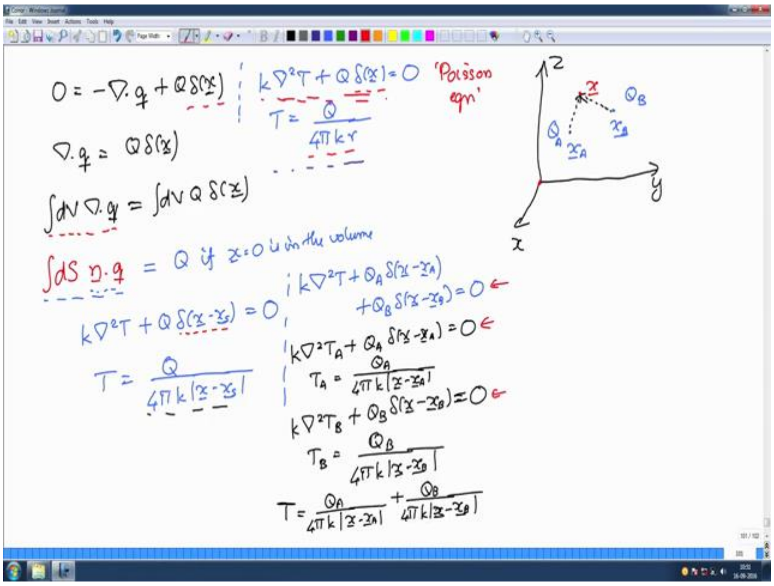
Transport Processes I: Heat and Mass Transfer

Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture - 58
Diffusion equation: Method of Greens functions

This is our continuing series of lectures on diffusion dominated transport and as I told you we were looking at the case where we had to solve the diffusion equation the Laplace equation, for either the temperature or the concentration field is equal to zero.

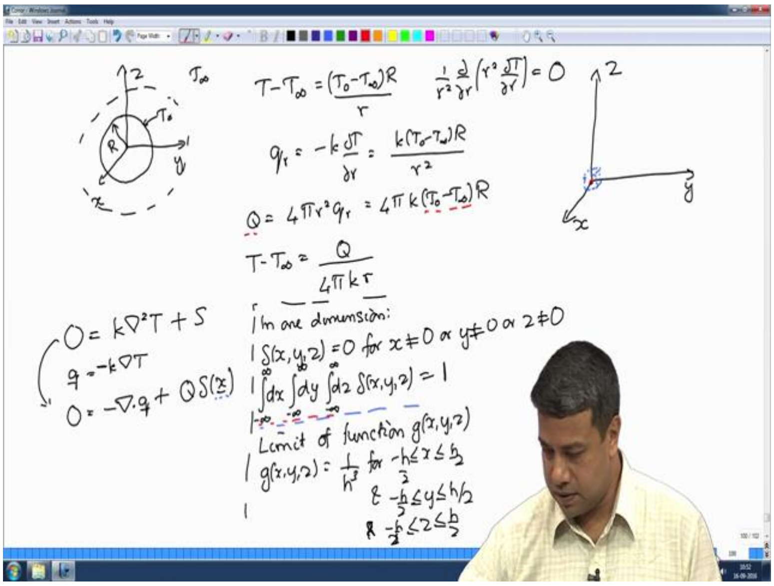
(Refer Slide Time: 00:38)



So, the equation is basically of the form K del square T, plus any sources or sinks is equal to 0.

We had considered the specialized kind of a source which is a point source, that point source it is located at a point the volume of that goes to 0, in such a way that the energy generated per unit time is a constant. So, if we have a point a source whose volume goes to 0 and therefore, if the energy generated is a constant, the flux has to go to infinity. You define the delta function previously, in this particular case the delta function is defined such that delta of x.

