Transport Processes I: Heat and Mass Transfer

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Lecture – 52
Diffusion equation: Heat conduction in a rectangular solid (continued)

Welcome to this, we were discussing diffusion dominated transport in our course on fundamentals of transport processes and I have told you the diffusion dominated transport takes place and the dimensionless number, the Peclet number for heat or mass transfer of the Reynolds number for momentum transfer is much smaller than 1. In that case you can neglect the convection terms in the conservation equation and the only terms that remain is the diffusion term and that diffusion term for heat and mass transfer has a common form; the Laplacian of T concentration or the temperature is equal to 0 and we were looking at how to solve problems where the Laplacian of the concentration or temperature is equal to 0. These are all based upon the separation of variables techniques, which we had already seen when we had solved unsteady problems.

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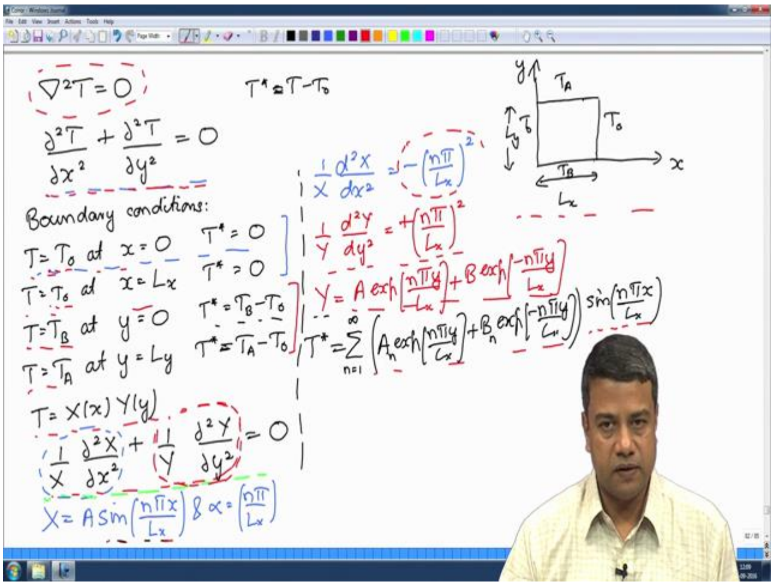

So, in this case we have an equation of the form the Laplacian of the temperature or the concentration is equal to 0. We had solved it first for the case where we had a rectangular block in a Cartesian coordinate system of extent L x in the x direction extent L y in the y direction. The temperature was equal on the left and the right phases and they were different on the top and the bottom phases. So, that was the problem that we were trying to solve, we wanted to find out how the temperature varies throughout this block. The conservation equation in a Cartesian coordinate system two dimensions d square T by d x square plus d square T by d y square is equal to 0, so that is the conservation equation. With boundary conditions T is equal to T naught at the left and the right sides and T is T v at the bottom T is T A on top.

Whenever we solve separation of variables problems, I had told you that we have to ensure that the boundary conditions are homogeneous in all directions except one. In this particular case we had managed to get homogeneous boundary conditions in the x direction because if we subtract T 0 from the temperature field then T star is equal to 0 on both the left and the right phases. Therefore, the boundary conditions are homogeneous on the left and the right phases, you can use separation of variables where we write the temperature is some function of x times some function of y; put that into the conservation equation divide throughout by x times y and we will get an equation in which one term is only a function of x, the other terms only a function of y. Therefore, both of these have to be constants, one has to be a positive constant; the other has to be a negative constant of equal magnitude, so that they sum to 0.

Which one should be positive and which one should be negative. Homogeneous boundary conditions are in the x direction therefore, you would expect the function x of x to be equal to 0 on both boundary, those are the homogeneous boundary conditions. That kind of a solution can be obtained only if the constant in the x direction is negative, if it were positive I would have obtained exponentially increasing and decreasing functions and if I were to try to sum both of them and get them to be 0 at two boundaries the only solutions will be when both constants are 0. If it is negative then I get (Refer Time: 04:16) oscillating functions which goes through 0 multiple times and therefore, I can get a sin function which ensures that the solution is 0; both at 0 and L x.

