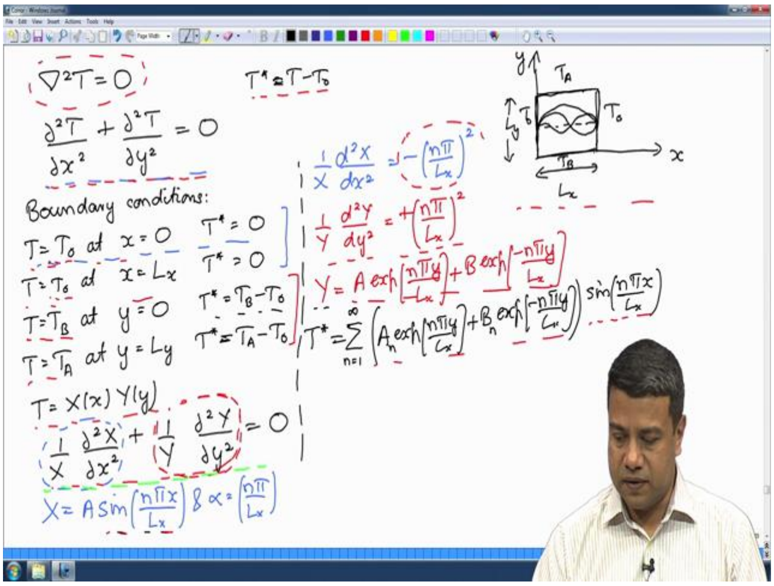
Transport Processes I: Heat and Mass Transfer
Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture – 53
Diffusion equation: Heat conduction around a spherical inclusion

Welcome to our continuation of lectures on diffusion dominated transport that we had started in the previous couple of hours. I had derived for you the conservation equations in a Cartesian coordinate system as well as the spherical coordinate system, the equations have the same form the operators are different, but when expressed in vector notation all equations end up having the same form, and in the case of a diffusion dominated system the equation basically reduces to the Laplacian of temperature equals 0 or the Laplacian of the concentration is equal to 0 for mass and energy conservation equations.

(Refer Slide Time: 01:02)

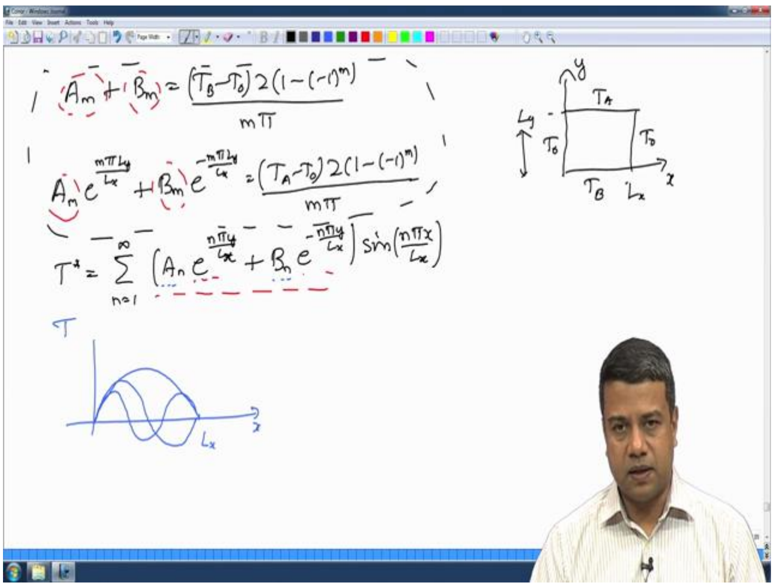


If you recall in the previous lecture, I had told you that in the case of equations for the Laplacian of something is equal to 0 is called a Laplace equation or if you have an inhomogeneous term, which is called the Poisson equation in both of those cases you can solve the equations using separation of variables and we had seen how to do that for the particular case of a rectangular geometry in a Cartesian coordinate system. Basically what you are doing is you are expressing the temperature field as a sum of basis functions all of which satisfy the boundary conditions, the homogeneous boundary conditions.

