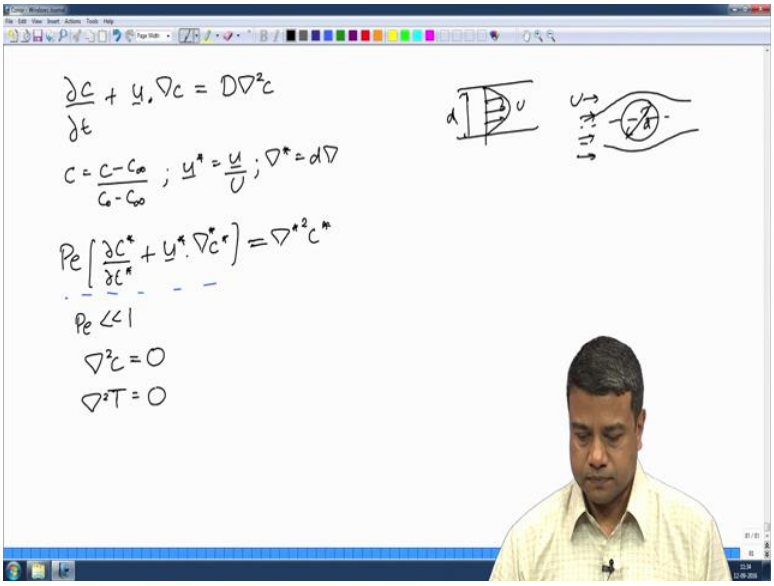
Transport Processes I: Heat and Mass Transfer
Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture - 51
Diffusion equation: Heat conduction in a rectangular solid

So welcome to this continuing series of lectures on transport processes. In the last lecture I had derived for you the conservation equations in vector notation for mass momentum and energy conservation; all these conservation equations have a common form.

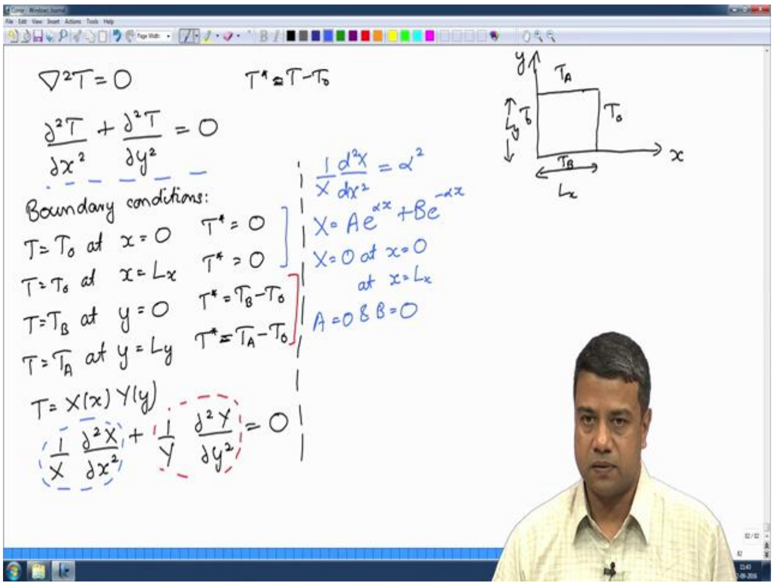
(Refer Slide Time: 00:45)



For the concentration field for example, d c by d t plus u dot grad c, this is only for incompressible flows where the divergence of the velocity is equal to 0; is equal to D del square c and if I scale c is equal to c minus c infinity where I have for example, in the case of internal flows; I have velocity u and a characteristic length d, for external flows this is d the characteristic length and this is the velocity u. I will get an equation of the form the Peclet number times partial c by partial T and when the Peclet number is small compared to 1; this left side is small compared to the right side. Therefore, if I neglect convective effects and consider the transport to be diffusion dominated, the conservation equation becomes del square c equals 0 for the concentration field, alternatively del square T equals 0 for the temperature field. These are diffusion dominated transport problems; they reduce to solving a Laplacian operator equal to 0 subject to boundary conditions. So, we will look at how to solve equations of this kind for diffusion dominated transport.

Now, let us take first a Cartesian coordinate system, so that is the simplest coordinate system that one can envisage search.

(Refer Slide Time: 03:07)



X y; the simplest problem that one can consider is for example, in two dimensions the transport in a square or a rectangular block. So, I have a rectangular block of a solid or a fluid which satisfies the conservation condition. Let us assume that the length in the x direction is L x the length in the y direction is L y and I will assume that the four phases are at different temperatures. The face at x is equal to 0 is the temperature T naught, at x is equal to L x is a temperature T naught. So, therefore, these two phases are at constant temperature; the top and bottom phases are at some different temperatures and I want to find out what is the temperature distribution throughout the system.

