Transport Processes I: Heat and Mass Transfer
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Lecture – 57
Diffusion equation: Conduction from a point source

Welcome to this our continuing series of lectures on fundamentals of transport processes, where we were trying to solve the diffusion equation in the limit of low Peclet number where the convection of mass or energy is much smaller than the diffusion the limit of low Peclet number.

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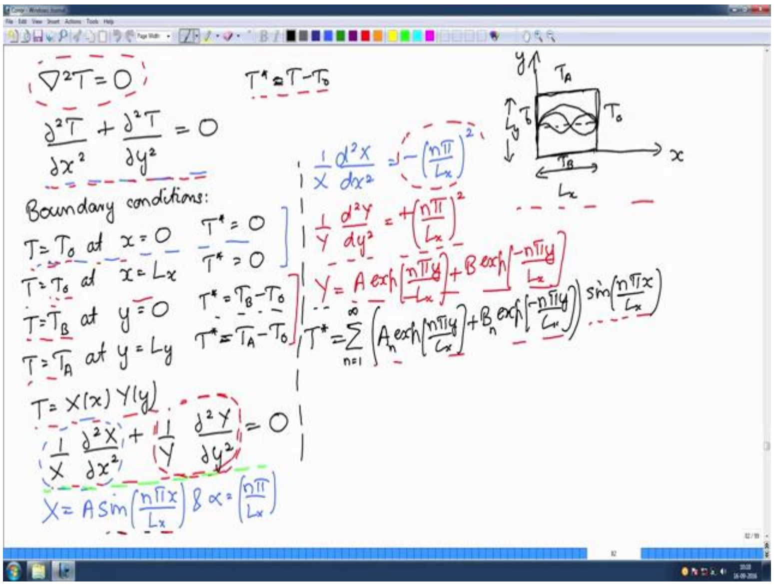

In that case the conservation equations provided the thermal conductivity or mass diffusivity are constants, the conservation equations all reduced to the Laplacian of the temperature field or the concentration field is equal to 0, in a domain where there are no sources or sinks and we were looking at ways to solve these problems. In all of these cases the solution is by a separation of variables, you have to identify homogeneous directions in the problem, in the homogeneous directions you have a natural basis set of functions in which any function can be expressed as in this Cartesian coordinate system for example, that set of basis functions was the sin functions for 0 temperature boundary conditions at the boundaries, if you had 0 flux it would have been cos functions because the cos of x for example, has 0 slope at x is equal to 0 and x is equal to l.

After expanding these we used orthogonality relations to determine the coefficients in the inhomogeneous direction and thereby completed the solution. This procedure is exactly the same regardless of whether you are using it for a transient problem or for a steady state problem in multiple dimensions.

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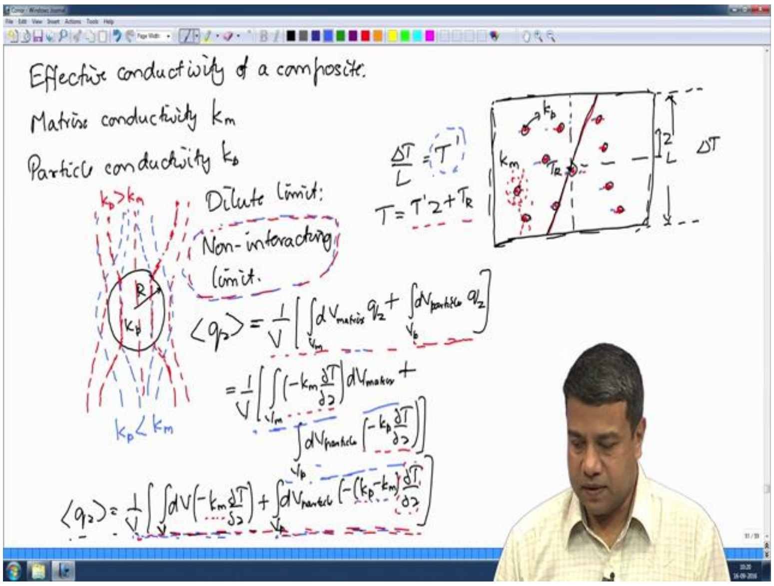

We had also solved the problem in a spherical coordinate system; if you recall we had derived the expression for del square in a spherical coordinate system and on that basis we have we had used it in order to find out solutions for the temperature field, for a sphere in a linear temperature gradient far from the surface of this sphere, if you have an inclusion of a different thermal conductivity within a material and the temperature gradient is constant far from the sphere, what is the distortion of the temperature field close to the sphere due to the presence of this material of a different thermal conductivity?