Since it is negative for the x direction, it has to be positive and of equal magnitude in the y direction. So, that the sum of these two terms is equal to 0 therefore, the equation for the y direction becomes something like this and the solutions are exponentials in the y direction, the boundary conditions in the x direction and the conservation equation are satisfied for any value of n; provided n is an integer, the constants; the boundary conditions in the homogeneous direction and the differential equation are satisfied for any value of n; provided n is an integer. So, the most general solution is one where I sum the solutions for each value of n, multiply them by contents; these constants are determined from the orthogonality relations.

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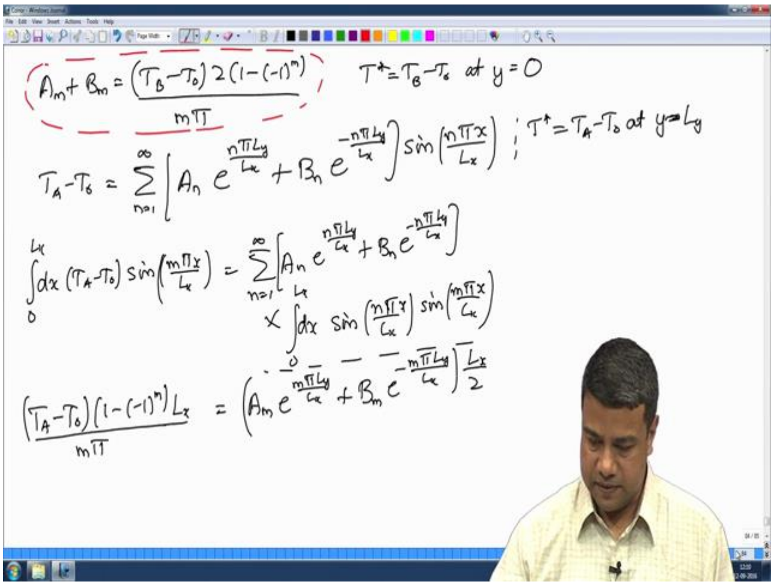

So, this is the general solution that I have, it is sinusoidal in the x direction because that is the homogeneous direction and it has exponentials in the y direction and I can use the orthogonality relation; in order to find out what those constants are and I get two equations for the two constants.

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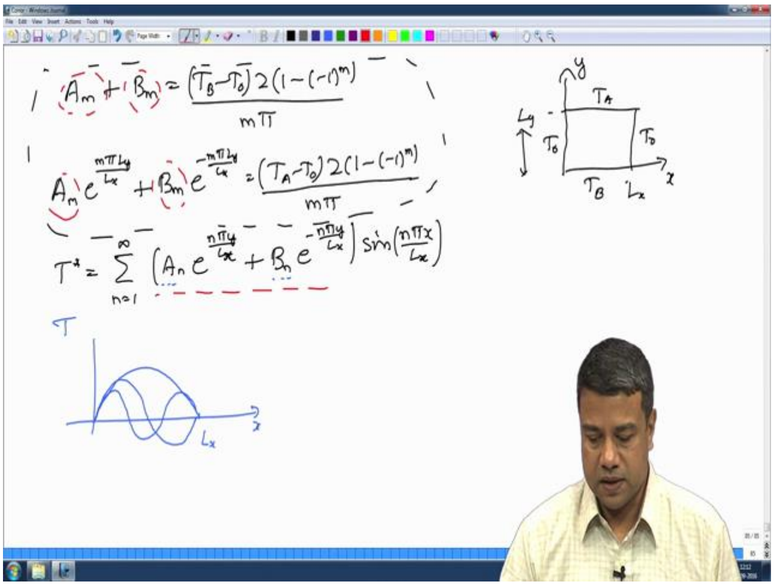

I get two equations for the two constants, I am sorry I should correct this one. So, I get two equations for these two constants I can solve them simultaneously in order to get A n B n, so to get A n, I have to multiply the first equation by e power minus m pi L y by L x and subtract the two equations and for B n; I have to multiply the first equation by e power m pi L y by L x and subtract the two equations and then I will get those two constants.

Insert it into the equation and get the temperature field; which is now a function of both x and y. So, basically what I have done is I have expanded the temperature field in the x direction in terms of sin function is 0 to L x; in the x direction the temperature field has been expanded in terms of sin functions because these are the basis functions, these are the natural solutions of the Laplace equation in a Cartesian coordinate system and what multiplies each of these natural solutions is a function that depends upon y. You can see that these functions which depend upon y, they will decrease.

