Transport Processes I: Heat and Mass Transfer
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Lecture – 49
Momentum balance: Incompressible Navier-Stokes equations

This is our continuing series of lectures on fundamentals of transport processes, heat, mass and momentum transfer, in the last few lectures; we have been looking at how to derive conservation equations for the concentration or temperature fields for different coordinate systems. So, the idea is that if we know what is the conservation equation for some fundamental set of coordinate systems, for each one of these you can derive the conservation equations and then apply them based upon the configuration of the problem that we are considering, so that was the basic idea.

We had looked at the balance equations first in a Cartesian coordinate system, where we had written for the concentration equation the change in mass in a time t is mass in minus mass out plus, the acceleration terms, accumulation terms and each of these has convective and diffusive parts, if you recall we had written down these equations for a volume whose surfaces are surfaces of constant coordinate perpendicular to the the coordinate axis.

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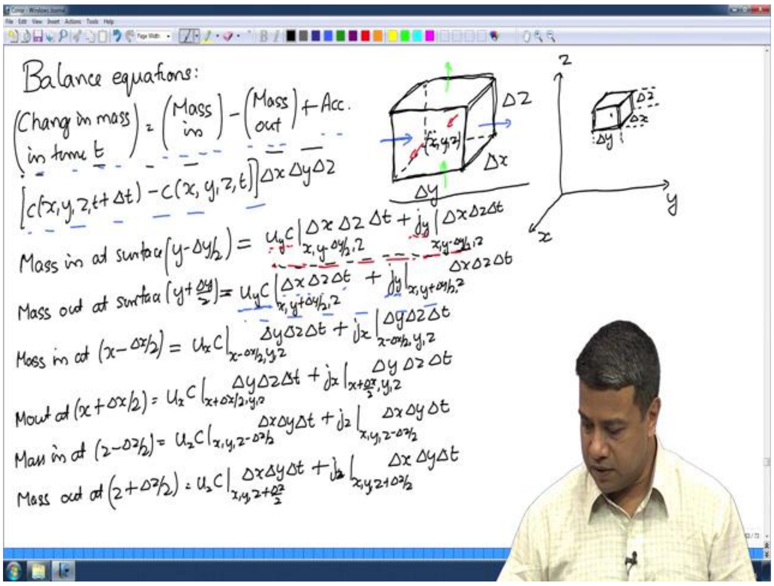

And by doing this we had written down balances consisting of mass in and out at different surfaces, each one consisting of a convective part which is the velocity times concentration and a diffusive part which is the diffusive flux.

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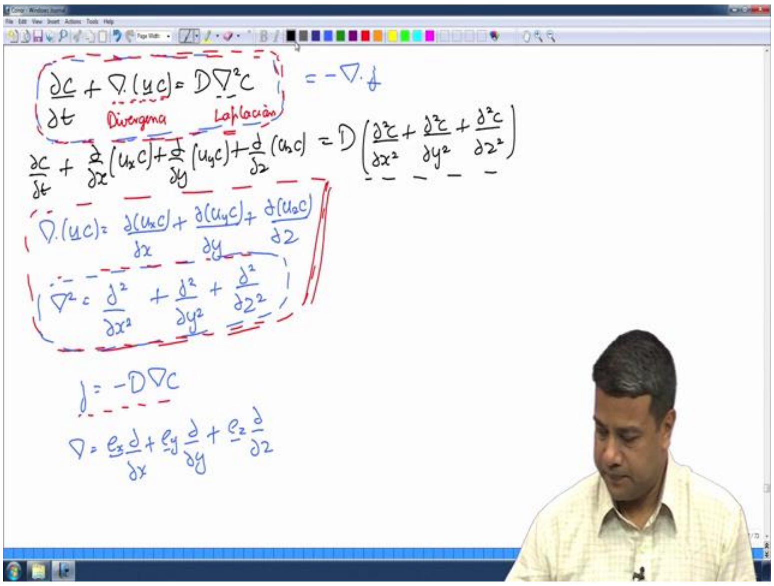

And we had put all of these together and got a differential equation, the differential equation consisted of two sides the left side had the time derivative and the convective part. The convective part was the divergence of a velocity times the concentration. So, that was the convective part and I had expanded out that convective part in a Cartesian coordinate system for you.