In this particular case when we had reduced the problem to a homogeneous problem in the x direction, we had got homogeneous solutions in the x directions we showed all of the form of sin functions, of all orders all the way from 1 to infinity and we had expressed the temperature as the summation of these sin functions with some 3 factors some constants An those constants were determined from the initial condition or the boundary condition in the y direction.

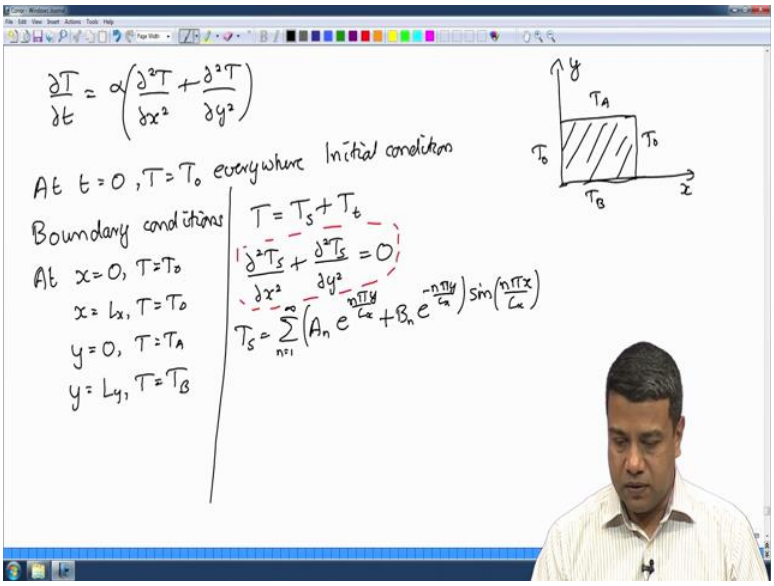
So, I had shown you how to do that step by step; if you have an inhomogeneous steady state problem identify the homogeneous direction and use sin functions in that direction that satisfy the homogeneous boundary conditions in that direction. The equations in the other direction you can solve in this particular case you get exponential solutions in the inhomogeneous direction and the constants in those equations are determined from the condition that the boundary conditions in the inhomogeneous direction have to be satisfied using orthogonality relations you can satisfy those conditions.

(Refer Slide Time: 03:00)



If in addition the problem happened to be inhomogeneous in time as well.

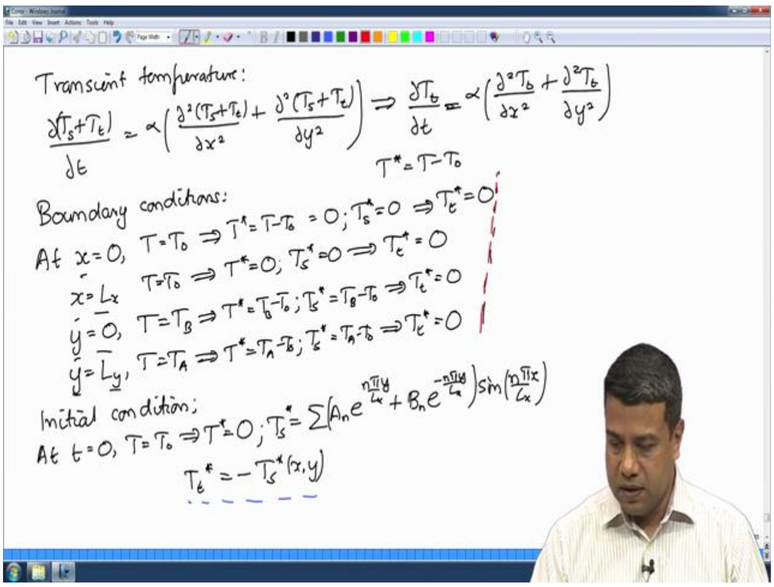
(Refer Slide Time: 03:15)



What I told you was we can separate it out first into a steady problem, steady problem solve it the way that I had solved it earlier.