Therefore this is a diffusion dominated problem; I need to solve the equation del square T equals 0. In this particular case, I will first consider the system to be at steady state therefore, there are variations only in the x and the y directions; partial square T by partial x square plus d square T by d y square; this is equal to 0 is the conservation equation and the boundary conditions T equals T naught at x equals 0 call that T B. So, those are the boundary conditions that I need to solve the problem and whenever you have differential equations of this kind, the natural procedure to use is the separation of variables procedure.

However as I had repeatedly emphasized when we were solving different separation of variables problems, it is necessary in a separation of variables procedure to ensure that there is homogeneous boundary condition in all directions except one; how much in this boundary condition means either the temperature is 0 or the derivative of temperature is 0 or some combination thereof 0, so those are homogeneous boundary conditions.

In this particular case, I can ensure that the temperature is 0 on the surfaces at x is equal to 0 and at x is equal to L x. I can do that just by writing T star is equal to T minus T naught; if I just write T star is equal to T minus T naught, I will ensure that T star is equal to 0 at x is equal to 0 and T star is equal to 0 at x is equal to L x. The inhomogeneous direction then becomes the y direction; T star is equal to T B minus T naught at y is equal to 0 and T star is equal to T A minus T naught at x is equal to L y, so that becomes the inhomogeneous direction.

So, have homogeneous boundary conditions along the x coordinate and the inhomogeneous driving is along the y direction because I have nonzero on temperature boundary condition at y is equal to 0 and y is equal to L.

So, now I have made one direction homogeneous and therefore, I can go ahead and try to solve the problem. So, the first thing I have to write T is equal to some function of x times some function of y; insert that into the conservation equation. So, I will get x times partials sorry equal 0 and I divide throughout by x times y in which I case I will get 1 over x d square capital x by d x square plus 1 over y.

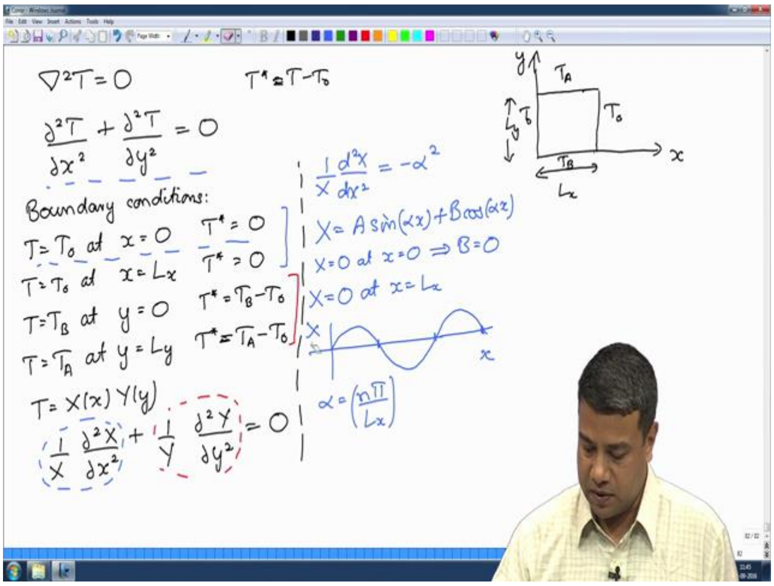
So, a substitute temperature is some function of x times some function of y insert into the equation, divide throughout by x times y. Once you do that, you have now one term which is only a function of y and the other term which is only a function of x. Therefore, both of these terms have to be equal to constant they have to be constants of opposite sign so that they can add up to 0; in this case they have to be constants of opposite sign so that the sum of these two is equal to 0.

Now, what constant should they be; should they be positive or negative; obviously, one is positive, the other has to be negative. How do we said which one to choose; that decision depends upon the boundary condition which is the homogeneous direction. I have one variation in x and the other variation in y, I have a homogeneous boundary condition at x is equal to 0 and x is equal to L x. Therefore, I would expect that in the homogeneous direction I should have sin cosine functions so that I can satisfy the boundary conditions in the x direction, the homogeneous boundary conditions. As you can see the solutions of this, so if I take the solution of this equation; this is a total derivative because x is only a function of x.