So there is an exponentially increasing function, it is an exponentially decreasing function and these inhomogeneous term multiplied the sin function to get the total solution; for this temperature field. So in this particular case, we had managed to get a solution by expanding for the steady the differential equation.

Now, you might ask what happens if the equation is not steady, so I will just briefly tell you outline, the way that you would solve this problem if you are solving for the unsteady equation.

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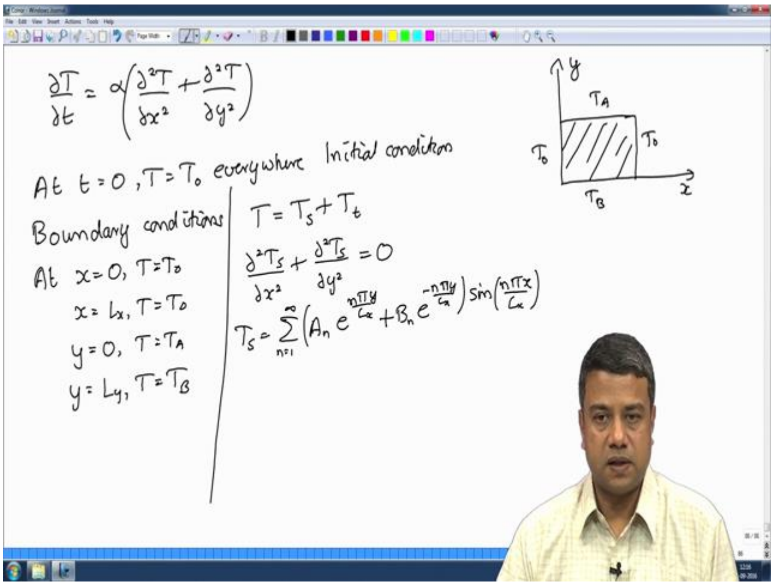

If I were solving this for an unsteady equation, the equation would be of the form; partial T by partial T is equal to alpha. If I was solving for the unsteady problem, the equation would be of this kind. Now for separation of variables, I require that the system be homogeneous in all directions except one. In this particular case if for example, I consider that the temperature throughout this block; at time T is equal to 0, the temperature is equal to T naught. So, I start with the block that is isothermal and at T is equal to 0, I bring the top surface in contact with the temperature T A and the bottom surface contact to the temperature T B and then I want to find out how the temperature involves as a function of time.

So, this is the condition at temperature T is equal to; at time T is equal to 0. When I have the boundary conditions, if this is the initial condition at x is equal to 0, T is equal to T naught, x is equal to L x, T is equal to T naught, y is equal to 0, T is equal to T A and y is equal to L y, T equals T B. So, those are the boundary conditions, but now I have a time dependent problem, which I have to solve as a function of x and y as well.

How would I go about solving this first thing, so I can write down the temperature is equal to a steady part plus a transient part. This we had done earlier as well when we solved in one dimension, we had separated the temperature into a steady part plus the transient part. We do that in the two dimensional case as well, for the steady part we are solving the equation partial square T steady by partial x square plus partial square T steady by partial y square is equal to 0. That is the equation we have just solved, if we just solve this equation to get the steady part a summation of A n; e power n pi, y by L x plus B n; e power minus n pi, y by L x into sin of pi x by x. We will just solve this in order to get the solution where the coefficients A n and B n were determined using the orthogonality relation.

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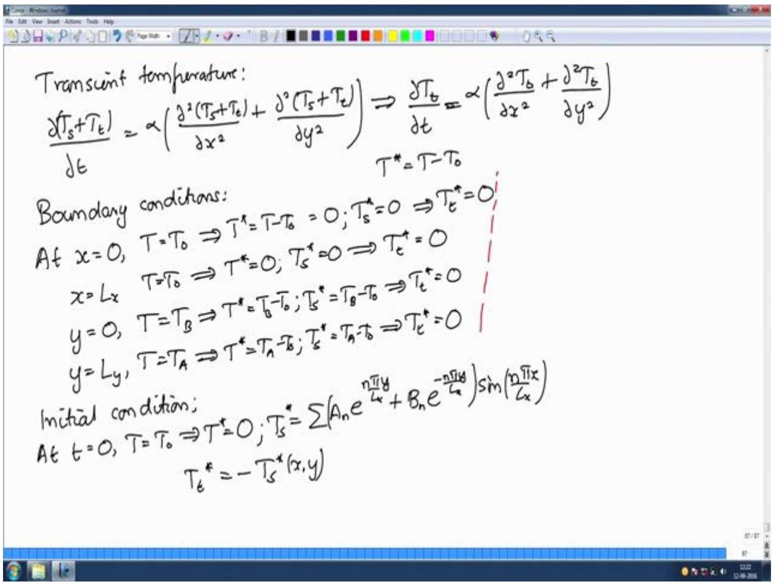