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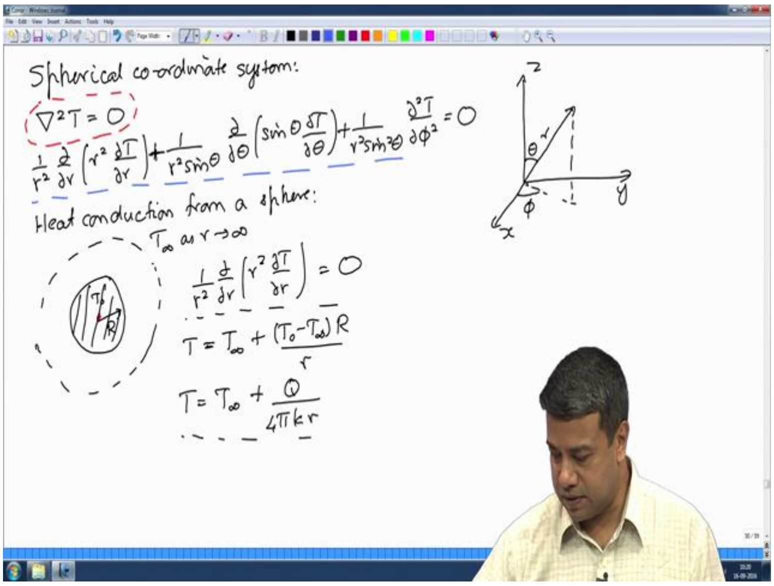

That is the problem that you had solved in the previous lecture in order to find out what is the effective thermal conductivity of a sphere.

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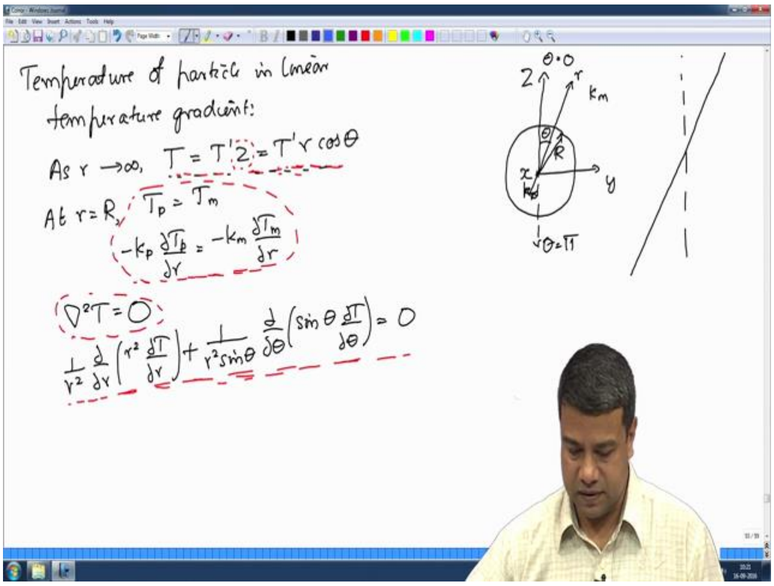

In that case the equation reduces the axis symmetric equation, del square T equals 0 and we had solved that in order to find out what is the effective thermal conductivity of this sphere.