There are two possibilities, one is if it is positive in which case x will be equal to Ae power some constant x plus B e power minus alpha x; a second order term equation which is positive the square root of these, so this will be of the form this is equal to alpha square a positive number, the solutions will be exponentials. These exponentials do not pass through 0, e power plus x is 1 at 0 and it goes to infinity, e power minus x is 1 at x equal 0 it goes to 0 as x increases. These are monotonic functions therefore, if I were to impose x is equal to 0 and at x is equal to L x; the only solutions that I will get will be a equals 0 and B equals 0. So, in the homogeneous direction if I assume a positive constant the solutions are exponentially increasing and decreasing functions and you cannot impose homogeneous boundary conditions, the only solution is for both A equals 0 and B equals 0.

We have seen that when we did the unsteady problems therefore, this does not give you solutions that decrease to 0 at both boundaries. Therefore, the only solution that I can get which will decrease to 0 at both the boundaries is if this constant is negative minus alpha square.

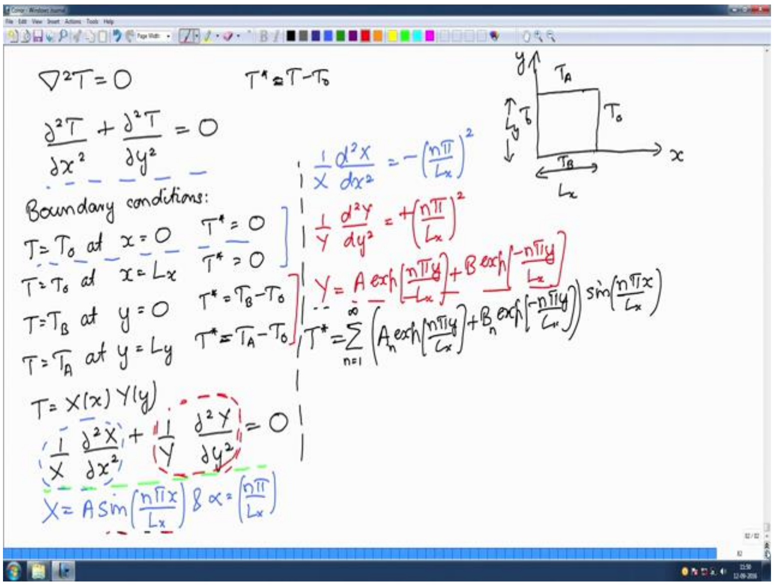
(Refer Slide Time: 12:12)



In this case, if I solve this I will get x is equal to A sin alpha x plus B cos alpha x. The requirement that x is equal to 0 at x equals 0; the first boundary condition here T star is equal to 0 at x is equal to 0 will give me that B equals 0, I also have x is equal to 0 at x is equal to L x and that gives me a value of alpha because I have the solutions are the form of sin functions. Therefore, alpha has to a specific values, so that the value of x becomes 0 at small x is equal to L x. So, solution for that is the alpha is equal to n pi by L x; you can see that when alpha is equal to n pi by L x, when x is equal to L x here when alpha is equal to n pi by L x; you have sin of n pi and that is 0.

So, therefore this is a valid solution the sin function it decreases to 0 on both the left and the right phase, if it satisfies both boundary conditions where n has to be an integer.

(Refer Slide Time: 13:53)



Therefore the solution for this is that x is equal to A sin n pi x by L x and alpha is equal to n pi by L x. So, we have got the form of the solution in the homogeneous direction these are the form of Eigen functions, as I told you earlier these Eigen functions are the natural solutions in a Cartesian coordinate system; the sin functions, we satisfy the boundary conditions of 0 temperature or 0 concentration on to fixed surfaces. So, there is a natural basis set which can be used for expanding these functions and since the Laplacian, so since the operator it is the same. The operator, the diffusion operators is a second derivative therefore you will always get the sin functions as solution.

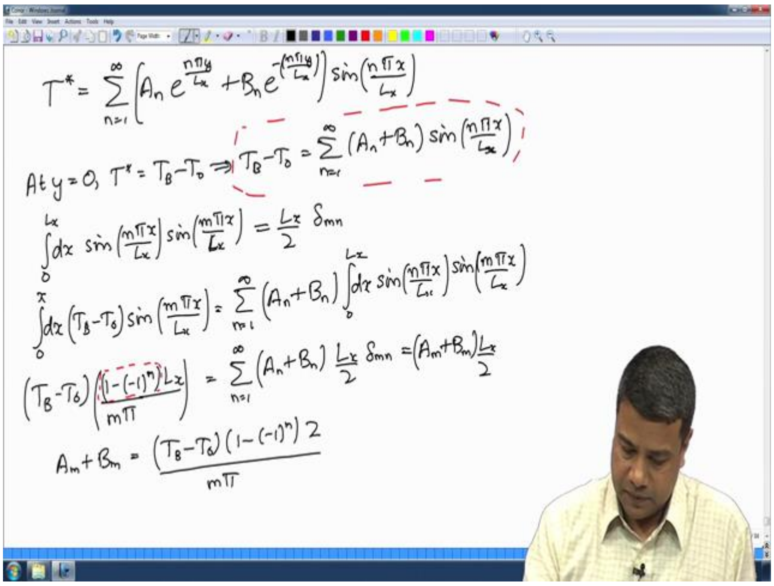
So, therefore, I have solved 1 over x; d square x by d x square is equal to minus n pi by L x the whole square; which means that the y derivative has to be of opposite sign; because the x term plus the y term they have to sum to 0 therefore, the y term has to be of opposite sign.