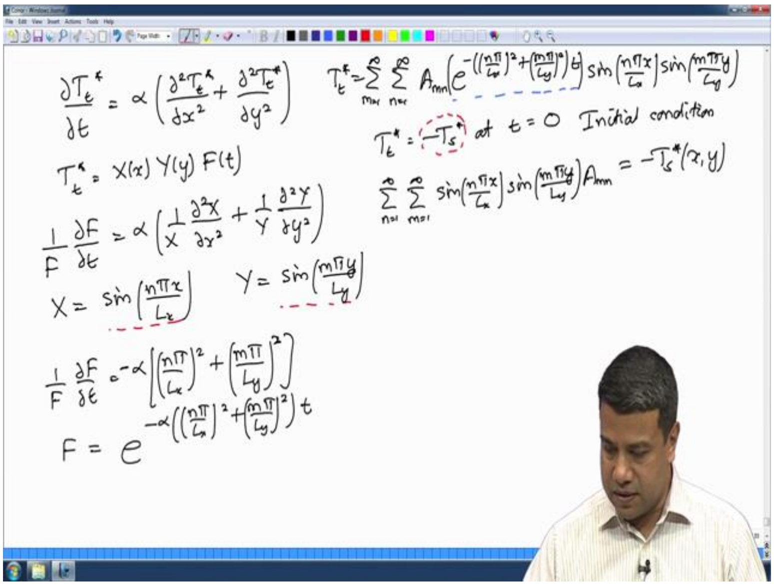
Then go to the transient problem for the transient problem it satisfies the same equation as the total temperature; the boundary conditions on all spatial boundaries are all homogeneous for the transient problem, we only in homogeneity is at initial time.

(Refer Slide Time: 03:43)



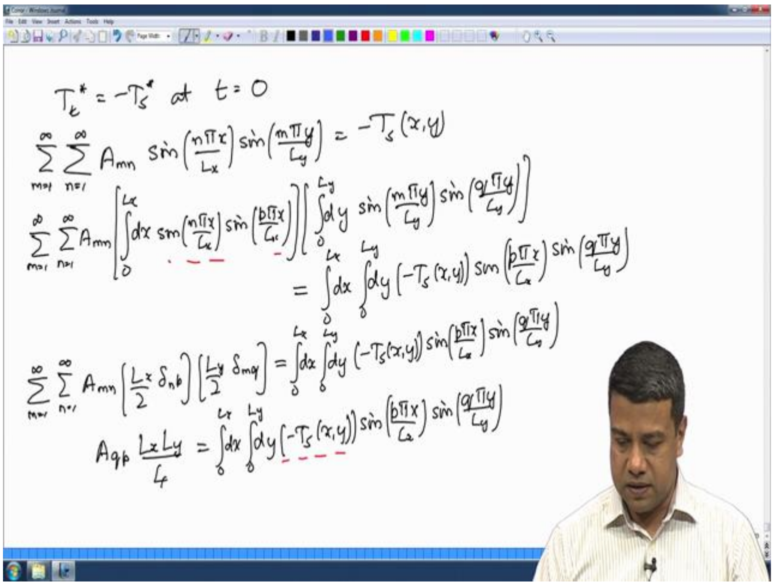
Therefore I can express the solution in terms of sin functions in both those directions; you can express the solution in terms of sin functions in both of these spatial directions.

(Refer Slide Time: 03:50)



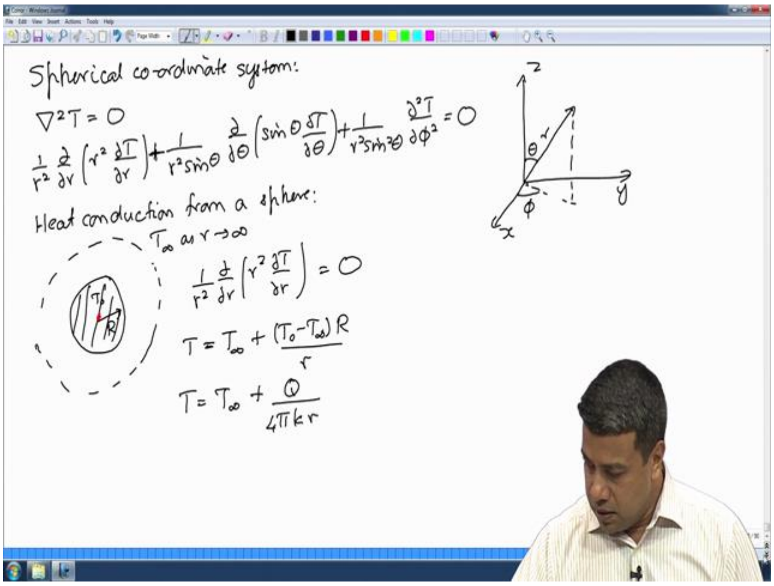
Because the solutions the solutions have to be homogeneous in both those directions and this inhomogeneous at initial time the coefficients are determined by doing two orthogonality relations, one over the x direction and the other over the y direction and if you have more number of coordinates, that you have to solve for the solution can be obtained in a similar way.