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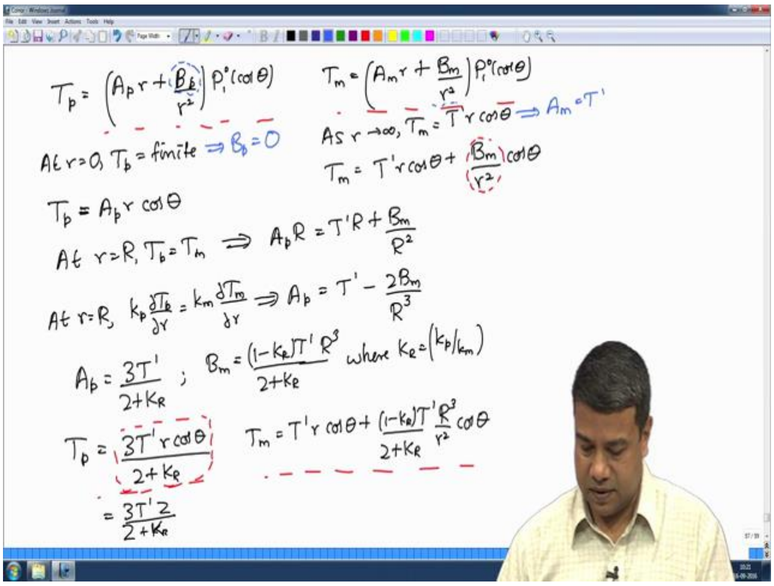

We had one is to get an expression in the dilute limit where the temperature disturbance due to one sphere is does not affect the temperature field around another sphere. So, the particles are sufficiently spaced such that the temperature disturbance around one sphere does not affect the temperature field around another sphere.

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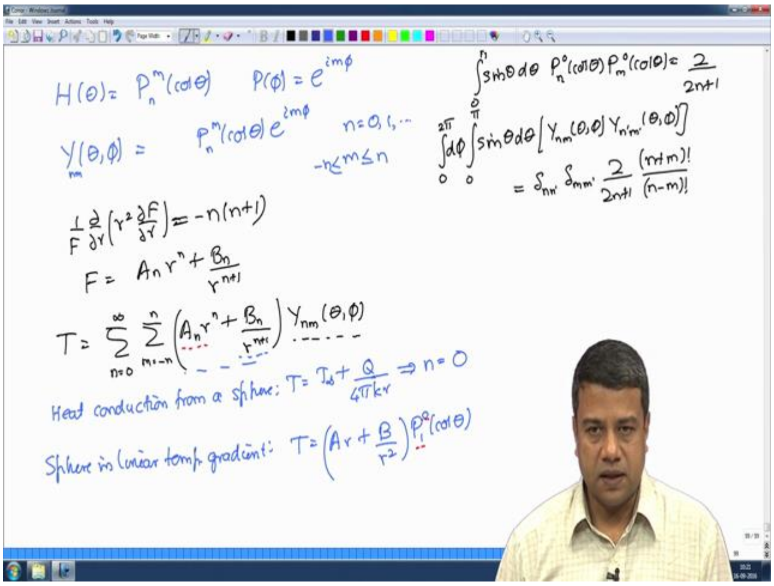

In the last lecture I had derived a rather I just shown you how to derive the separation of variables in a spherical coordinate system consisting of the r theta and phi coordinates. This was the expression that we had got in the end, An times r power plus n, plus B n by r power n plus 1 into Y nm of theta and phi, where Y nm are the spherical harmonics which basically consists of a legendre polynomial in the theta direction and cos and sin functions in the phi direction. I showed you that for n is equal to 0, the solution corresponds to sphere in a temperature field which is generating some particular heat per unit time, this was the solution for n is equal to 0 heat conduction from a heated sphere; n is equal to 1 the same solution that we got for the a sphere in a linear temperature gradient.

So, in this lecture I will try to give you some physical understanding of what these functions mean. You can see that for n is equal to 0 you have a constant term, plus a term that decays as 1 over r; for n is equal to 1 there is one term that increases proportional to r another term it decreases proportional to 1 over r square.

Similarly, for n is equal to 2 n and so on. For n is equal to 0, this coefficient m it is just equal to 0 so there is just 1. For n is equal to 1 m is minus 1, 0, 1 and plus 1 there are 3 solutions. For n is equal to 2, m can have values from minus 2, minus 1, 0, plus 1 and plus 2, there are 5 solutions what do these solutions physically mean? I will try to give you some understanding of what is the physical meaning of these solutions, but first we will look at the solution of the diffusion equation in another way and that is to consider the temperature field due to a point particle.