(Refer Slide Time: 01:22)

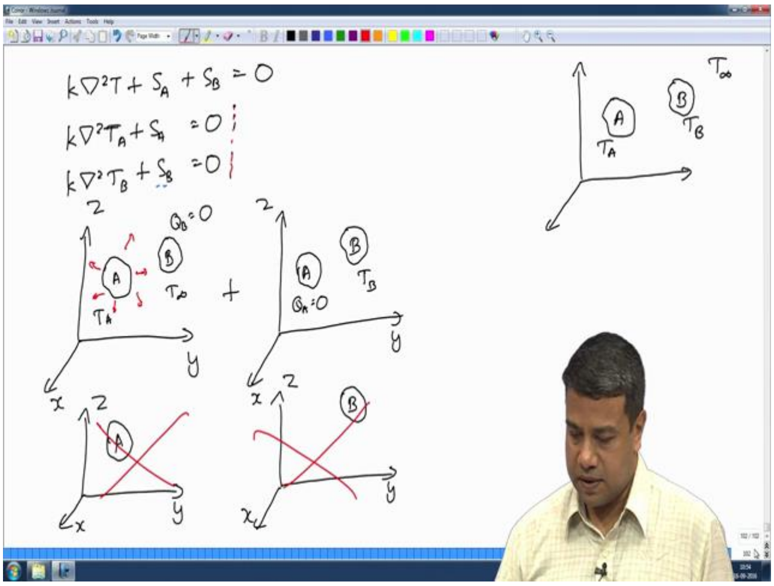


Such that delta of x is non zero only at x is equal to 0, and it is equal to 0 everywhere else the integral of the delta function over the entire volume has to be equal to 1. So, it is an idealization of a function which is non zero only in the at the origin where X is between minus h by 2 n plus h by 2 and Y is between minus h by 2 and plus h by 2 and Z is between minus h by 2 and plus h by 2 and within this region it has an attitude of 1 over h cubed. There are other ways of expressing the delta function as well, this is the simplest way; however, regardless of how you express the delta function the solutions that I get for this delta function will end up being the same.

So, I showed you that if we have an equation of the form k del square T plus Q delta of x is equal to 0, the solution of that is temperature is equal to Q by 4 pi k r. So, this temperature field is the solution for the Laplace, rather the poison equation; a Laplace equation with an in homogeneous term. So, I had shown you that when the sources at the origin you get Q by 4 pi k r, there are is the distance from the origin; when the sources at some other location you just have to take the distance of the observation point from the source point the vector distance, that in this case was equal to the magnitude of the vector x minus x s where x is the observation point and excess is the source point.

I had shown you super position; if you have two different sources at the observation point, you can find out the temperature due to each of those sources individually and add them up and you will get the source the temperature due to the two sources together, that works only when the sources are in the form of delta functions when you have point sources of infinitesimal volume.

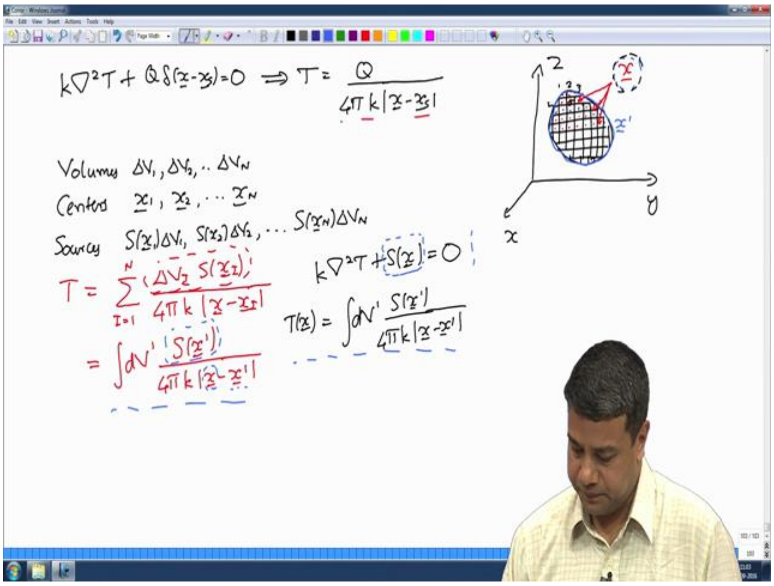
(Refer Slide Time: 03:40)



In general if you have sources of finite volume, that principle does not apply in gentle you have to keep the solid boundaries the same therefore, the source of the delta function type is a special source and that is the reason that it is useful for getting these equations.

So, I showed you that for two sources you can get a solution by a superposition.