(Refer Slide Time: 04:10)



Next I would like to go into the solution of some problems in a spherical coordinate system.

(Refer Slide Time: 04:34)



Recall the differential equation in a spherical coordinate system, the distance from the origin is r the angle made with respect to the z axis is theta, the angle made by the projection onto the x y plane with the x axis is phi and the solution the equation del square T equals 0 in a spherical coordinate system is 1 by r square d by d r of r square that is the conservation equation in the spherical coordinate system; we had already solved a simple problem in a spherical coordinate system that is for heat conduction from this sphere.

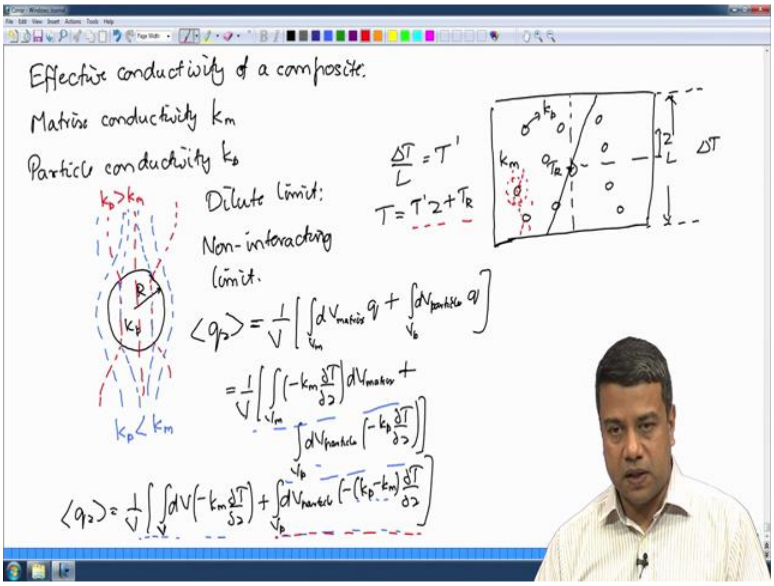
so we had a sphere which was a temperature T naught of radius R, in a medium which was at T infinity as r went to infinity, very far from the sphere the temperature was T infinity at the surface the sphere temperature was T naught and we had wanted to find out what is the temperature profile outside of this sphere. In this particular case the temperature depends only upon the radial distance, it does not depend upon the azimuthal or the meridional quarters, in other words at a certain distance from the origin if I place my origin at the center of the sphere, at a certain distance from the origin the temperature does not depend upon the orientation of the radius vector.

Therefore there is no dependence on theta and phi and we get back the equation that we had originally got for a spherical coordinate system equals to 0; this we had solved to get the temperature is T infinity plus T naught minus T infinity R by r. The temperature decreased as 1 over r if you recall, the temperature gradient decreased as 1over r square and since the surface area at any location increases proportional to r square, the total heat coming out is a constant at any surface any surface at any radius r that you pick, the flux has to decrease as 1 over r square. So, that when it is multiplied by the area which increases as r square, you get something that is a constant.

Alternatively I just told you that this can be expressed as T infinity plus the total heat coming out by four pi k r. As I told you when you express it in terms of the heat that is coming out of this sphere, the total heat coming out per unit time of this sphere which is q, the temperature turns out to be independent of the size of the sphere and it decreases as 1 over r. So, if you are sufficiently far away, that the radius is the radial coordinate is much larger than the sphere radius, the temperature field depends only upon the total heat generated not upon the radius of this sphere or the flux at the surface of this sphere, that I told you that even if you take the limit of capital R going to 0.

