Transport Processes I: Heat and Mass Transfer

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Lecture - 50
Balance equation: Convection and diffusion dominated regimes

Welcome to our continuing series of lectures on fundamentals of transport processes. In the previous few lectures we had derived conservation equations in different coordinate systems for both concentration fields as well as the momentum fields. For the concentration, we had actually gone through and derived the equations in some detail.

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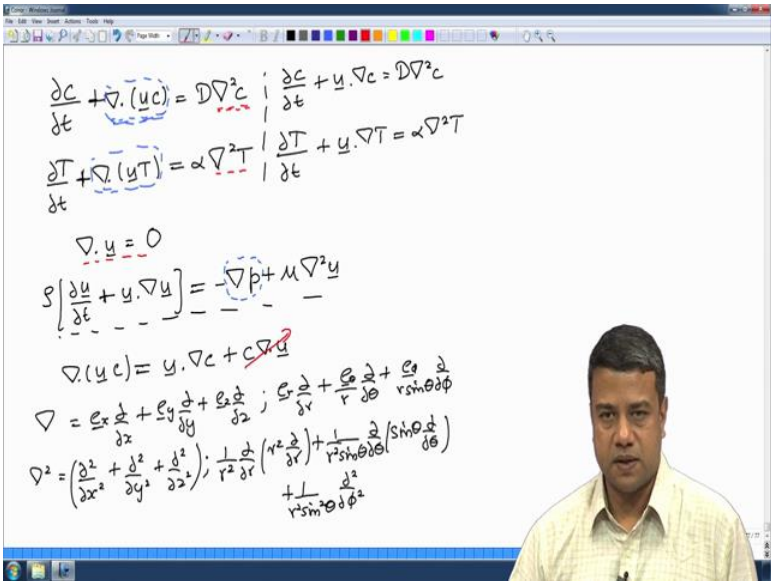

So, the equation was of the form partial c by partial t plus the divergence of the velocity times c is a dot product here is equal to D times the Laplacian of the concentration and for the energy equation, you will get a similar equation where alpha is the thermal conductivity and both of these are valid when the diffusion coefficient or the thermal conductivity and the specific heat are independent of position. All of these have a common structure, there is the divergence of the velocity times the concentration of the velocity times the temperature on the left side and there is a Laplacian of the concentration, the Laplacian of the temperature on the right side.

For momentum conservation equation, there were two equations because we have to write an equation for the total mass. The total mass equation was of the form the divergence of the velocity is equal to 0 and the momentum conservation equation was of the form; I should note that, if the fluid is incompressible these terms here is the blue terms can be alternately written, I can write down del dot u c is equal to differentiating using chain rule; u dot the gradient of the concentration plus concentration divergence of velocity and if the fluid is incompressible because of this mass conservation condition, the second term is 0.

Therefore, I can alternately write for incompressible fluids alone I can alternately write the concentration and the temperature equation as; written in this form the equation is exactly the same as the momentum conservation equation except as I said that I have this additional pressure gradient. Here this momentum conservation equation is for the three vector components of the momentum. So, there are three equations in here included within this one vector equation, I can take the three components of the velocity as u x, u y, u z or u r; u theta u phi and I can get the different scalar equations, for the different component in the different coordinate systems.

And I defined for you, in Cartesian and in spherical coordinate system or in a spherical coordinate system; it is e r; d by d r plus e theta by r. So, that is the expression for the gradient in a Cartesian coordinate system and the spherical coordinate system and the Laplacian was of the form; d square by d x square or in a spherical coordinate system, 1 by r square d by d r of r square. So, those were the operators that we had to write in Cartesian spherical coordinate systems, you just have to put in those operators into these equations to get the equations the different coordinate systems.

Note that this divergence operator or the gradient operator has dimensions of 1 over length, if the divergence of the gradient operator have dimensions of 1 over length, Laplacian has dimensions of 1 over length square because you are taking two derivatives with respect to the spatial position.

