendicular to the boundary, flux is along the z direction because there is no variation in temperature along the plane. So, the entire flux at all of these points has to be perpendicular to the surface.

In the previous case when there was a 0 flux condition, the flux lines had to be parallel to the surface in this case because the temperature is a constant along the surface, there is no variation along the surface therefore, the flux has to be perpendicular to the surface. How do I solve this problem? I have a point source here Q at the location source location, I can replace it by an alternate problem where this is z is equal to 0 rather than using a point source, I use minus Q. So, within this source strength is Q, so the heat flux lines are all outwards on this. In this case it is minus Q, so the lines are all inward and just from the symmetry of the problem it is easy to see that the flux lines in between these two have to all be perpendicular to the surface.

because I have a source at one point symmetrically I have a sink therefore, the energy that is being coming out of that source has to go back into the sink because both of them have equal strength: one is generating Q per unit time, the other one is absorbing Q per unit time. These two have equal strengths therefore; this corresponds to a case where the flux lines are always perpendicular to the surface at the surface itself.

Which is exactly the boundary condition that I require for the original physical problem; so rather than having a source with a constant temperature boundary, I have a source and a sink located at the image of that source in the boundary. For this second problem the temperature is quite easy, is equal to Q by 4 pi k into x minus x s that is due to this first the source function. For the sink it is minus Q; minus Q by 4 pi k into x minus the image. So, therefore, this combination of the source and sink essentially mimics the constant temperature profile condition at the mid plane between these two. So, therefore, for any boundary condition whether it is 0 flux which means that the derivative temperature has to be 0, or constant temperature which means that the temperature minus T infinity has to be 0, in either of these cases you can replace the original problem by a second problem where you replace the physical boundary by a source or a sink located at the image of that boundary.

Solve that problem for the image of that boundary and you will get the correct temperature field in the physical space which is basically the upper half space. So, that procedure this method of images can be used effectively to represent boundaries. In this case they are just flat boundaries, but; however, they can be used even in more complicated cases. So, what I have shown you is what all we can do by this delta function representation and the extension of the delta function representation to finite objects

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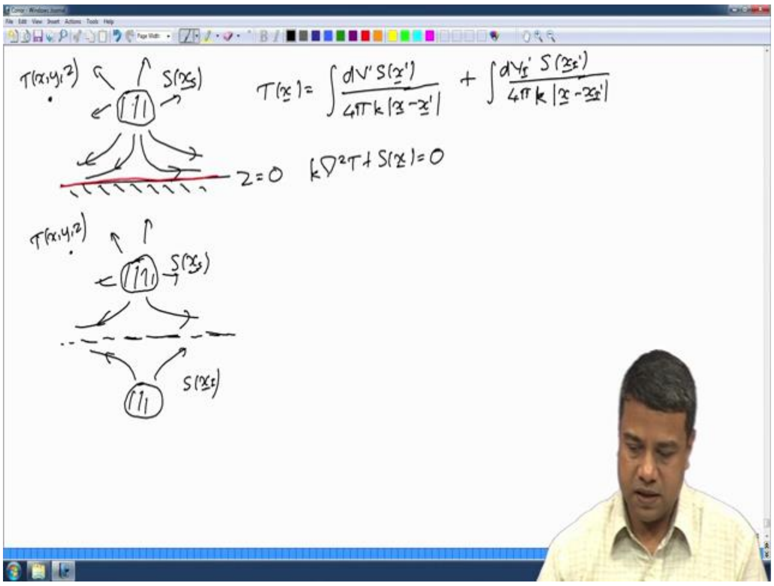

The extension of the delta function generation representation to finite objects by the method of images; by using images in order to represent the same boundary conditions that you would get at a physical boundary.

So, these are the solutions that I got for you by the method of images, by the writing the equation as a delta function.

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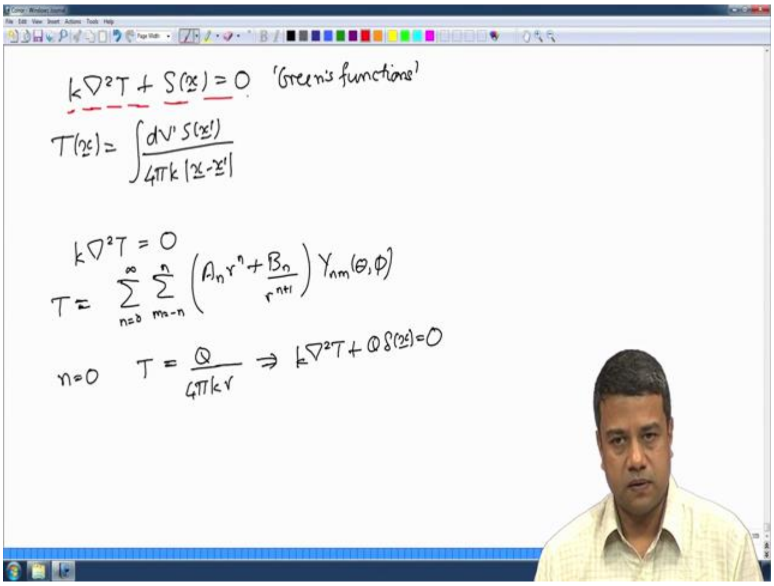

This is called the method of greens functions in the solution for that T of x is equal to integral d V prime S of x prime divided by 4 pi k into x minus x prime; where I had shown you how to express this distance in terms of the x y and z coordinates: x prime is the source point x is the observation point you are interested in the temperature at a given location x as a function of that the sources that are there in this differential volume.

In the spherical coordinate system, I had also shown you if I have k del square T is equal to 0, the temperature is equal to sigma n is equal to 0 to infinity; for n is equal to 0 I had shown you that the temperatures is equal to Q by 4 pi k r (Refer Time: 27:46) for n is equal to 0 the solution is the same that I got for the delta function. So, this is the solution for k del square T plus Q delta of x is equal to 0. A specialized form of this particular more general equation that corresponded to n is equal to 0 in which case m was equal to 0. What would the other terms mean or do the higher terms mean, how can they be understood in this context? That we will continue in the next lecture, I will try to show you the relationship between note we had solved for delta function sources and what we got from the spherical harmonic expansion.

So, we will continue that in the next lecture. I will see you then.