So, if the distance the radial distance is much larger than this sphere radius, you still get the same temperature provided the total heat coming out per unit time is the same. In this lecture we will solve a slightly more complicated problem and that is the effective conductivity of a composite.

(Refer Slide Time: 09:35)



So, what you have is a composite material it consists of a matrix and spiracle particle intrusions. Across which there is a certain temperature gradient there is a certain temperature gradient that is enforced. So, if the length of this composite is L and the difference in temperature is delta T between the top and the bottom surfaces, the gradient of the temperature the average gradient of the temperature from the top to the bottom it is going to be equal to delta T by L, which is T power one this gradient is specified. So, this gradient is specified so that the temperature if the material were homogeneous if they had only one conductivity, we know that the temperature would be a linear function of distance and the temperature would basically be given by T prime times z, plus some reference temperature where the z is this coordinate. So, if I place my reference at the location of the temperature passes through 0, the absolute temperature at this point is T R and about this point there would be a linear variation if the material were homogeneous.

However the material is not homogeneous, the matrix as a conductivity K m and these particles have conductivity K p. So, if I have a matrix conductivity K m and a particle. So, that if I just looked at the region around one particle, the particles are considered to have a radius R and the conductivity K p in the matrix that is conductivity K n. Now if I look at the flux lines if the particle conductivity would greater than the matrix conductivity, the flux lines would much rather go through the particle because this as a greater conductivity and therefore it conducts heat more efficiently. So, the flux lines would go through the particle, if the particle conductivity was greater than the matrix conductivity.

On the other hand if it were the other way round, the flux lines would actually go around the particle I am sorry. So, depending upon this the total heat flux through the material is going to change depending upon the ratio of conductivities, if the matrix conductivity is greater you would expect the average flux to be lower because the particles transport heat less efficiently than the matrix and vice versa. So, this is what we have to find out; what is the effective conductivity of the material given that there are spherical particles within a matrix and I will consider this problem in what is called the dilute limit. What I mean by the dilute limit is that the disturbance of the temperature field due to one particle, does not affect the temperature field around another particle we will come back and see under what conditions that is valid.

What that basically means is that the particle number density is sufficiently small, the particles are sufficiently well separated from each other that the disturbance to the velocity temperature field due to one particle does not affect the temperature field around the other particle, this is what is called the non interacting limit; that is I can consider each particle to be placed in a temperature field which is linearly varying with the z position individually, the presence of another particle does not affect the temperature field around this particle. So, I can consider each particle individually to be placed in a temperature field, which is linearly which is varying as this far from the particle so that is the basic idea.

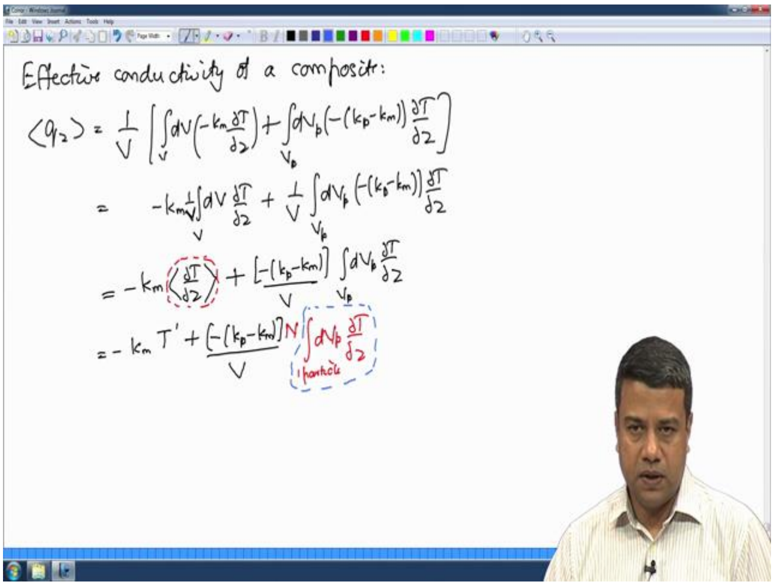
How do I calculate the average flux? There are a few different ways to do it from our perspective since we are doing a dynamical calculation the average flux in the z direction can be written as a volume averaged 1 over the total volume times integral over the volume of matrix times q, plus integral over d v over all the particles times q. So, I take the total flux as the volume average 1 over the total volume, times an integral over the matrix of the flux within the matrix, plus an integral over the particles of the flux within the particles. If I were to expand this out since each one satisfies the conduction equation, I will get integral over V matrix minus k matrix partial T by partial z over the matrix, plus the contribution over the particles so that is the average flux rather than writing it this way, it is more convenient for us to write it as 1 over the total volume, I integrate over the total volume minus k m partial t by partial z, plus integral over the particles of d v particle minus of k p minus k partial T by partial z; you can see that I have just rewritten these two terms.