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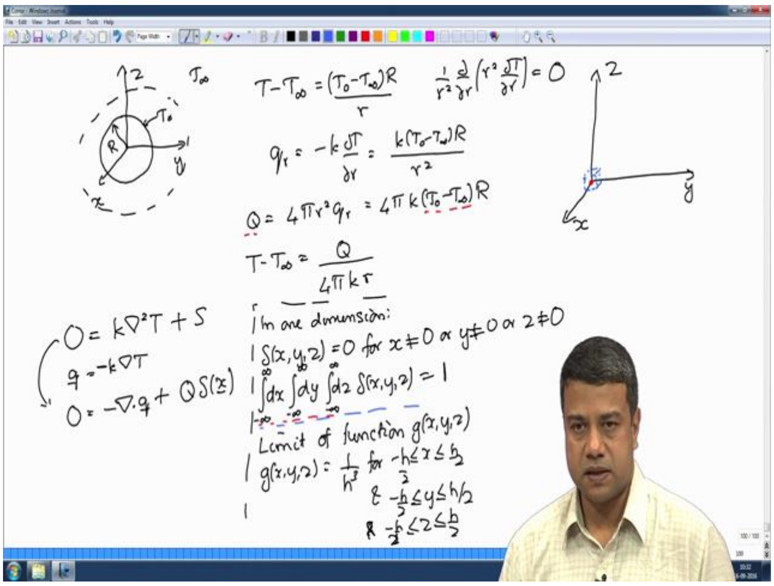

If you recall we had solved this temperature field around a spherical particle with temperature T naught at the surface, and T infinity far away and the temperature disturbance that I had got, I would got it as T minus T infinity, the disturbance to the temperature field due to this presence of the surface with temperature T naught was equal to T naught minus T infinity times R by r, where r was capital R was the radius of this sphere; this satisfies the differential equation the spherically symmetric differential equation because in this case there is no variation in theta and phi. So, this satisfies this spherically symmetric conservation equation, well del square T is equal to 0 and a spherical coordinate system, when there is no variation in the theta and the phi directions.

I also showed you that if you express this in terms of the total heat generator from this you can get the heat flux q r is equal to minus k partial T by partial r, which is minus k into T naught minus T infinity into R by r square with a positive sign here, when you take the derivative you get a negative sign because you have taken the derivative of 1 over r. The total heat generated is at any surface at any radius R, the total heat that is coming out per unit time at any surface it is going to be equal to 4 pi r square times q r, which is equal to 4 pi k into T naught minus T infinity into R. So, if I use this to express this T 0 minus T infinity in terms of q, what I get is that this temperature disturbance is equal to q by 4 pi k into r.

So, that is the temperature disturbance due to a spherical inclusion, which is generating heat capital Q per unit time. This solution is independent of the radius of this sphere r therefore, it should be valid even when the radius of this sphere goes to 0, when you have a point which is generating heat of 0 radius the radius goes to 0, but the amount of heat being generated per unit time is still q. So, I taking the limit where the radius goes to 0; as the radius goes to 0 if the total amount of heat being generated they are still q; that means, that the flux goes to infinity. So, it is in that limit that this is a solution.

Now, this solution can also be considered as a solution of the inhomogeneous equation in which you have a point source as the delta function. I will just show you briefly how that can be formulated. First of all if we have sources and sinks, the equation becomes k del square T plus the source is equal to 0, that is the equation when there is a source of heat. Alternatively since grad q I am sorry, k grad T is the heat flux, you know that q the heat flux is equal to minus k times grad T, this comes out of heats Fourier’s law of conduction the Fourier's law for heat conduction, the heat flux is equal to minus k times the gradient of the temperature.

