Transport processes I: Heat and Transfer
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Lecture – 54
Diffusion equation: Heat conduction around a spherical inclusion

So, welcome to our continuing discussion on diffusion dominated transport. As I had told you in the previous lecture, diffusion dominated transport refers to the case where the Peclet number is small or the Reynolds number is small so that effectively you are solving for a Laplace equation or a Poisson equation for the temperature field or for the concentration field.

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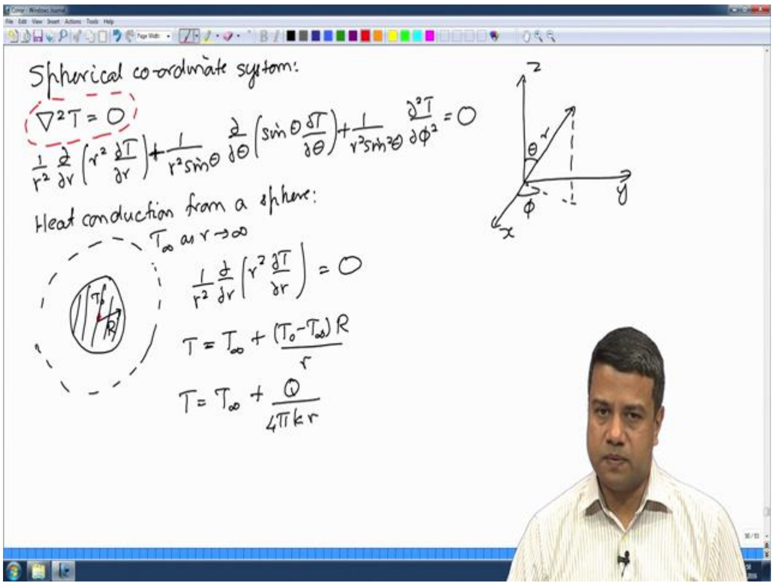

We had solved for the temperature field around a rectangular object in the previous couple of lectures and last class we were looking at a spherical coordinate system. The heat conduction from this sphere is a simplest such problem where the temperature just decreases as 1 over r and last time we were looking at a slightly more complicated problem and that was the effective conductivity of a composite.

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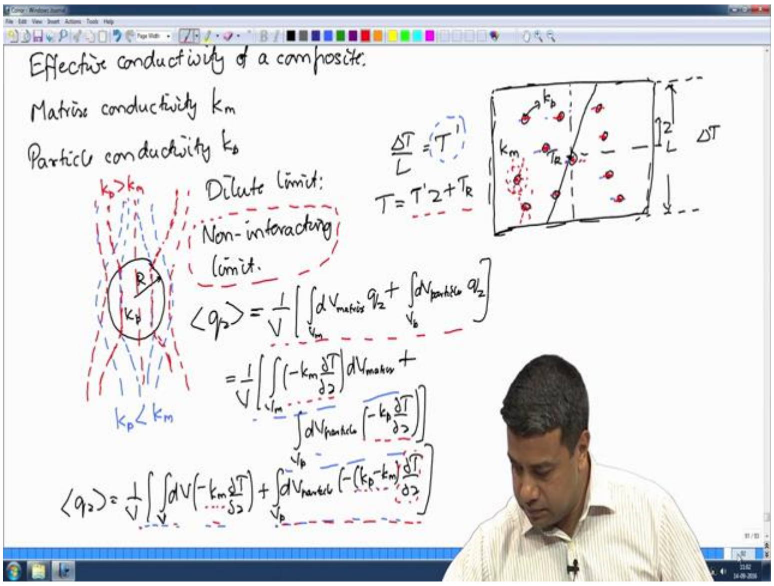

In this case you have spherical inclusions within a matrix, composites are often used because their properties are different from; and better than just single-phase materials; a small amount of added material can result in a significant change in properties such as the strength, the electrical and the magnetic properties and so on. In this case we were looking at the thermal properties, so we wanted to find out, if the matrix has a certain conductivity k m and within this matrix, you included particles with a different conductivity k p. How does the effective thermal conductivity of the entire system change due to the inclusion of these particles? You of course, expect that if the matrix has a lower conductivity than the particles, then the particles will conduct faster therefore, the effective conductivity would increase whereas if it is less, it will decrease.