(Refer Slide Time: 04:04)



Now, what if you had some volumetric source? You had some region in space where let us say some reaction was generating a certain amount of heat, that heat being generated per unit volume per unit time and considered as Q of x prime, they should be careful here because I have used capital Q for an energy per unit time, so rather than using that I will use a different notation for the energy per unit volume per unit time. So, I will consider this as some source S of x, that is the energy generated per unit volume per unit time and I want to find out what is the temperature at some observation point x, that observation point could be within the source or outside does not matter. So, how do I do that? I already have a solution for a point source, that solution is T is equal to Q by 4 pi k can be write this as source point.

So, there is the temperature field for this. So, what I can do is, I can take this object; I can take this object divided into a large number of small volumes, the volumes can be defined with centers at delta V1, delta V 2 etcetera delta V N divided into a large number of volumes a label them as 1, 2, 3, 4, 5 etcetera the volumes our delta V 1, delta V 2 etcetera up to delta V N, the centers are at locations x 1 vector, x 2, x N.

So, the sources within this volume I am going to be of the form S at x 1 delta V 1, S at x 2 delta V 2, S at x 3 and so on. So, those I can consider to be point sources at the centers of each of these volumes these I can consider to be point sources at the centers of each of these volumes. If the volumes are small enough the fact that you are combining the entire source at it is center does not affect the problem because the extent of the volume is much smaller than the distance from the observation point of each of these. The extent of the volume is much smaller than the distance from the observation point so this is effectively a point source.

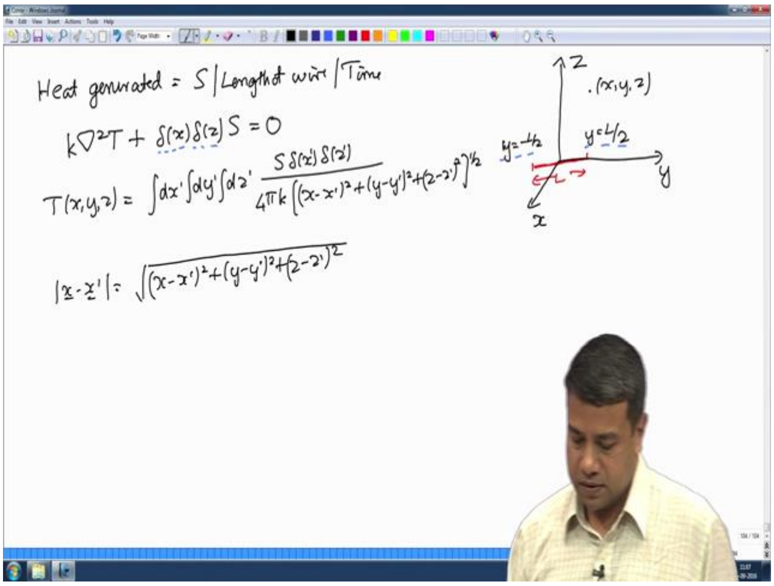
So, once I have defined it this way I can straightaway write down what is the total temperature field. I just have to add up the temperatures due to each one of these points sources. So, therefore, the total temperature field will be equal to summation over all i is equal to 1 to N ok of delta V i into S of x i, I am use a capital index here, times the source at the location x I divided by 4 pi k into x minus x I. So, I just added the temperature due to each one of these point sources.

The source strength in this case delta V times the source, that divided by 4 pi k times the distance between the source point and the observation point, that is my expression for the total temperature. In the limit as delta V goes to 0 I can equivalently write this as an integral over the volume; this is integral over the source volume of the source strength at that location by 4 pi k into x minus x prime, where x is the observation point, x here is the observation point x prime is the source point x prime is the location on the source and your integrating over x prime which is over the entire volume of the source.

So, this one here is the source point and this S is a function of the source point and the integral is over the source point and from that you get the value of the temperature at the observation point x. So, you can do that for any distributed source for any finite object, provided the source is specified you can get the temperature field at any observation point by using this expression. So, this effectively, recall that we started off trying to solve the temperature k del square T plus S of x is equal to 0, that was the differential equation for the temperature that we were trying to solve the poison equation with an in homogeneous term and using this delta function construction, we have managed to get a solution for that T is equal to integral d V prime, S of x prime by 4 pi k into x minus x 1; this is an integral relation this gives us on the left side this gives us the temperature at the observation point x the temperature at the observation point x, as a function of the source location and the source strength distributed source.