The diffusive part was the diffusion coefficient times the laplacian of the concentration that once again, the laplacian is the second derivative in the 3 directions the diffusion flux is equal to minus d times the gradient of the concentration the gradient is a vector gradient operator the gradient is written as gradient is equal to e x d by d x plus e y d by d y where e x e y and e z are the unit vectors in the 3 coordinate directions.

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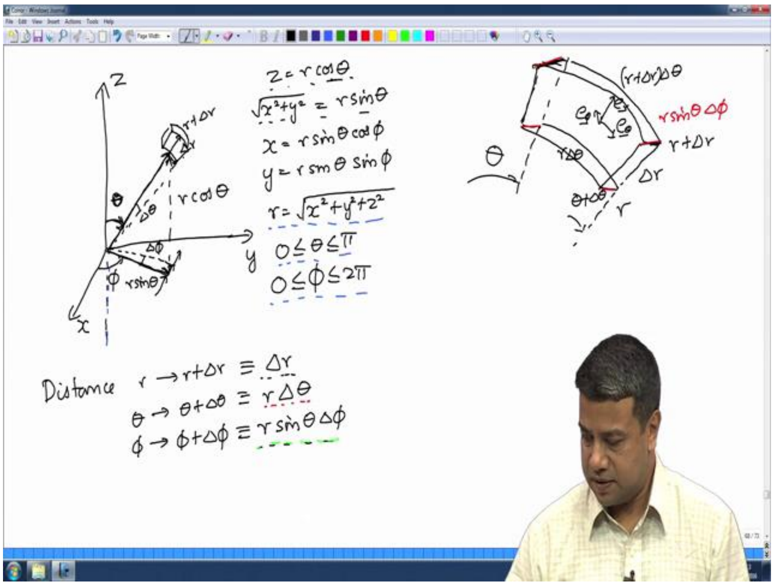

We had done a similar derivation for a spherical coordinate system where the coordinates are the radius distance from the origin, the azimuthal angle, the angle from the z axis, the north south axis, if you will and the third one was the meridional angle; the angle around the z axis the angle in the x y plane made by the projection of the radius vector with the x axis and for each of these 3 coordinates, we can choose a differential volumes such that you have surfaces perpendicular to the coordinates as the surfaces of this volume that gives you 6 surfaces. You write down balance equations, the flux consisting of you have surfaces at different locations of in this coordinate system.

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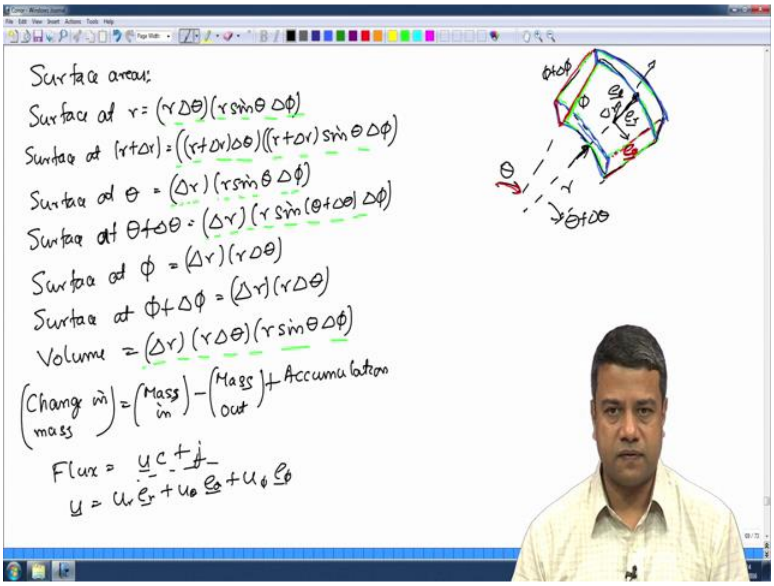

The flux is written as the sum of a convective flux due to the velocity field and the diffusive flux and based upon this we got the same differential equation.