So, I can also write this equation as 0 is equal to minus the divergence of cube plus source. So, I just read it in the heat conduction equation. Now if I were to assume that this source is equal to Q times delta of x vector, this delta function we have seen before in one dimension. Delta of x is equal to it should in one dimension delta x 0 for x is not equal to 0 and integral minus infinity to infinity d x delta x is equal to 1. So, if you recall we had idealized this as a function x delta of x, if you take a function which is nonzero only between minus h by 2 and h by 2 and the height is 1 over h and then you take the limit and you call this sum g of x, if you take the limit of h going to 0 of g of x is equal to this delta function and since you have this integral condition, this delta function has dimensions of one over length.

We have seen this delta function in one dimension before; we have also seen the extension to 3 dimensions. If you have a three dimensional space this can be written as delta of x y z is equal to 0 for x not equal to 0 or y not equal to 0 or z not equal to 0. So, it is nonzero only when all three x y and z are all 0; it is nonzero only when all three x y and z are all 0 and the equivalent of this normalization condition is that integral minus infinity to infinity d y delta of x y z is equal to 1. So, it is nonzero only when x y and z are 0 and it is integral is equal to 1. So, as I told you it is you can take this as a limit of a function g of x y z; g of x y z is equal to 1 over h cube for minus h by 2 less than x, less than h by 2 and minus h by 2 less than x I am sorry y and minus h by 2 less than or equal to z, less than or equal to h by 2.

So, when we only in a cube of side h at the origin, this is nonzero and this as a value of 1 over h cubed over there, in such a way that you can see this integral will be equal to 1. So, that is the idealization of the delta function in three dimensions and rather than writing it as delta of x y and z, I would much rather just write it as delta of x vector; where x vector is the vector location x e x, plus y e y, plus z e z.

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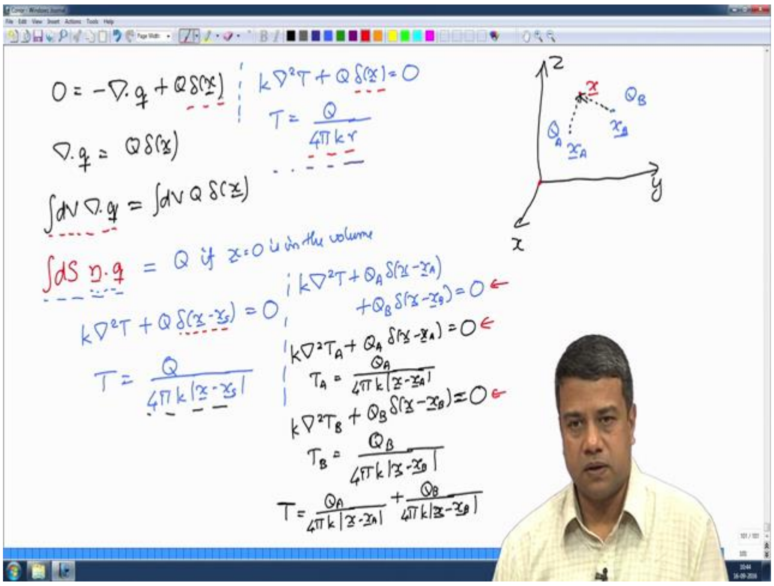

So if we have the heat conduction equation 0 is equal to minus del dot q, plus Q delta of x.

So, this is nonzero only within this infinite as a volume at the origin. I can write this as the divergence of q is equal to Q times delta of x and if I take the integral of this over any volume if I take the integral of this that the left side and the right side over any volume that volume may include the origin, it may not include the origin; if I take this over any volume you know that integral over the volume of del dot q is equal to Q integral over the volume of Q times delta of x.

Now, I will go to a slightly more advanced concept here that is something that you must knowing, I will just write it for you anyway. This integral by the divergence theorem it can be written as integral over the surface of that volume, integral over the surface of that volume of the unit normal at each location the outward perpendicular, of the outward perpendicular times q. The integral over the surface of the outward perpendicular times q, this from the right side integral over a volume of the delta function is equal to Q if x is equal to 0 is in the volume; if this location is within the differential volume then this integral of delta of x times d x has to be 1, that was the property of the delta function.

You integrate the delta function over the volume if that volume contains the location of the delta function, that integral is 1 if it does not into 0. So, over this volume the integral is just q. So, what; that means, is that the total heat that is coming out of this surface; the total heat that is coming out of this surface is going to be equal to the heat flux times the surface area.