So, this is effectively the integral equation that corresponds to this differential equation, there is a solution of this differential equation and you can use this formulation to find out the temperature due to any field. So, let us briefly look at how we have do it for a simple problem, and that is the heat generation due to a wire.

(Refer Slide Time: 12:23)



So, if I have a wire of total length L, extending from Y equals minus L by 2, 2 y equals L by 2; this is generating some heat this is the finite wire, you can think of it something like an immersion wire for example, which is immersed in a fluid is generating a certain amount of heat and you would like to find out what is the temperature field around this due to the presence of this wire. The heat being generated is equal to S per unit length of wire per unit time that was the heat that is being generated. This wire of infinitesimal thickness, if it has 0 thicknesses in the x direction and 0 thicknesses in the z direction and it is a certain amount of heat generated per unit length.

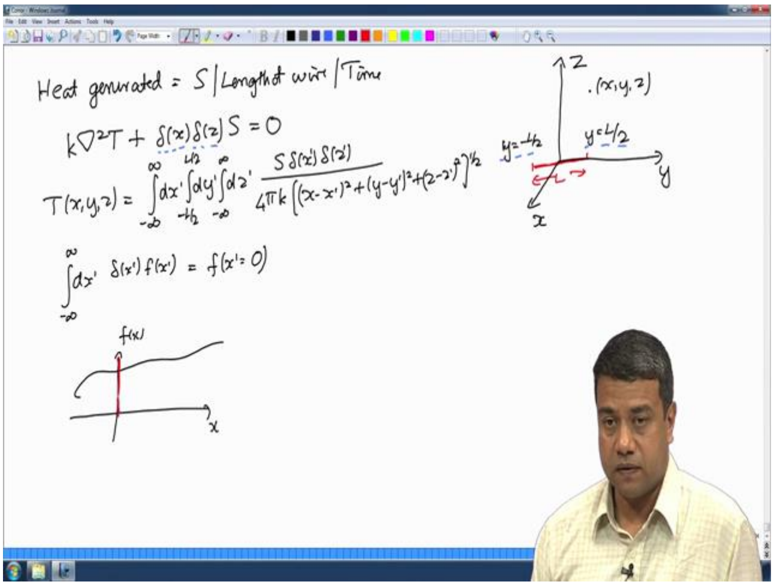
So, you have to solve the equation k del square T plus this as infinitesimal thickness in the x and z direction. So, this is delta of x, delta of z times S ok is equal to 0; where s is non zero only for y is equal to minus L by 2, 2 y equals plus l by 2 ok. So, S is sourced per unit length of wire, it is confined only to the y axis; that means, it is non zero only when x is equal to 0 and z is equal to 0 otherwise it is 0, it is non zero only when both x and z are non zero therefore I have delta of x, a function that is non zero only when x is 0 and delta was z that a function that is non zero only when z is equal to 0 and I want to find out the temperature at some observation point x, y, z.

So, therefore, I need the inverse of this equation which I had told you T of x, y, z is equal to integral over the source point that is integral d V prime that I had in the previous example. The source S times delta of x, delta of z primes here; these are the source locations, recall that in the previous class in the previous lecture, I have k del square T plus S of x is equal to 0, the solution for that is integral over d V prime of S of x prime divided by 4 pi k into x minus x prime ok divided by 4 pi k into what is x minus x prime?

This is going to be the distance between x and x prime, this will be square root of x minus x prime the whole square, plus y minus y prime the whole square, plus z minus z prime the whole square. So, that is the distance between the two points that is what is meant by the magnitude of the vector x minus x prime. So, this will be x minus x prime the whole square, whole square, who is taken the square root of. So, that is going to be the expression.

So, x goes from minus infinity to infinity y from minus L by 2 to L by 2 and z goes from minus infinity to infinity.