These two terms the first one was integral over the matrix and integral over the particles; the second term what I have done is that I have taken it as integral over the entire volume of minus k m times d T by d z that counts both the particles and the matrix therefore, in the second term I have subtracted out the matrix contribution alone; you can easily the see that the sum of these two terms gives you back the original heat flux, the advantage would doing it the second way is that this second term is gives you effectively the temperature disturbance due to the particles; if the particles were not there or if the particle conductivity were equal to the matrix conductivity, this second term would be effectively equal to 0.

So, therefore, I have written it as one integral over the entire matrix and two the disturbance due to the particles alone. So, I have written it as the first term which is an integral over the entire volume of the matrix conductivity times the temperature gradient that includes the volume of the particles plus the volume of the matrix. The second term is an integral over the particles alone and that is equal to I have written as the difference in conductivity between the particles and the matrix.

So, this is an alternate way of writing is at exact same relation.

(Refer Slide Time: 19:17)



So, now effective conductivity is equal to 1 over the total volume, integral now over the total volume of k m times it should have a negative sign there because if heat flux as a negative sign over the particle volume alone of the integral of the particle volume alone of. In the first term, the matrix conductivity is a constant so I can take that out, so I will just put the volume over here, this first term is an integral over the entire volume and the second term is going to be 1 over the total volume integral over the particles of; note that the first term is just the volume average of the temperature gradient the first term is the volume average over the temperature gradient, the second term is an integral over all the particles of the particle the temperature gradient within the particle times the difference in the thermal conductivity between the particle and the matrix.

Now, this first term here as the average thermal gradient as to be equal to the imposed temperature gradient, the average temperature gradient over the entire volume this is an average temperature gradient over the entire volume, this as to be equal to the temperature gradient that is actually imposed on the composite material and that imposed temperature gradient is T prime that imposed temperature gradient is T prime.

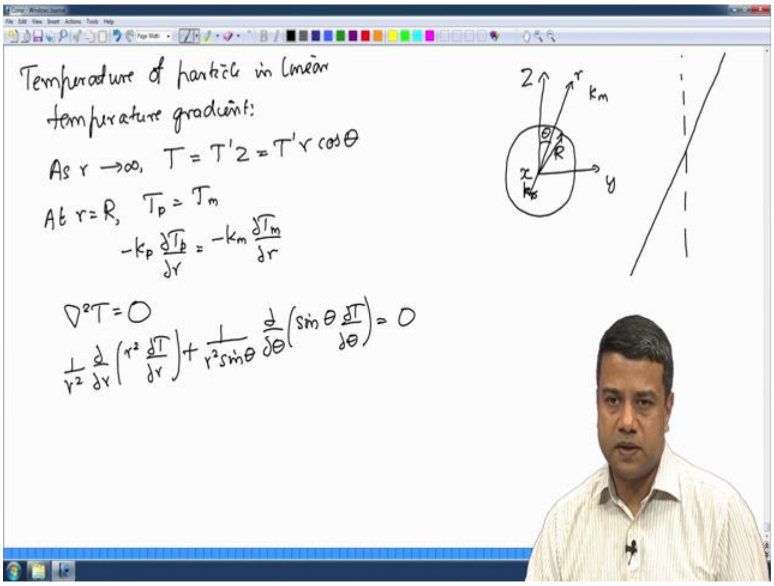
Therefore this just gives me minus k m into T prime, which is the imposed temperature gradient and the second term which is minus of k p minus k m divided by the total volume, I have an integral over all of the particles. So, basically what I am doing the integral of the particle volume is integral over all of these particles.