The heat flux in the direction perpendicular to the surface is the heat flux dotted with the unit normal. So, the total heat coming out of the surface is just equal to q, and that is what precisely we started off with we said that the total heat that is generated within this volume per unit time is equal to q and that has to be the total heat flux that is coming out of the surface. So, therefore, what I have just shown you is that the solution of this equation written in terms of the flux this way, alternatively if I write in terms of the temperature field I will get k del square T plus Q delta of x is equal to 0; this equation the solution that I have got for the temperature field, where Q is the amount of heat generated per unit time note that in three dimensions, the delta function has dimensions of 1 over volume; it is 1 over h cubed where h is a distance, so there is dimensions of 1 over volume.

The solution for this is T is equal to Q by 4 pi k r; where r is the distance from the origin, so that is the temperature field due to a point source. Now how can we use this? So, this solution was the temperature field due to a point source at the origin it is called the greens function solution, but I will just look at look at it as the solution due to a point source. If the source were located at some other location x s, and x s is the vector to the source point x s is the vector if the source were located at this location x s and I wanted to know what is the temperature at some other location x, that temperature depends only upon this distance, if the distance r is the distance from the source point to the observation point, so therefore, the temperature depends only upon this distance. So, for this look problem where you have a source at some location x s; k del square T plus Q delta of x minus x s is equal to 0, delta of x minus x s means that this delta function is nonzero only at the location x is equal to x s, there is the location of the source, it is 0 everywhere else and the integral of this is equal to 1.

So, this is the heat conduction equation if the source were located at the location x s rather than at the location the origin if x is equal to 0. So, we have take the source point at the location x s rather than location 0, then this would be the heat conduction equation and for that the temperature field is going to be exactly the same that I have derived here, except that r it is now the distance between the source point and the observation point. In the previous case the source point was at the origin, so the r was just the distance from the origin. In this case the source point is not at the origin it is at some other location x s therefore, the temperature solution is going to be of the form T is equal to Q by 4 pi k into the distance x minus x s. So, that is going to be the temperature field.

If I had two sources if I have two sources in this problem, let us say that there were two sources that were located at two locations x A and x B and I wanted to find out what is the temperature at some location x; then I have to solve the problem k del square T plus Q A delta of x minus x A. Let us say that the heat coming out of this was Q A, the heat coming out of this was Q B; plus Q B delta of x minus x B is equal to 0 that is the problem that I would have to solve. In order to find out the temperature field at some location x some observation point x due to these two sources of heat.

Now, how do I solve that? I could write this as the sum of two problems: one is due to the source at the location of x A the other is due to the source at the location x B. So, the first problem I will solve is k del square T A, plus Q A delta of x minus x A is equal to 0. The second that is with only one source present at the location A, the other is to solve k del square T B plus Q B delta of x minus x B is equal to 0. You can see that if I add up these two equations, I get back the original equation if I add up these two equations I will get back the original equation therefore, if I add up these two solutions for T A and T B; I will get back the original solution.

So, solution for T A is equal to 1 by 4 pi k into x minus x A the distance of the location x from this source x A. The solution for T B is equal to one I am sorry this is equal to Q A divided by just at the solution for T was equal to Q by 4 pi k times x minus x s here, same thing I have two sources: the first one I taken the distance from the first source, the second one I take Q B by 4 pi k into x minus x B.

Where x minus x B is now this distance of the observation point from that second source point and therefore, I can get the total temperature as just the sum of these and I can do the same regardless of how many source points I have, independent of the number of source points that exists, I can get the temperature is the observation point just by adding up all of those source points. So, it sort of seems too good to be true that you can just do it this way in terms of you can, but you can do it only if the source is in the form of a delta function; you can do it only in if the source is in the form of a delta function, it is not applicable generally for any source and the reason for that is as follows.