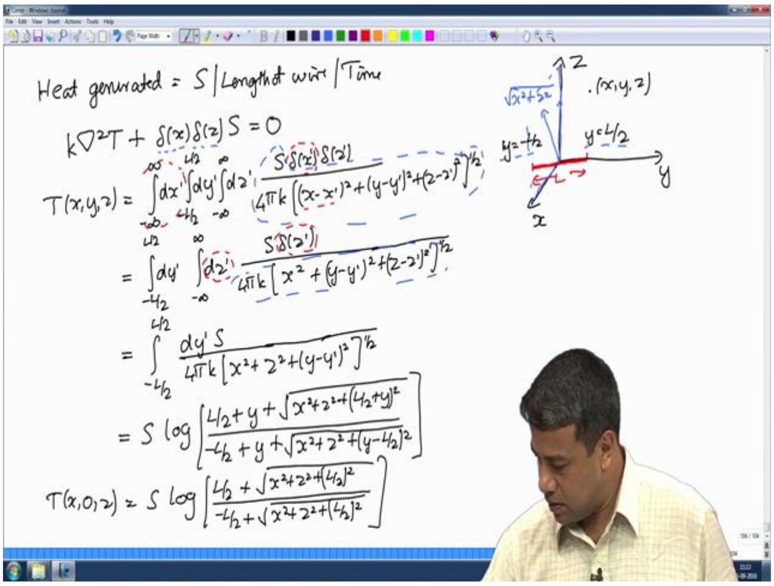
(Refer Slide Time: 17:00)



So that is the extent that is the temperature field. Now we know that integral minus infinity to infinity d x prime, times delta of x prime, times any function of x prime is equal to the value of that function at x prime equals 0; that was one of the properties of the delta function, it is 0 everywhere except where x is equal to 0 integral is 1 and if you multiplied this delta function by any function sum function f of x, and you integrated from minus f to infinity, the delta function is non zero only at x is equal to 0, so this will pick out the value of the function at x is equal to 0. So, that is one of the properties of the delta function.

So, in this case I have the integral over x of delta of x prime, integral over x prime of delta of x prime, times some function of x prime.

(Refer Slide Time: 18:17)



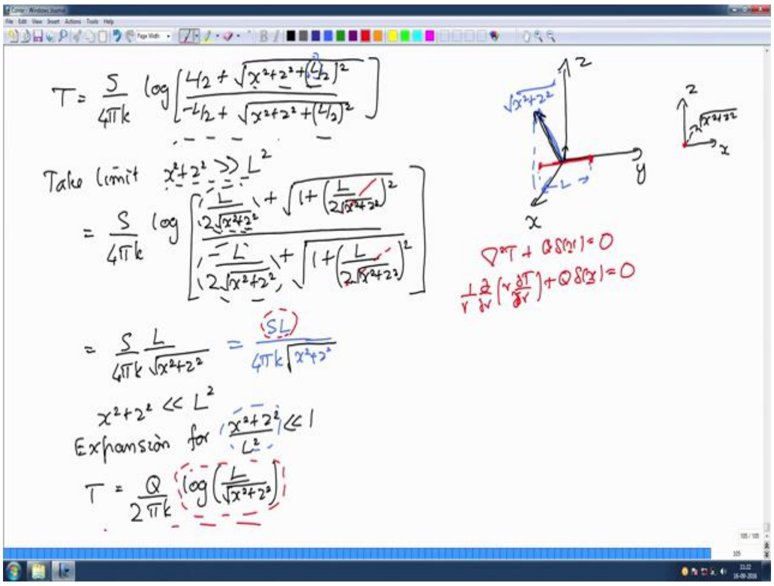
The result is going to be the value of that function at x prime is equal to 0; the value of that function at x prime is equal to 0 which will be equal to minus L by 2 to L by 2 d y prime, integral minus infinity to infinity, d z prime, S delta of z prime divided by 4 pi k times the value of the function at x prime is equal to 0 x square, the value of this function at x prime is equal to 0.

And then I have the second integral over z prime of delta of z prime times this function of z prime, once again it will pick out the value of the function at z is equal to 0.

So, this is the expression for the temperature, but I have to integrate over the y coordinate over the length of this wire the y coordinate, I have to integrate this over the length of this wire. You can do this integral I would not go into the details of for the integration is done, but in the end what you will get is S times log of L by 2 plus y plus root of x square plus x square, plus the whole square.

Now, this is a rather complicated expression what we can do for example, is to simplified by taking the value at the locations z is equal y is equal to 0 along the central plane, I will take the value of the temperature along the central plane at the location y is equal to 0, at the central plane where y is equal to 0 in that case if the temperature becomes T of x, 0, z is equal to S into log of L by 2 plus root of. So, this is the temperature field, we do not get very much physical insight from this temperature field, we can get physical inside; however, if we consider two limiting cases: the first is where the distance from the source point there is this distance is of course, square root of x square plus z square, that distance you could consider limiting cases where that is large compared to the length of the wire L and where that is small compared to the length of the wire L.