Now, therefore, I have to now solve for the transient part; how do I solve for the transient part. For the transient temperature, you know that the total equation is partial of steady plus transient by partial T is equal to alpha times. For the steady part alone, we have already solved the equation where this is equal to 0 and the steady part is independent of time. Therefore, the equation for the transient part will end up, so that is the equations of the transient; boundary conditions at x is equal to 0; T star T is equal to T naught which implies that T star is equal to T minus T naught is equal to 0.

Let me express everything in terms of the temperature (Refer Time: 13:56) express everything in terms of T star is equal to T minus T naught. We know that the steady part; T steady is equal to 0 that it is all they know homogeneous boundary condition T s is equal to 0 which implies the transient part is also equal to 0 because the sum of these has to be equal to the total temperature. Similarly at x is equal to L x, T is equal to T naught which means that T star has to be equal to 0, the steady part is equal to 0 which implies that the transient part is equal to 0.

What are the boundary conditions in the y direction; at y is equal to 0, T is equal to T B; which means that T star is equal to T minus T B minus T naught. The steady part also we had solved with the same equation; T steady is also equal to T B minus T naught. This implies the transient part has to be equal to 0 because a transient part plus the steady part has to add to the total temperature. Similarly at y is equal to L y, T is equal to T A which implies that T star is equal to T A minus T naught.

We know that the steady temperature is equal to T A minus T naught; which implies that the transient temperature has to be equal to 0. So, when expressed in terms of the departure from the steady temperature, the transient part; that transient part has 0 temperature boundary conditions on all spatial boundaries, it does not have 0 temperature boundary condition at time T is equal to 0. So, if I write the initial condition; I know that at T is equal to 0, the temperature is equal to T naught everywhere which implies that T star is equal to 0 everywhere; the steady part is independent of time. So, the steady part is going to be equal to this summation of A n; e power n pi y by L x plus B n; e power minus n pi y by L x into sin n pi x by L x, where these coefficients are known; the steady part is given by this.

The steady part which means that the transient part has to be equal to minus the steady part, which is a function of x and y, so for this transient problem; the boundary condition once I have expressed the temperature in terms of a steady part plus the transient part. For the transient part, the boundary conditions are homogeneous boundary conditions on all four spatial boundaries, the only in homogeneity is that initial time.

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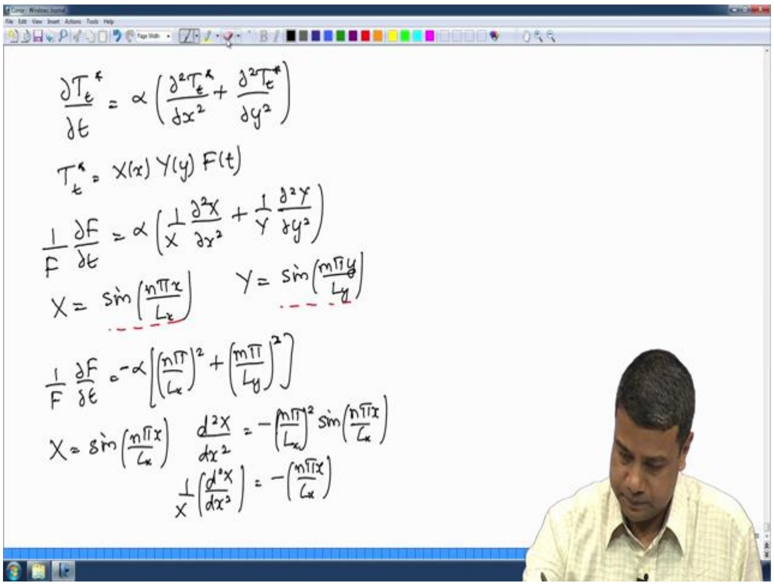

Therefore, the equation that I have partial T transient by partial t, I can write the temperature field in terms of T star because I just subtracting out a constant value and they subtract out a constant value, all the derivatives will all be 0. Separation of variables T star transient is equal to x of x, y of y times from functional t. Substitute into the equation, divide throughout by x times; y times F, on the left side I will get 1 over F, partial F by partial T is equal to alpha into 1 over x, partial square x by partial x square plus 1 over y partial square y by partial y square

The left side depends only on time, first term on the right side depends only on x, second term on the right side depends only on y. Therefore, all of these have to be constants; for the two terms from the right side, the spatial boundary conditions are homogeneous; at x is equal to 0 and x is equal to L and at y is equal to 0 and y is equal to L y, should we call for the transient part. All boundary conditions are homogeneous; at x is equal to 0 and x is equal to L x and at y is equal to 0 and y is equal to L y.