However I also told you that we would consider it in the non interacting limit, in the dilute non interacting limit where the temperature gradient around one particle does not affect the temperature around another particle. So, if you are considering it in the non interacting regime the dilute regime, each particle effectively sees a linear temperature gradient far from the particle; it is embedded in the temperature field which as a linear temperature gradient far from the particle. So, each particle is effectively identical as far as the temperature gradient is concerned for each particle of course, the average temperature at it is center is going to be different. The average temperature at it is center is going to be different because it is located at different z locations and the temperature varies with z.

However the gradient that it sees far from the surface it is going to be the same for each particle therefore, the temperature gradient within the particle will also be the same for each particle therefore, this summation I can write it as the number of particles within this differential volume, times the integral over one particle of. This comes out of the non interacting assumption that the temperature gradient far from each particle is exactly the same, each particle is in exactly the same environment apart from a constant shift in the reference temperature along it is central plane. So, therefore, the temperature gradient around each particle is exactly the same therefore, if you solve the problem the temperature gradient within the particle will also exactly be the same therefore, rather than integrating over each particle individually, I can just take the number of particles and do the integral over one particle. So, there is a non interacting limit and now we are reduced to finding out what is this integral d T by d z over one particle.

So, let us look at that.

(Refer Slide Time: 25:40)



So, I have one particle of radius R in a temperature field which is linear far from the particle; however, the particle conductivity is different from the matrix conductivity therefore, close to the particle there is going to be a disturbance to the temperature field, within the particle as well there is going to be a disturbance to the temperature field because if the particle conductivity is different from the matrix conductivity, it is either higher or lower and therefore, the flux lines will distort around the particle whereas, if I just a linear temperature gradient the temperature is linear, the flux is all in the z direction in this case it will not be so, and our task was to find out what is the temperature field within the particle and outside the particle? This is a temperature of a particle in linear temperature gradient.

Therefore the boundary conditions as r goes to infinity, the temperature is equal to T prime times z. So, now if I put in a coordinate system here this is z, this is y and this is x x y z coordinate system, particle radius is r far from the particle the temperature is equal to t prime times z at any location r, at any observation point and we know that z is equal to in a spherical coordinate system z is equal to r cos theta, you know the temperature T z is r cos theta in a spherical coordinate system. At the particle surface itself r is equal to r, we require that the temperature is the particle is equal to the temperature of the matrix equal temperature on both sides you cannot have two different temperatures at that same location and the flux is also have to be equal; whatever flux is leaving the particle has to enter the matrix by heat balance therefore, I require that minus k particle times partial T particle by partial r is equal to minus k matrix times the partial T matrix by partial r.

Note that I have applied the flux balance in the radial direction in the direction perpendicular to the surface, because it is only the flux in that direction that changes the energy on either side of this spherical surface. So, these are the boundary conditions what is the conservation equation? Del square T is equal to 0, in this particular configuration we can simplify the problem if the temperature does vary with the theta coordinate, this is not like our particle in our heated sphere in a matrix at constant temperature at infinity, where the temperature is only a function of r in this case it does depend upon theta it does not however, depend upon phi; it does not depend upon the angle around the z axis, for a given r and theta no matter what we angle around the z axis is the temperature has to be the same therefore, the temperature is only a function of r and theta and it is not a function of phi.

Spherical coordinate system is more convenient in this case because the sphere itself has spherical symmetry. So, in that case my conservation equation will be. So, this is the conservation equation that we have to solve subject to these boundary conditions to find out what is the temperature within the particle. Once we know what is the temperature within the particle, we can then find out what is the flux within the particle and thereby calculate what is the average flux through the material and that gives us the effective conductivity of the composite.

We will continue this in the next lecture how to solve separation of variables in a spherical coordinate system now that we have two coordinates r and theta I will see you in the next lecture.