Therefore the equation for y will be of the form 1 by y d square y by d y square is equal to plus n pi by L x whole square. So, that it satisfies the conservation equation that the sum of these two terms has to be equal to 0. Recall that the y direction is the inhomogeneous direction, this can now be easily solved y is equal to A exponential of n pi y by L x plus B exponential of minus n pi y by L x, so those are the two exponential solutions in the y direction.

So, a general solution for this will be that the temperature is equal to A exponent of n pi y by L x plus B exponent of minus n pi y by L x. This whole thing; this is capital Y, this has to be multiplied by x; if I to multiply this by sin of n pi x by L x. So, this is the solution, this solution satisfies the equation, you can easily substitute that into the equation and see, this solution satisfies the equation it satisfies the homogenous boundary conditions in the x direction, it satisfies those for any value of n. So, the most general solution is the summation n is equal to 1 to infinity of A n plus B n; this is the most general solution because the product of these two terms, the y as well as the x satisfies the equation for any value of n. So, the most general solution is when you multiply these two and you can sum up the series from n is equal to 1 to infinity.

So, this satisfies the equation; it satisfies the homogenous boundary conditions, it does not yet satisfy the inhomogeneous boundary conditions. How do we satisfy the in homogeneous boundary conditions as usual we do it using the orthogonality relations, so let us do that.

(Refer Slide Time: 18:52)



The solution that I had for the temperature; field summation n is equal to 1 to infinity A n e power n pi y by L x plus B n e power minus n by y by L x into sin of n pi x by x and the inhomogeneous boundary conditions are that at y is equal to 0; T star is equal to T B minus T naught. Recall that at y is equal to 0, the value of T star is equal to T B minus T naught clear, it is that was the boundary condition which implies that T B minus T naught is equal to sigma n is equal to 1 to infinity of A n plus B n sin.

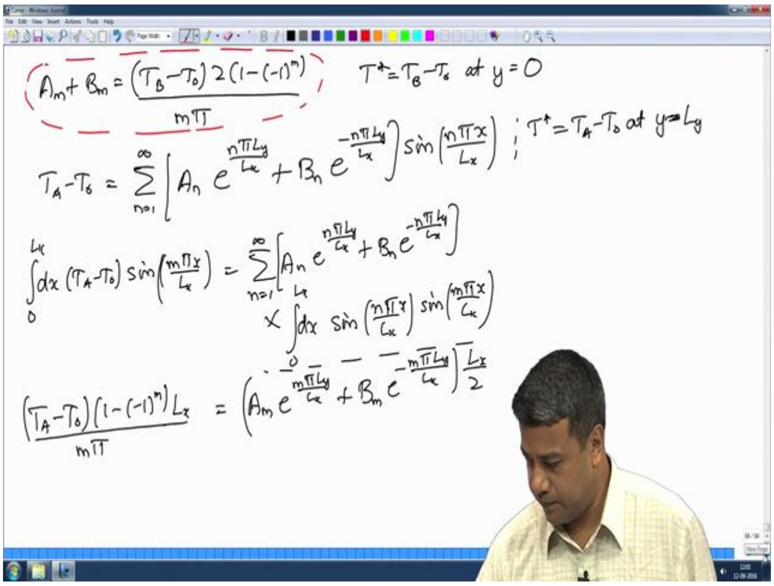
So, that is the equation and now I need to get an equation for A n and B n and how do I do that; I use the orthogonality relation integral d x from 0 to L x; sin of n pi x by L x. We had previously expressed it in terms of x star, so it was basically 1 over L x, in this case since the length in the two directions are different I have left it as length itself. So, so that is a slight simplification, but nevertheless you can do this integration and this will just give you L x by 2 times delta m n. So, this is non zero only when m is equal to n and when m is equal to n; this is equal to L x by 2.