Therefore I need to have Eigen function in each of these directions. So, x will be of the form sin n pi x by L x; where n has to be an integer, y has to be of the form sin n pi y by L y be careful here, this integer can in general be different; it has to be an integer nevertheless. In the x direction the solution has to go to 0, at x is equal to 0 and x is equal to L x. That will happen only if the solution is of this form and n is an integer.

Similarly in the y direction it has to go to 0 at y is equal to L y and at y is equal to 0 and that will happen only if it is of this form and once you put that in you will get 1 by F; partial F at partial t is equal to alpha; the thermal diffusivity times n pi by L x the whole square plus n pi with a negative sign; why the negative sign, if I take this function x is equal to sin of n pi x by L x, I take the second derivative d square x by d x square is equal to minus n pi by L x the whole square sin, therefore 1 by x; d square x by d x square; just a property of the sin function.

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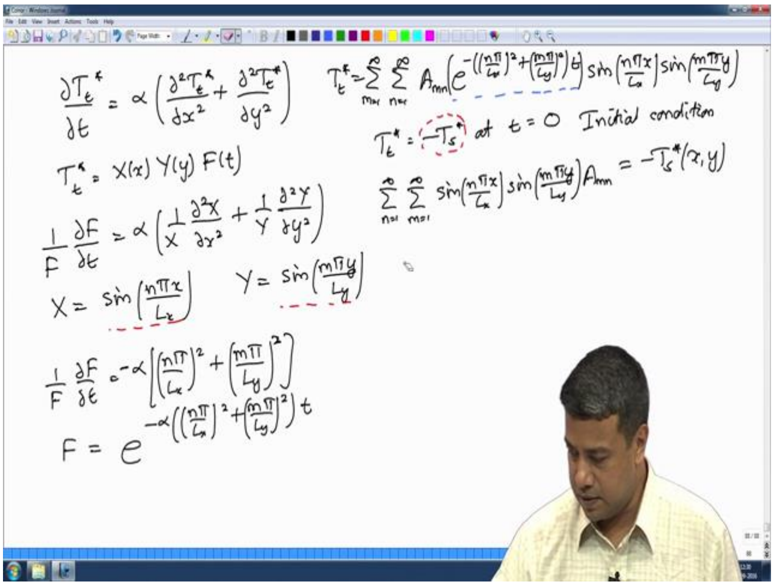

And with that I will get F is equal to e power minus alpha into t. So, that is going to be the solution for F; the total temperature T; t star is equal to this function F e power minus t, this into sin and this solution is valid, it satisfies the equation, it satisfies the spatial boundary conditions at x is equal to 0 and l and y is equal to 0 and L y for any value of n and m, so long as they are integers.

So, the most general solution is going to be this times some constant which is a function of m and a function of n; that is going to be the most general solution; how is the constant determined? That is determined from the initial conditions; the solution as it is already satisfies the boundary condition. So this is determined from the initial conditions, how do I enforce this initial condition, I already know what is this steady solution and we know what is the steady solution therefore, I have at time T is equal to 0; this entire exponential is just equal to 1 because time is equal to 0; a e power 0 is 1. So, I just get sin; n pi x by L x; sin m pi y by L y into A m m is equal to the minus the steady solution and I have to use orthogonality relations, I know the steady solution is function of x and y have to use the orthogonality relations in order to determine the coefficients A m n; how do I do that; I integrate both sides I would multiply both sides by sin p pi x by L x sin; let me do that on the next page that is clear.