(Refer Slide Time: 22:46)



So, the temperature as I have written it for you is equal to 1 by 4 pi k. So, there is S by 4 pi k and may have missed that S over here, I should have a 4 pi k in both of these that comes out of the 4 pi k over here into log of L by 2 plus square root of whole square.

If we take the limit where the distance from the source x squared plus z square is much larger than L square the length of the wire, this distance is much larger than the length of the wire and I have to do a series expansion in the small parameter L by 2. I have to expand this entire term in the series expansion in L by 2. I will get x by 4 pi k, I divide both the numerator and the denominator by x square plus z square the square root of. So, I will get L by 2 root of x squared plus z square plus root of 1 plus whole square.

And now I can do a series expansion because this L is much smaller than root of x squared plus z square. So, these numbers are small parameters compared to 1, these numbers are small parameters compared to 1 and if you do that you can actually neglect this term because it is quadratic. I get 1 plus x by 1 minus x the result of that just turns out to be equal to S by 4 pi k into L by root of x square plus z square; that is what I would get by expanding out this log function in a binomial series.

Recall now along the plane that we had in the x z plane, along this x z plane at the wire was not the along the y axis, along this plane we had done an expansion where the distance was root of x squared plus z square, this was large compared to the extent of the wire L, that was the limit that we were considering. In that case we get a solution which is basically equal to the total heat generated per unit time; S was the heat generated per unit length per unit time of the wire. So, S times L is the heat generated per unit time divided by 4 pi k into this x square minus x plus z square, which is basically the distance from I should write this as square root my apologies, the square root of x squared plus z squared is basically the distance from the original that was the solution that we had got for the point source, Q by 4r pi k r. So, if the distance from the source point is much smaller larger than the extent, if this distance is much larger than this length the source point actually looks like a point source of source strength is tainted.

So, as the distance from the source becomes much larger than the spatial extent of the source, it basically looks like a point source, that is the idea here. On the special extent is very large, you get back the same result that you would have got for the point source. You can consider the opposite limit x squared plus z square is much smaller than L square that is the distance from the source is much smaller if I look this in the x z plane, the distance here x square plus z square, square root of it is much smaller than the length of the wire is they are doing only a small distance compared to the length of the wire.

In that case you have to take the opposite limit, in this expression x square plus z square is much smaller than L square and I need to do an expansion in the case where for x square plus z square by L square much smaller than 1.

I want go through the details you can work out the expansion, what you have to do is rather than dividing throughout by x square plus L square here; at x square z square here I have to divide throughout by this L square, I have to divide throughout by this and if I do that I will get the temperature is equal to Q by 4 pi k log of L by root of I am sorry Q by 2 pi k, L by root of x square plus z square therefore, the temperature field goes as the logarithm of a distance, what happens when the distance is small compared to L, the wire effectively looks like a wire of infinite length because the distance from the wire is much smaller than the length of the wire.

So, the wire effectively looks like a wire of infinite length the problem effectively reduces to a two dimensional problem, where in two dimensions the x and z dimensions you have a wire which is generating a certain amount of heat per unit length in the third direction. So, this is the equivalent of the delta function source the point source in two dimensions. The point source in two dimensions has a solution which is logarithmic in the radius.

You can solve explicitly the point source in two dimensions by taking the two dimensional version of the equation del square T, plus Q delta of x is equal to 0. In two dimensions if it is x is symmetric you get 1 over r, d by d r of r partial T by partial r plus Q delta of x is equal to 0 the solution set for this turns out to be a logarithmic function, you can solve that quite easily and verifies that you will get a solution that is of this type. So, the two limiting cases we get the correct solutions for this case of the wire and of course, this can be generalized in this case we took a two dimensional wires you can take a surface or you can take a volumetric object, for all of these you will get solutions just by inverting this equation, this may taking the inverse of this equation, the inverse of this equation the differential equation is basically the this integral equation and we saw that by going by expressing it first as a point source and then generalizing it to a volumetric source.

We will see a little bit more about this what do you do this was all in infinite domain, what do you do when there are boundaries; we look at that in the next lecture then come back to the original point that I had said, what do those spherical harmonic expansions mean that we will continue in the next lecture. I will see you then.