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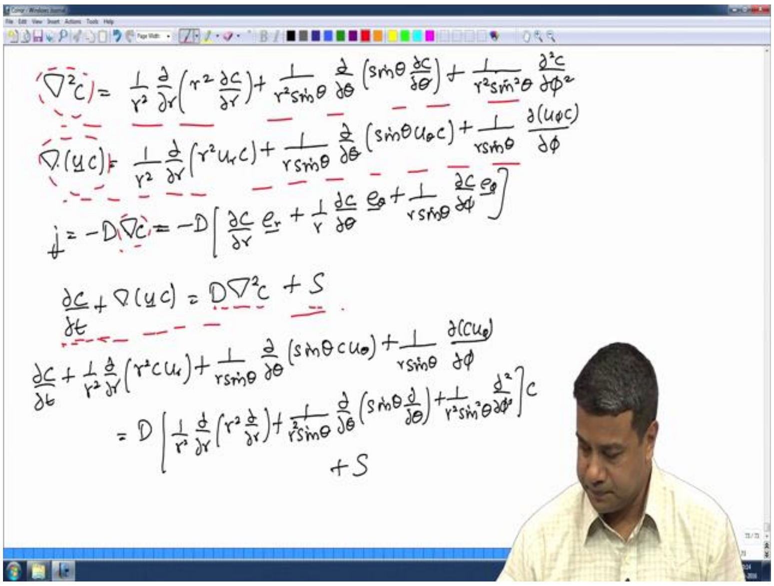

The differential equations written in vector notation was the same the definitions of the divergence the laplacian are now different because in this spherical coordinate system. The surfaces of perpendicular to the coordinate axis are not surfaces of constant area the area changes as the coordinate changes and that has to be taken into account in the definition of the divergence and the laplacian. Other than that the procedure is exactly the same we had got the conservation equations is the concentration field.

Now, the only equation that I have not yet derived for you is the momentum balance equation. The momentum balance equation is a little more complicated because the momentum itself is a vector and therefore, you have two vectors - one is directions two vector directions one is the direction of the momentum itself the other is the direction of the transport of momentum which is why the stress tensor that we had was a tensor it was it had 9 components, the first one corresponding to the direction of the momentum the second one corresponding to the direction of the transport momentum.

We have to, if we want to derive momentum conservation equations we have to derive momentum conservation equations along 3 different directions, 3 scalar equations that is a little tedious to do in any coordinate system with it is Cartesian or spherical or others therefore, what I will do here is to just write down the momentum conservation equation for you in vector notation. Now that we know what are the definitions of these operators the divergence, the gradient and the laplacian, we can use those definitions in the different coordinate systems we had already derived those definitions based upon our understanding of the mass and energy conservation equation. So, those operators can now be used in the momentum conservation equation as well.

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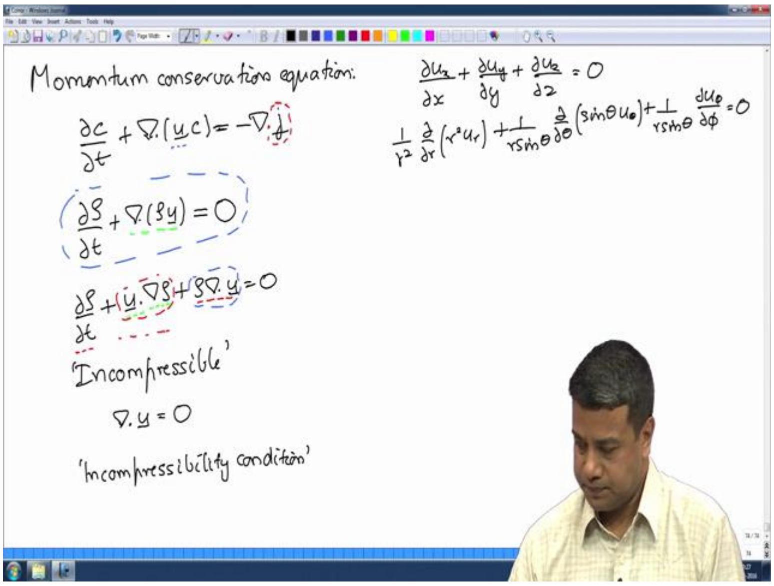

But first, let me write down the conservation equation for the total density for you. I had written down the concentration equation as d c by d t plus the divergence of the velocity times c is equal to minus the divergence of the mass flux. Now if I were to write an equation for the total density this equation is for the concentration of one particular species in the fluid, if I were to write the conservation equation for the total density the total density conservation equation that is the total mass conservation equation not just the mass conservation for a particular species, but the total mass will be of the form d rho by d t plus del dot rho u. The left hand side you will get exactly the same if you do the balance over the 6 phases of a cube in a Cartesian coordinate system or a spherical shell in a spherical coordinate system the left hand side is exactly the same.