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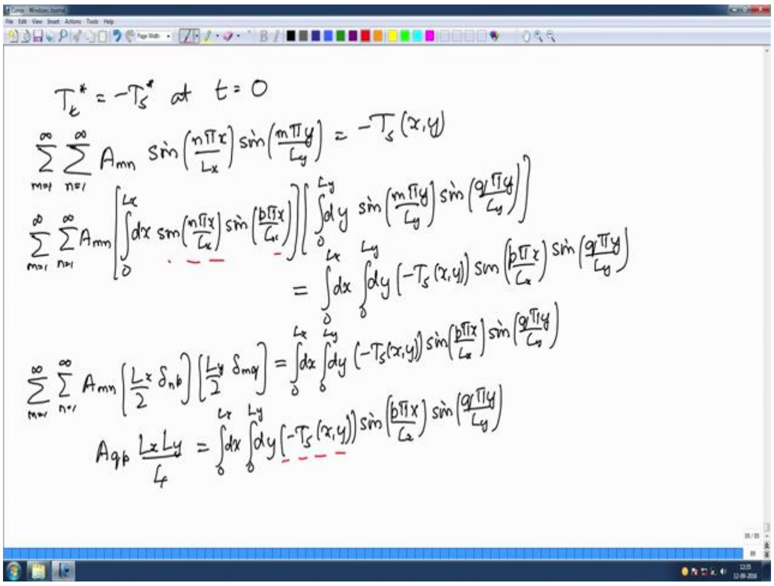

So, I have T t star is equal to minus T s at T equals 0 as summation of A m n, sin n pi x by L x, sin is equal to this function of x and y; I just leave it as a function of x and y; this n is equal to 1 to infinity; summation m is equal to 1 to infinity.

Now, I have two functions in the x and the y directions therefore, I need two orthogonal T relations; I need two orthogonal relations. The first in the x direction, so integral d x from 0 to L x; sin n pi x by L x sin, some other integers p pi x by L x; that is one orthogonality relation in the x direction, in the y direction 0 to L y; d x, d y sin m pi y by L y sin q pi y by L y; let us choose another integer. So, multiplied by sin p pi x by L x, sin q pi by L y integrated over x and y therefore, I have to do the same thing on the right hand side. On the right hand side I will have integral 0 to L x; d x integral 0 to L y; d y minus the steady solution of x and y, sin p pi x by L x sin.

So, just multiply both sides by sin p pi x by L x integrate from 0 to L x; sin q pi q by L y integrated from 0 to L y and once I do that, I can use the orthogonality relation m is equal to 1 to infinity; A m n. The first one is L x by 2 delta m p, this term is non zero only when n is equal to p; therefore, I will get delta n p into L y by 2 into delta m q, the second term is non zero only when m is equal to q. This is the right side 0 to L x; d x integral 0 to L y; d y into minus the steady temperature of x and y multiplied by these two functions and since delta n p is non zero only when n is equal to p, delta m q is non zero only when m is equal to q; I effectively get A q p; L x; L y by 4 is equal to integral 0 to L x.

So with this knowing the form of the steady temperature, you can now determine what these coefficients are. So basically we have to do separation of variables two times; the first to determine what is the steady temperature, in that case we just solving the Laplace equation for the temperature field. Once you have done that, we have got the steady temperature, we subtract out the steady temperature to get the transient temperature and for that transient temperature, you get the solution; the boundary conditions are homogeneous in all spatial dimensions. Therefore, you get sin and co sin functions in all spatial dimensions x and y, the homogeneity is in at time T is equal to 0. So, I need to use two orthogonality relations in both the x and y directions to determine what the constant is.

So, by this series of steps you can use the separation of variables procedure to get the solution for any kind of problem, whether it is steady or unsteady. I have shown you in this case how to do it for a Cartesian coordinate system, in the homogeneous direction the natural solutions r sin and co sin functions. If the temperature is 0 on both walls, if the flux is 0 then you get a different kind of boundary condition, but when the temperature is 0 on both walls, you get these sin functions and you can construct the solution as a function of time; times the sin functions in x and y. So, this is the solution in the Cartesian coordinate system for both steady and unsteady in multiple dimensions and this can be done even if you have variations in three directions and variations in time.

Next class, we will look at how to do it in a spherical coordinate system. The solution of the spherical coordinate system is slightly more complicated because you have the operator itself is more complicated but you can get solutions even in the spherical coordinate system. So, I will start that in the next lecture how do you get solutions in the spherical coordinate system and more importantly in a spherical coordinate system, there is a physical interpretation for these solutions; they will try to give you what is the physical interpretation. We will continue that in the next lecture, we will see you then.