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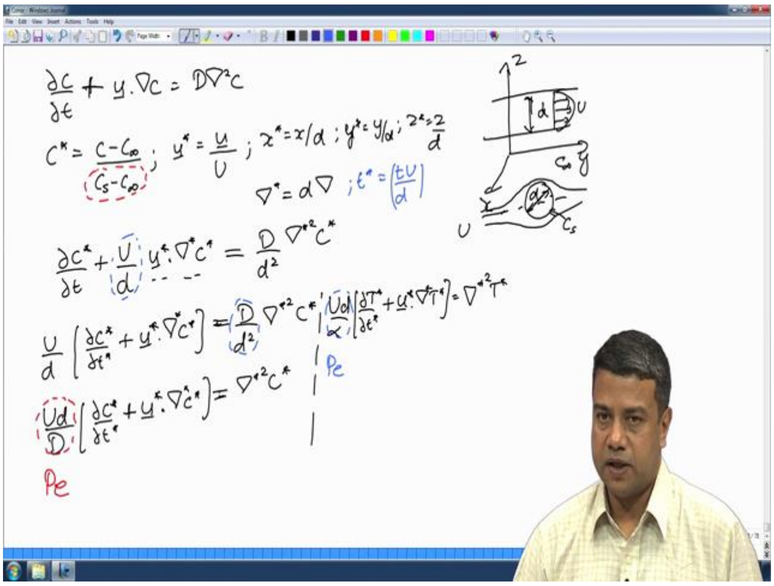

So, now if you look at these equations and try to scale them, I have an equation of the form d c by d t plus, if it is incompressible u dot grad c is equal to D del square c and if I have some configuration; such as for example, the flow through a pipe of some diameter d with some velocity u; average velocity u or the flow past a spherical particle of some other particle the characterized dimension in this case is the particle diameter d and the free stream velocity far from the particle has some velocity u and of course, if I have a mass transfer problem, where I have concentration c infinity far away and on the surface I have a concentration c s, as we have been doing in other such situations in these problems; one can write down a scaled concentration.

One can define the scaled velocity u star is equal to u by the characteristic velocity capital u which is the flow through the pipe; average velocity of the flow through the pipes for internal flows or flow around the object for external flows. I can also scale the length scales, in general I would scale the length scales as x star is equal to x by d; y star is equal to y by d and z star equal to z by d. Since the gradient operator has dimensions of inverse of length, it is d by d x times d x plus d by d y times e y plus d by d z times e z. Our similar expressions in other coordinate systems, I can write a scaled gradient operator as D times (Refer Time: 09:13) this is dimensionless because the gradient operator has dimensions of 1 over length.

So, if I do that the equation is linear in the concentration field therefore, I will get partial c star by partial t plus I have a velocity here times d, times u star dot grad the scaled gradient operate and c star. So, basically since I am scaling all length by d this operator also gets scaled by the inverse of d. So, defined this way this time is dimensionless and on the right side, I have two derivatives; that means, I will get two factors of D in the denominator. So, I will get d by D square del star square c, when I go from c to c star I get this factor coming out and I change the variable from c to c star I get the factor of c s minus c infinity in all the derivatives, but since the equation is linear in c that cancels out on all terms. Now this gives me a way of scaling that time scale, the time scale for variation; a natural time scale in this case would be this u by d, the time scale for the variation.

And if I do that; what I will get is t times u by d is now dimensionless therefore, I will get u by d into partial c by partial t and if I divide throughout by this factor here, what I will get is and this is the Peclet number for mass transfer. This is the general structure of the equations of motion; this is how the equations of motion generally look.

If I were to do the same for heat transfer, I get exactly the same. So, exactly analogous I substitute t for c and alpha for d and what you would get is u d by alpha into partial t by partial t. So, this is the Peclet number for heat transfer and I could do the same for momentum transfer, in this case the only variable that I have is the velocity.

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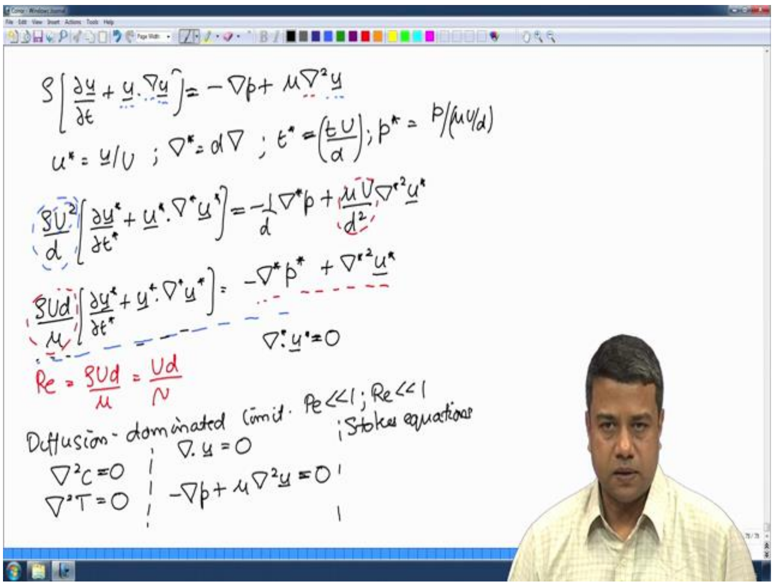

If I were to do that for momentum transfer, the equation is slightly more complicated, but you can scale it nevertheless. The equation is rho into partial u by partial t plus u dot grad u and if I define u star is equal to u by the characteristic velocity; grad star is equal to d times grad; t star is equal to t u by d, what I would get on the left side is going to be equal to rho u square by d. You can work it out, there are two factors of u here which gives you a factor of u square and there is one gradient which gives you 1 over t.