The presence of the particles in the matrix results in a disturbance to the temperature field around the particles. As I said, as I showed you in the previous class in the red if the particle conductor is greater than the matrix conductivity, the flux lines would prefer to go through the particle rather than through the matrix and therefore, they would be stock inwards around the particle. On the other hand if the matrix conductivity is greater than the particle conductivity, the flux lines will go around the particle; this is going to result in a change in the effective conductivity of the whole system that if the particles plus the matrix and we were considering that in the non-interacting regime, the dilute regime where the particles are spaced sufficiently far apart that the temperature field around one particle does not affect the temperature around the surrounding particles.

So, as far as each particle is concerned that particle is embedded in a temperature field which has a linear temperature gradient far from the particle surface. So, the average temperature, so if I have an effective material; the average temperature at each particle location will of course, be different because there is a temperature gradient, but; however, the gradient that the particle sees far from its surface is going to be exactly the same for each one of those many particles.

The average flux, I had written as the volume average integral over the matrix of the flux through the matrix, the flux is of course we have only worried about the flux in the z direction; that is the flux along the direction of the temperature gradient and within the particles and within the matrix, the Fourier law of heat conduction is obeyed, but in each case the thermal conductivity is different. If I have different conductivities in the matrix and in the particles and I have written this as a summation and integral over the entire volume of the matrix conductivity times d T by d z plus integral over the particles alone.

In the second case, I have taken the difference between the optical and the matrix conductivity. You can see that the second term here is just the difference in the flux due to the fact of the conductivity is different, the disturbance to the flux due to the difference in the conductivity. So, written it as an average conductivity of the entire system, I had taken the matrix conductivity and taken that average over the entire system and over the particles alone, I take the difference in conductivity that of course, gives me back the same expression that I had earlier and I had simplified this. The first term is just the matrix conductivity times the integral of the temperature gradient over the particle.

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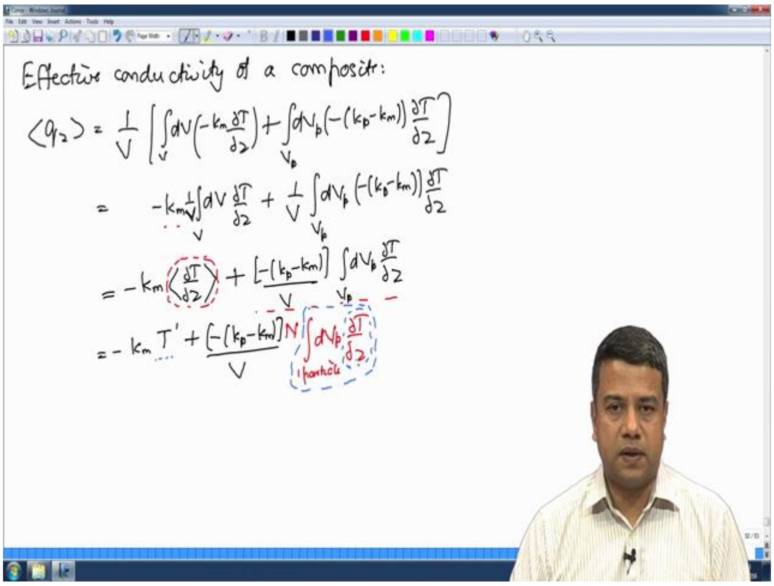

So there is an average volume average temperature gradient; that volume average temperature gradient has to be the same as what is an imposed temperature gradient. If you recall in the previous slide, I had said that T prime is the imposed temperature gradient therefore, if I take the volume average temperature gradient; I should get the same as the enforced temperature gradient. The second term is an integral over the particles alone of the difference in conductivity times the temperature gradient within the particle.

Since the particles are non-interacting, the temperature field around one particle is not affected by the temperature around another particle. The temperature gradients within each particle will be exactly the same, the baseline temperatures at the center will of course, be different depending upon whether the particle is located closer to the hot surface or the cold surface. However the temperature gradient that each particle sees is exactly the same.

Therefore apart from an additive reference temperature, the temperature field is exactly the same except for an additive reference temperature at the center of the particle location. That reference temperature does not change the temperature gradient; so therefore, the temperature gradient will in each article has to be exactly the same.