Note that u vector is the center of mass velocity u vector is the center of mass velocity it is the velocity moving with the center of mass with the fluid, in a reference frame moving with the center of mass of the fluid there can be no diffusion of mass - diffusion is a flux of one particular constituent relative to the center of mass. If I have one particular species of fluid in I am sorry, one particular species within the fluid the flux basically tells you how that particular species the mass of that particular species is moving relative to the center of mass of the fluid, after all u is the center of mass velocity. So, in the reference frame moving with the center of mass there should be no relative motion of the total fluid, but you get the mass of the whole fluid itself.

Therefore on the right hand side, this will just turn out to be 0 because the velocity u as defined is already the velocity of the center of mass therefore, if you are writing a conservation equation for the total mass there should be no motion relative to the center of mass therefore, the mass conservation equation just reduces for the total density to this form. I can expand this out plus you can think of this expansion in a Cartesian coordinate system for example, if I take this term del dot rho u this is basically d by d x of rho u x and I can differentiate using chain rule.

First I will get u x partial rho by partial x plus rho into partial u x by partial x and this first term is this one this first term is this one, what I get by expanding by chain rule and the second term is the second term here so that is just an expansion of this first term, here using chain rule for differentiation and a similar procedure can be done in spherical coordinates or cylindrical coordinates or any other coordinate system as well you can always expand it out. And it is important to note that this operator here is a gradient operator, it is acting on a scalar that first operator is basically grad rho is equal to e x partial rho by partial x plus e y partial rho by partial y plus e z partial rho by partial z, it is acting on the density, it is a gradient operator when you operate on the density, you get a vector that vector is now being dotted with the velocity which is also a vector to give you a scalar. So, you have the dot product of two vectors basically giving you a scalar.

On the other hand, the last term on the right, it is actually a divergence operator the last term on the right is actually a divergence operator if I write it in a Cartesian coordinate system l dot u is equal to partial u x by partial x plus partial u y by partial y plus partial u z by partial z. Both the gradient here and the velocity are vectors and you are taking the dot product of those two vectors so that gives you a divergence which is a scalar. So, that is a distinction between these two operators. The first is a gradient the gradient of the density gives you a vector and you are then dotting it with a vector velocity to get a scalar.

On the other hand, in the last term on the left side the divergence of the velocity the dot product of the gradient operator and the velocity vector that gives you a scalar now in most practical applications that we encounter the density is approximately a constant, these are called incompressible flows incompressible essentially means that the density does not change either with time or with position the density is a constant the assumption of incompressibility is valid when the velocities in the flow are much smaller than the speed of sound or when the mach number is small because density waves are transmitted at the speed of sound these are what are called pressure waves which are transmitted at the speed of sound.

Whenever the velocity is much smaller than the speed of sound you can consider the system to be incompressible that is the density is a constant. This approximation of incompressibility is valid for all liquid flows and for almost all gas flows unless the application is at supersonic speeds where the velocity exceeds the speed of sound. So, in all such cases where the velocity is smaller than the speed of sound we can assume that the flow is incompressible and what that means, is that there are no variations in density either with time or with position.

And in that case, you can neglect these first two terms in the equation because there is no variation in the density and the mass conservation equation basically reduces to the density times the divergence of velocity is equal to 0. I can divide both sides by the density and therefore, the incompressibility condition this is called the incompressibility condition, basically reduces to divergence of velocity is equal to 0. So, that is the mass conservation equation for Newtonian fluid, in a Cartesian coordinate system for example, you would write this as in a Cartesian coordinate system you would write this as partial u x by partial x plus partial u y by partial y plus partial u z by partial z is equal to 0 divergence of velocity is equal to 0. In a spherical coordinate system this is slightly more complicated as we had seen in the previous lecture when we derived the conservation equation in a spherical coordinate system it becomes of the type one by r square d by d r of r square u r.

We had derived it in the previous lecture for the flux; we had derived the expression for the divergence of the flux in the previous lecture. Similarly for the divergence of the velocity, you just have to substitute the components of the velocity for the components of the flux and this is the mass conservation equation for an incompressible fluid that you will get. What about the momentum conservation equation? As I told you earlier momentum is a vector and therefore, we have to write down a vector equation for the momentum conservation equation.