So, this is partial u star by partial t star minus 1 over d, I have not yet scaled the pressure I will do that shortly and on the right side, I have mu; I have one factor of velocity due to the velocity and there are two gradients. So, that gives me a 1 over d square; so that is the scaled equation. Now I can divide the equation throughout by either of these coefficients, either the viscous coefficient or the inertial coefficient; which one you choose will depend upon what is dominant. For the present, I will choose to scale it by the viscous scale and if I do that; on the left side I get rho u d by mu; choose to scale it by the viscous scales.

And of course, I can use this to scale the pressure and define p star is equal to p by mu u by d, then this equation the pressure term just becomes minus the gradient of the scaled pressure and on the left side, we have the dimensionless number, the Reynolds number divided by the kinematic viscosity or the momentum diffusivity. So, all of these equations have a common structure, on the left side you have the Peclet number or the Reynolds number times the time derivative and the convective terms. On the right side, you have the diffusive term in all of these equations one can consider two limits; one is where the Peclet number is very small, when it is very small diffusion is dominant and basically you are solving an equation which basically states the Laplacian of the concentration of the temperature field is equal to 0.

In the case of the momentum conservation equation, it is a little more complicated because you have the pressure gradient as well and this has to be coupled with the mass conservation condition; del dot u is equal to 0, but in both cases the diffusion operator is a Laplacian operator. On the other hand in the limit where the Reynolds number or the Peclet number is large basically these convective terms will balance the pressure term in the case of the momentum equation or in the case of the mass and energy equations, the convective term on the left side is much larger than the diffusion term on the right side.

So one can understand two limits; one is the diffusion dominated limit, this happens when the Peclet number is much small compared to 1 or when the Reynolds number is much smaller than 1. In that case diffusion is dominant; convection is negligible compared to diffusion and in that case the conservation equations for concentration or temperatures basically reduced to del square c is equal to 0 or del square t is equal to 0. So, both the concentration and the temperature field in this case have to be solutions of the Laplace equation. For momentum transport, it is slightly more complicated; del dot u is equal to 0 is the mass conservation condition and if I neglect the inertial terms in the limit of the Reynolds number being small, the conservation equations reduce to minus grad p plus.

So, these have to be solved subject to boundary conditions, these momentum conservation equations they call stokes equations in the limit of low Reynolds number, in the limit where convection is dominant.

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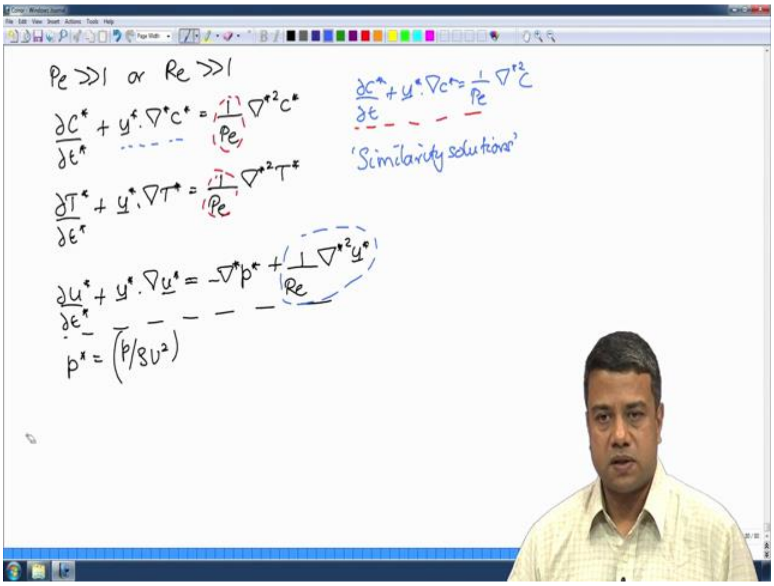

When the Peclet number is large compared to 1 or Reynolds number is large compared to 1; I could divide the equation for example, the concentration equation throughout by the Peclet number. So the concentration equation if I divide by the Peclet number, I will get an equation of the form. So, that will be the conservation equation in the limit of high Peclet number for the concentration field and an analogous equation for the temperature field.