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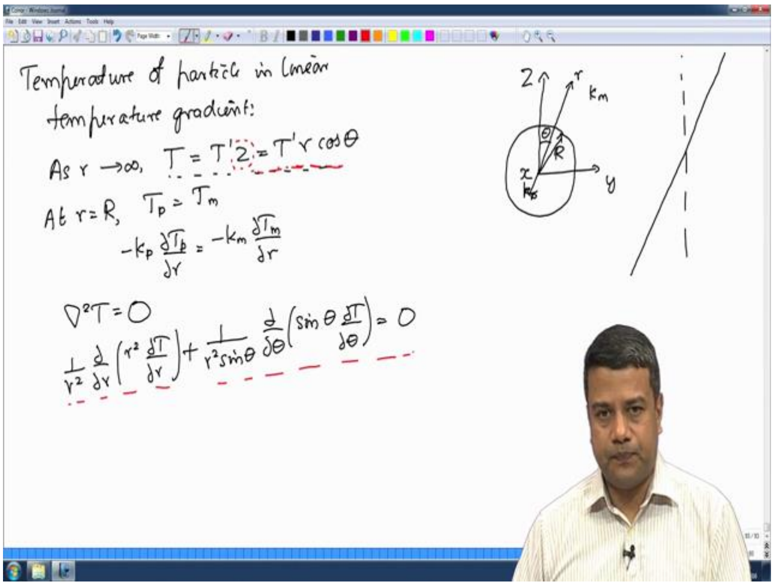

And in the last lecture I had forced this problem of the temperature field around one particle. I had said that we have a spherical particle in a temperature field in which the temperature is linear function offset. It is preferable to use a spherical coordinate system here because we have a spherical surface for the boundary between the particle and the matrix and if you use a spherical coordinate system, the spherical surface becomes a surface of constant coordinate. There is a complication of course, that this coordinate z now becomes a function of r and theta, but we will see how to deal with that as we go through this lecture.

So, the condition as r goes to infinity is that the temperature is equal to is a linear function of z; T prime time z where T prime is the imposed temperature gradient. This can effectively be written as T prime times r times cos theta because there is r cos theta in this case. At the particle surface, at r is equal to capital R; T P is equal to T m the temperatures are equal the fluxes are equal and we have to solve the Laplace equation for the temperature field. In this particular case, the temperature does depend upon r and it does depend upon theta because the boundary condition says that temperature is equal to r cos theta it does not; however, depend upon phi. So, therefore, it becomes a two dimensional problem and this is the conservation equation.

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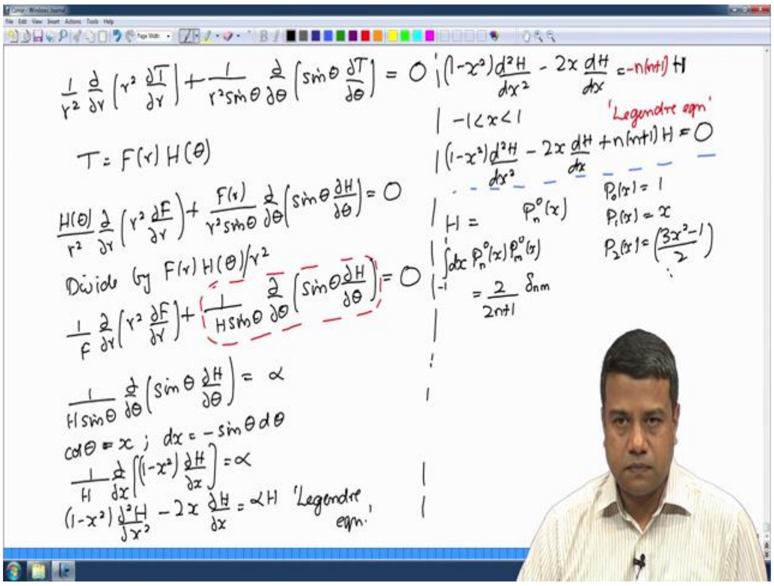

So how do we solve this conservation equation, separation of variables write down T equal to sum function of r times sum function of theta, substitute into the equation divide throughout by r and theta. Now the equations second term on the left still depends upon r and theta, but that situation is easily remedied you could multiply the entire equation by r square and now I have a situation, so I can multiply throughout by r square. So, I have divide by 1 over r square and now I have a situation where the first term on the left; is only a function of r, the second term on the left is only a function of theta. Therefore, both of these have to be constants otherwise one term will change r and keep theta constant; one change the other does not change, so this is the separation of variables here.

Now, let us take the second term first; is equal to some constant, what should that constant be; we do not know that yet, but let me just call it some constant alpha. Now the way to simplify this equation is to write cos theta is equal to some coordinate x which means that d x is equal to minus sin theta; d theta. If I substitute this, if I change variables; I will get 1 over H d by d x of sin square theta; d H by d x is equal to alpha and this sin square theta I can write it in terms of x as square root of 1 minus x square sin square theta is 1 minus cos square theta. So, sin square theta I can write it plus 1 minus x square therefore, I have an equation which is of the form 1 minus x square; d square H by d x square minus 2 x d H by d x is equal to alpha into H; this equation is called the Legendre equation.