So, what do I do multiply both sides of this equation by sin of n pi x by L x and integrate. From the left side you can do the integration; this will be equal to T B minus T naught into 2 L x by m pi, let me just confirm that for you; 2 L x by m pi and this is 2 L x by n pi for odd m and it is equal to 0 for even m.

So, you can actually write it simply as; so this factor here 1 minus; minus 1 to the power of m; if m is odd, this is equal to 2, if m is even; this is equal to 0. So, this integral is non zero only for odd m and is equal to 0 for even m. So, that is the value of the integral and on the right side I have A n plus B n into L x by 2 delta m n and this is of course, non zero only when n is equal to m; the right side is non zero only when n is equal to m. Therefore, I can write this effectively as A m plus B m into L x by 2 and this can now be solved therefore, A m plus B m is equal to T B minus T naught into 1 minus minus 1 power m; the L x will cancel out on both sides and get a factor of 2 here divided by m pi

So, I have got one equation that involves A n and B n; I have a m plus B m is equal to T B minus T naught into 2 into 1 minus; minus 1 power m by m pi. This is from the boundary condition T star is equal to T B minus T naught at y equals 0. I have one more boundary condition; the boundary condition at y is equal to L y.

(Refer Slide Time: 25:54)

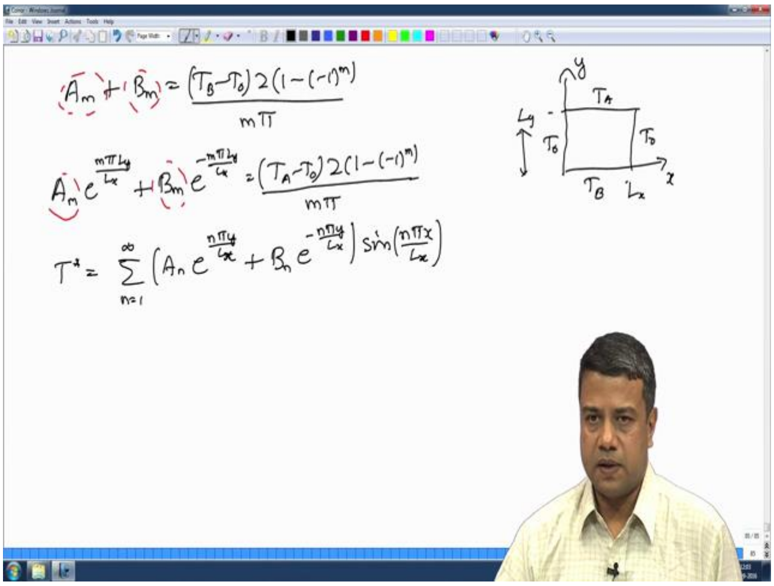


So, it enforces boundary condition at y is equal to L y, the same boundary condition at y is equal to L y and if I do that what I will get is; T A minus T naught is equal to summation n is equal to 1 to infinity into A n times e power; m pi L y by L x because this is at y is equal to L y, so instead of y I have to substitute L y plus B n e power minus n by L y by L x into sin n pi x by L x and now once again I have to substitute in terms I have to use the orthogonality relations; this is the boundary condition that T star is equal to T A minus T naught at y is equal to L y; so this is the second boundary condition.

So, once I can have to use the orthogonality relations; integral d x; T A minus T naught; sin m pi x by L x 0 to L x is equal to summation n is equal to 1 to infinity A n e power n pi L y by L x plus B n; this whole thing into integral 0 to L x; d x sin and you can simplify this and what you will end up with is; on the left side you will get T A minus T naught into 1 minus; minus 1 power n by m pi. Similar to what we had got previously and I should put a fact of L x, let me do the integral and I here I will get A m; e power m pi L y by L x plus B m; e power minus m pi L y by L x into L x by 2.

So, that is the equation that I get from the orthogonality relations by using the fact that this is non zero only when n is equal to m and with that I will get two simultaneous equations; the first one is this one, so if I write it once again.

(Refer Slide Time: 29:34)



And the other one is at A m; e power and these are two simultaneous equations that can be solved in order to calculate these constants A m and B m. The constants are a little complicated and form so I will not go through the details, but you can solve these equations both simultaneously and determine what the two constants are and once I have that, the temperature is just determined as summation n is equal to 1 to infinity of I am sorry. So, those are the solutions for this equation; for this heat conduction problem in which I had a distance L x in the x direction and distance L y in the y direction and this is y and I had different temperatures on the four different phases T naught. So, that is the procedure for getting this solution for this separation of variables problem. I will continue a little bit revise this once more and then we will proceed to looking at separation of variables in a spherical coordinate system in the next lecture; I will see you then.