In this case, you would think simplistically Peclet number is large therefore, this number is very small. So, I can just neglect the right side of the diffusion equation and solve an equation of the form partial c by partial t plus u dot grad c equals 0. You just go ahead and solve that to get the temperature or the concentration field, a similar equation for the temperature field; however, when you do that the original equation that you had was the second order differential equation; the highest derivative is in the diffusion term.

Therefore, the original equation was a second order differential equation in each of the spatial coordinate; that means, that to completely specify the original problem. You need two boundary conditions in each special coordinate, if you assume Peclet number is large and just neglect that second order term; the equation just reduces to a first order differential equation along the velocity direction u dot grad, the variation along the direction of the velocity is first order.

And you can no longer satisfy both boundary conditions, therefore, one cannot solve an equation like this and still satisfy all the boundary conditions in the original problem, you will end up with an inconsistency. We saw that when we were solving the oscillatory flow in a pipe, when you reduce the order of the equation boundary conditions can no longer be completely specified. That is the mathematical reason why a solution of this kind is not correct. Physical reason is that, this convective term transports mass or energy only along the direction of the fluid flow. At solid surfaces or liquid gas surfaces due to the no penetration condition; there is no fluid velocity perpendicular to the surface; therefore, transport perpendicular to the surface cannot take place due to convection.

Transport perpendicular to the surface has to take place only due to molecular diffusion. Therefore, even in the limit of high Peclet number; it is still necessary to include the effect of diffusion within thin boundary layers close to surfaces in order to be able to predict what is the transport rate across those surfaces. As the Peclet number increases, the thickness of those boundary layers decreases, the gradients increase in such a way that within that boundary layer; the concentration field when scaled bothered by the boundary layer thickness if you scale the coordinate perpendicular to the surface by the boundary layer thickness, the concentration field is independent Peclet number even in the limit as Peclet number going to infinity.

So what you need to do is to include the effect of diffusion very close to surfaces by rescaling. The distance from the surface within the boundary layer of thickness whose thickness is determined from the requirement that convection and diffusion have to be of equal magnitude within this boundary layer even as the Peclet number goes to infinity and the boundary layer thickness goes to 0, even as the Peclet number becomes larger and larger the boundary layer thickness will become smaller and smaller in such a way that convection and diffusion of equal magnitude within that boundary layer. So, we will be using similarity solutions here.

A similar situation holds for momentum transfer; the equation if I scale it by the inertial scales. In this case; in contrast to what I had defined earlier in the earlier case I had scaled the pressure by the viscous scales. In this case vetting the limit of high Reynolds number, the pressure has to be actually scaled by the inertial scales. I will not go through the derivation, if you scale the pressure by the inertial scales and simplify the equation; you will get an equation of this kind. Once again in the limit of high Reynolds number, you would think that you can neglect this viscous term in the conservation equation; that is true in most of the bulk of the fluid flow. You could solve this equation without take into account the viscous term; however, close to the surface; the retardation of the fluid close to a surface near the solid surface; the velocity has to be equal to 0.

Close to the surface, it is the stress exerted by the surface that actually slows down the fluid and enforces the no slip boundary condition at the surface, if we neglect it; the viscous term you cannot enforce the 0 tangential velocity boundary condition at the surface because the velocity comes to 0 because of momentum diffusion from the surface. So, one cannot enforce the 0 tangential velocity boundary condition, in order to enforce the 0 of tangential velocity boundary condition; one has to postulate a boundary layer once again; a momentum boundary layer close to the surface, whose thickness decreases as the Reynolds number increases in such a way that there is a balance between convection and diffusion in the limit of the Reynolds number going to infinity within this boundary layer.

In this course, I will deal exclusively with mass and heat transfer. In the next lecture start looking at diffusion dominated transport as I said, the Laplacian of the concentration of the temperature is equal to 0. So, these are basically ways of solving equations for the Laplacian of something equal to 0 and I will go through the ways of solving that starting in the next lecture for diffusion dominated flows. In all these cases, whenever the Laplacian is equal or some function is equal to 0, the method of solution is by separation of variables. We had seen that for unsteady diffusion into in one dimension, similar procedures are applicable even for steady diffusion in multiple dimensions.

I will first go through an example of how you solve a problem for steady diffusion in two dimensions using separation of variables and then we will look at separation of variables in a spherical coordinate system. So, that is going to be the broad program for the rest of this series of lectures. First how do you solve the limit of 0 Peclet number or Reynolds number when diffusion is dominant and then how do you solve in the limit of high Peclet number when convection is dominant. We will continue; we will start diffusion dominated flows in the next lecture, I will see you then.