So, the equation is of the form 1 minus x square; t square H by d x square minus 2 x; t H by d x is equal to some constant times H. We had got similar equations in a Cartesian and the cylindrical coordinate system if you recall. In the Cartesian coordinate system the equation was of the form d square H by d x square is equal to alpha times H and that satisfied the boundary conditions when alpha is equal to minus of n pi by L x the whole square, so that the solution reduced to 0 at both boundaries, so those are the Eigen functions.

This equation is more complicated; in this particular case theta goes from 0 to pi. If you recall in a spherical coordinate system, theta goes from 0 at z is equal to 0, this is theta equals 0 and along the minus z axis; theta is equal to pi which means that this function x goes from minus 1, less than x, less than 1. At theta is equal to 0, x is 1, at theta is equal to pi x is equal to minus 1. In this particular case, there are no physical boundaries there are; however, coordinate boundaries; the coordinate boundaries are at x is equal to plus and minus 1. If you recall in the case of a Cartesian coordinate system, there were physical boundaries; on those physical boundaries we required that the temperature has to go to 0 and that gave us a very specific form for these Eigen functions, they had to be negative and then to be equal to n pi by l the whole square.

In this particular case also, the requirement that the solution for H has to be finite at theta equal to 0 and theta is equal to pi imposes restrictions on what exactly is the form for this constant alpha. I will not be able to go through the details of how that is derived in this course; however, the procedure is very similar to the procedure for Cartesian coordinate systems. The requirement that you have homogenous boundary conditions that the function has to be 0 at both boundaries imposed what is the value of this.

You had only a discrete set of values n pi by L x the whole square, where n was an integer. In this case as well, the regularity condition that the temperature has to be finite at theta is equal to 0 and theta is equal to pi, imposes restrictions on the value of this function alpha. It turns out that this function alpha has to be minus n into n plus 1 where n is an integer; value of alpha has to be minus n into n plus 1, where n is an integer value, for 0 it is of course, 0 but then n can be 1, 2, 3 etcetera any integer value. If n is an integer value then the solution for this equation is finite at both theta is equal to 0 and theta is equal to pi; with that the equation becomes 1 minus x square this is what is called the Legendre equation and the solutions for these are what are called Legendre polynomials P n 0 of x; these are called Legendre polynomials.

The most general solution is just a summation or all of these solutions where P n 0 of x is a solution of this equation for each individual value of n; I should put an plus 1 here; I am sorry each individual value of n and you can get these solutions by putting n is equal to 0, n is equal to 1 so on and solving this equation P 0 of x is just equal to 1; P 1 of x c equal to x P 2 of x equal to 3 x square minus 1 by 2 and so on. So, you can get the solution, substitute the value of n put it 0 get one solution.

You can see when the n is equal to 0, P 0 of x is equal to 1 identically satisfies equation for n is equal to 1 of course, you have to solve this and you will find that the solution is x n is equal to 2, the solution is x square and so on. More importantly, these Legendre polynomial solutions also have orthognality relations and that orthognality relation is that integral from minus 1 to 1; d x, P n 0; x, P m 0 x this will be equal to 0 if n is not equal to n and this will be equal to 2 by 2 n plus 1 if n is equal to m, it is going to be equal to 0 if n is not equal to m n is equal to 2 by 2 n plus 1, if n is equal to n. So, therefore, I can write this as 2 by 2 n plus 1 delta n m.

I will come back and try to give you physical understanding of what these mean, a little later. So, from this (Refer Time: 20:14) polynomial solution where have you got so far.

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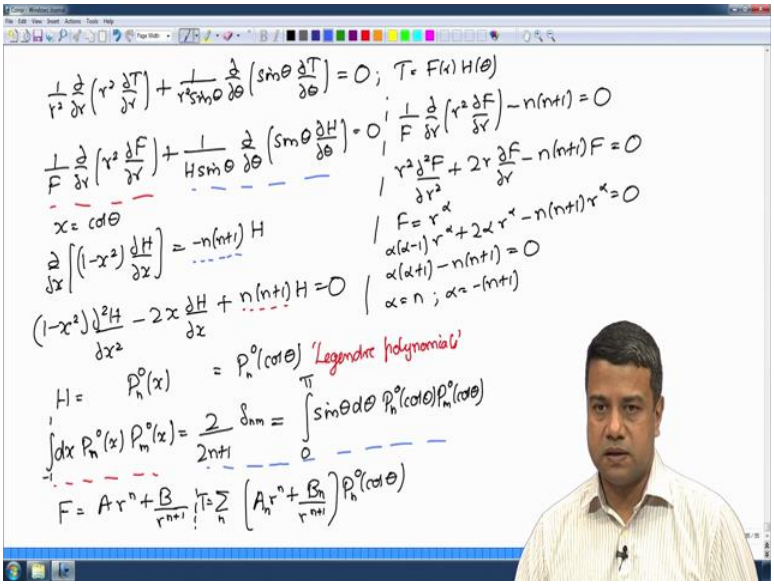