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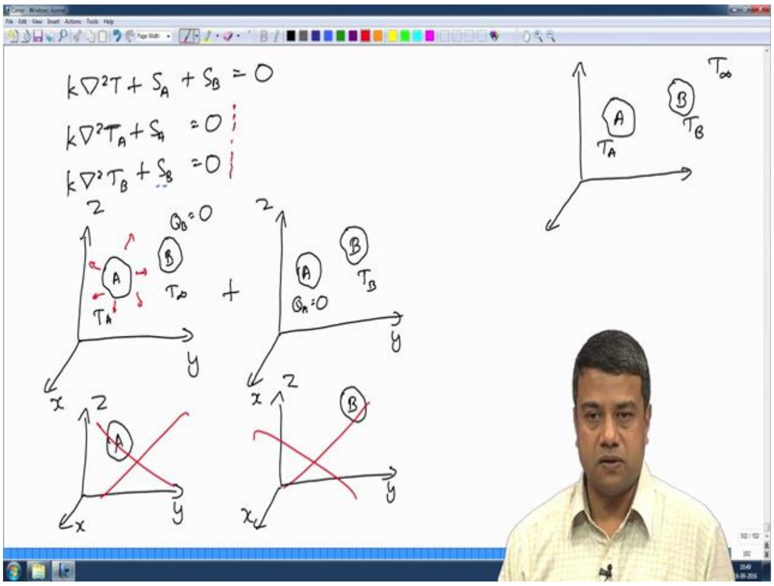

If I had two objects that were a finite extent with some temperature, T A and T B on the surface, then I can write the temperature equation as k del square T, plus the source due to object A, plus the source due to object B is equal to 0. I cannot separate it out into two parts, k del square T A plus S A is equal to 0 and k del square T B plus S B is equal to 0.

So, when I do this first thing, I have to enforce that the temperature on the surface is the second object is T infinity; that the second object this is at the same temperature as these surroundings and when I solve this problem I have to assume that the temperature on the first object. So, I can separate this out into two problems: one is where I have object A with temperature T A, and object B with temperature T infinity plus the second one.

So, the second object the net flux coming out of the second object has to be equal to 0; the next source coming out to the second object has to be equal to 0 when I do this decomposition and the second problem, but I have object A and object B; now object b is at temperature T B, but you are required that the net flux coming out Q A has to equal to 0 whereas, in the first problem I required that Q B has to be equal to 0. So, this is a valid decomposition I can add up two cases: the first one where the object A exerts the flux the object B does not.

In the second where the object B generates a certain heat and the object A does not; what I cannot do is to separate this out into two problems: the first one where A is present and B is not and the second one where B is present and A is not present. This decomposition is wrong you cannot do it this way; the reason is because when I do this decomposition, the physical boundaries all have to be the same; when I do this decomposition the physical boundaries all have to be the same. So, I cannot do the decomposition in the second way, just remove one object and keep the other one present this is not possible. How I can do it this way where I consider only one object to be generating the other object not generating and the complementary part.

Now, when I do it this way when one of these objects is generating energy, there is still going to be a distortion to the temperature field due to the presence of this other object because it may not have the same conductivity as the matrix in general and vice versa therefore, you still have to solve the couple problem, just as we have done in the heat flux case the average conductivity of a matrix, you still have to have the problem where two physical boundaries are present and you are solving when one is generating and then where the second is generating and that it says it is a difficult solution.

The reason it works in delta functions is because delta functions as the 0 volume; the delta function as a heat source which is generated at the point in three dimensional space therefore, it has 0 volume. So, where that volume is there or not it makes no difference therefore, this superposition principle will work only when the sources in the form of a delta function and that is why the solution for a delta function source as a special place in our description of the heat conduction.

The equation that we have here is what is called the Poisson equation, the Laplace equation with and inhomogeneous source term is what is called a Poisson equation, in this particular case it is a Poisson equation with a delta function source; we look at a little bit in further detail how we can get solutions for this for a more general case, where there are multiple sources and multiple sinks as well as further the physical boundaries.

So, we will continue this in the next lecture, I will look a little further at these delta functions and recall our original objective was to get a physical understanding of the legendre polynomial expansions. So, after going through this in some detail I will come back and show you what those expansions actually mean we will continue this in the next lecture, I will see you then.