To recall: the equation that I had was 1 by r square t by d r of r square; d T by d r plus 1 by r square sin theta is equal to 0. I had substituted t is equal to F of r; H of theta that is the substitution that I had made divided by throughout H times r and what I got was 1 over F; d by d r of r square, d T by d r plus 1 over sin theta and you put this and I had said that the first term depends only upon r, the second term depends only upon theta. So, both of these have to be equal to constants, taking the second term first; I had substituted x is equal to cos theta, the equation becomes of the form is equal to some constant alpha times H this is in the form of a Legendre equation and I said that this has solutions only if this constant, has a specific value and that is minus n into n plus 1 and if I do that the equation becomes 1 minus x square; partial square H by partial x square minus equal to 0 and I told you that this one was equal solution of this verse H is equal to P n 0 of x. So, it will be equal to P n 0 of cos theta. For any value of n, note that within this equation n has to be an integer otherwise the solutions are not finite at theta is equal to 0 and theta equals pi.

So that is a basic requirement; n has to be an integer and these satisfy the orthogonality relations. Integral d x from minus 1 to 1; P n 0 of x; P m is equal to 2 by 2 n plus 1 delta n, these are what are called the Legendre polynomials and if I have to write this expression back in terms of theta; d x will be equal to sin theta d theta, it is actually minus sin theta d theta, but then the limits of integration then become pi to 0 because at minus 1; it is pi and 0 to 0. So if I take the limit as 0 to pi I get sin theta d theta P n 0 of cos theta.

So, that is the orthogonality relation that I have for this particular direction. This direction, the theta direction does not have any physical boundaries, but; however, there are coordinate boundaries at theta is equal to 0 and theta is equal to pi and then those coordinate boundaries; I require that the solution should be regular, the solution should be finite; those are the homogenous boundary conditions and those homogenous boundary conditions impose a restriction on the form of this constant; the Eigen value, but it has to be equal to n into n plus 1; where n is an integer.

So, now the other part of that you solved for the theta part we are now solve for the r part. I have for the radial coordinate; I have 1 by F d by d r, so r square partial F by partial r. This whole thing was equal to minus n into n plus 1, so I get minus n into n plus 1 is equal to 0 or alternatively I will get, if I multiplied throughout by F and simplify; I will get r square, d square F by d r square plus 2 r; d F by d r minus n into n plus 1 into F equal to 0. This is an equation that is equi-dimensional in r, each term here has r to the 0th power. Therefore, the way to solve it is to assume a function of the form F is equal to r power alpha and if I assume that form and substitute into the equation, what I will get is that alpha into alpha minus 1; r power alpha plus 2 minus n into n plus 1 alpha, alpha equals 0.

So this becomes of the form alpha into alpha plus 1 minus n into n plus 1 is equal to 0 and this of course, has solutions; one solution is sort of obvious, alpha is equal to n; the other solution is that alpha is equal to minus of n plus 1 because alpha is minus of n plus 1, then alpha plus 1 is minus n and once again this goes to 0. So, therefore, the solutions for F; out of the form F is equal to A r power n plus B by r power n plus 1, one of them increases as r increases; the other decreases as r decreases and this implies the source total solution for the temperature is going to be of the form A by r sorry P n 0 of cos theta; that is going to be the total solution for the temperature. This solution satisfies the differential equation for any integer value of n, it satisfies the boundary condition in the theta direction for any integer value of n. Therefore, the most general solution is the solution which is the summation of these; overall n of A n times r power n plus b n by r power n plus 1 times P n 0 of cos theta, this is the series of solutions.

So, we have got a solution which satisfies the homogeneity condition at the coordinate boundaries at theta is equal to 0 and theta is equal to pi. It satisfies the differential equation, now in the theta direction the solution is in the form of orthogonal Eigen functions because this function P n has orthogonality relations. Using these now we have to find out what is the solution in the radial direction that satisfies the boundary conditions. So, that solution we will continue in the next lecture, I will see you then.