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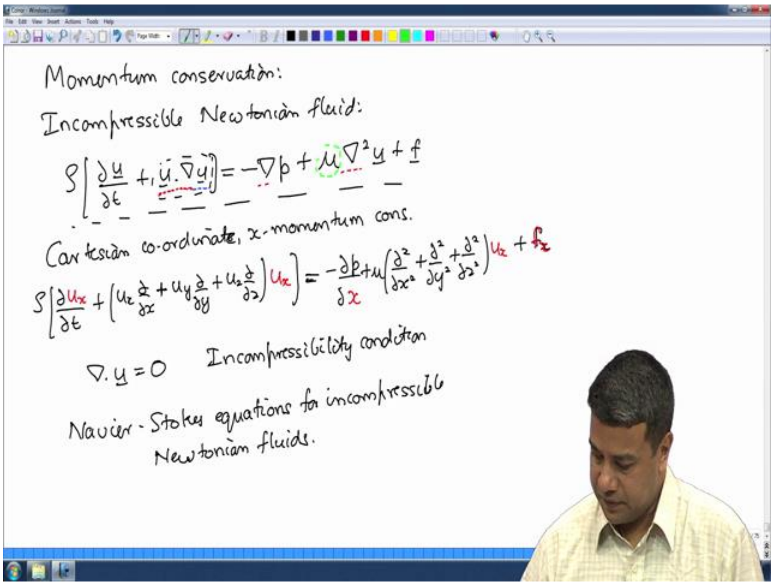

I will not go into the derivation of this I will only give you what is the final result for the momentum conservation equation for an incompressible fluid.

Then momentum conservation for an incompressible Newtonian fluid, this is a vector equation as I told you and it has the form the density into partial u by partial t plus u dot grad u is equal to minus gradient of the pressure plus mu del square u plus any forces. So, this is a vector equation there are 3 equations in there this term here - if I were to write it in a Cartesian coordinate system note that the dot product is between the velocity vector and the gradient the dot product is between the velocity vector and the gradient therefore, if I were to write the x momentum conservation equation for example, the Cartesian coordinate in x momentum conservation.

The first term is partial u vector by partial t the x component of that it is going to be equal to partial u x by partial t. The second term there is a dot product between the u vector and the gradient operator. So, if I take the dot product between u vector that is u x e x plus u y e y plus u z e z and the gradient operator what I am going to get is u x partial by partial x plus u y partial by partial y times this operates on u vector this operates on u vector. So, if I am taking the x component of u vector this becomes just u x.

The first term on the right is minus the gradient of the pressure the x component of that is equal to minus partial p by partial x the last one is mu times del square of u, u vector is the vector. Del square the laplacian is a scalar the laplacian is a scalar. So, if what I will get is t square by d x square plus d square by d y square plus d square by d z square u x plus f x. So, this is the momentum conservation equation for a Newtonian incompressible fluid.

Note that the direction comes in at different locations here the first term here the direction comes in the direction of the velocity itself. The first term the direction comes in the direction of the velocity, so if it was the y momentum this would have been u y. The second term the direction comes in this term here if the velocity vector itself therefore, the direction comes in here in the velocity u x this u dot grad this scalar operator are operates on u x. In the pressure gradient the direction comes into the direction of the gradient in this viscous term here and I should apologize I should put in a viscosity there I forgotten the viscosity the direction comes into the direction of the velocity.

I am taking the laplacian of the x component of the velocity and in the force of course, the direction comes in the forcing term when you go to the y equation, these are the 3 that you would change the red colored ones are what you would change from u x to u y the rest remains the same and similarly from u x to u z so that is the momentum conservation equation. If you wanted to go to a spherical coordinate system all you would have to do is to change the velocity vector from u x u y u z to u r u theta u phi the gradient operator from u x u y u z to u r u theta u phi and this laplacian operator from u x u y u z to u r u theta u phi and the components of the velocity will also change and there are standard formulas that are available in textbooks for each component of the momentum conservation equation so this combination of the incompressibility condition.

This combination of the incompressibility condition and this momentum conservation equation are what are called the Navier stokes equations for incompressible Newtonian fluids, I should caution here that in the derivation of these equations, we are assuming one that if the density is a constant it is independent of time it is independent of space and the other we are also assuming that the viscosity is actually a constant when we derive these equations we are also assuming that the viscosity is a constant and finally, for a Newtonian fluid I will also briefly write for you what is the stress tensor.

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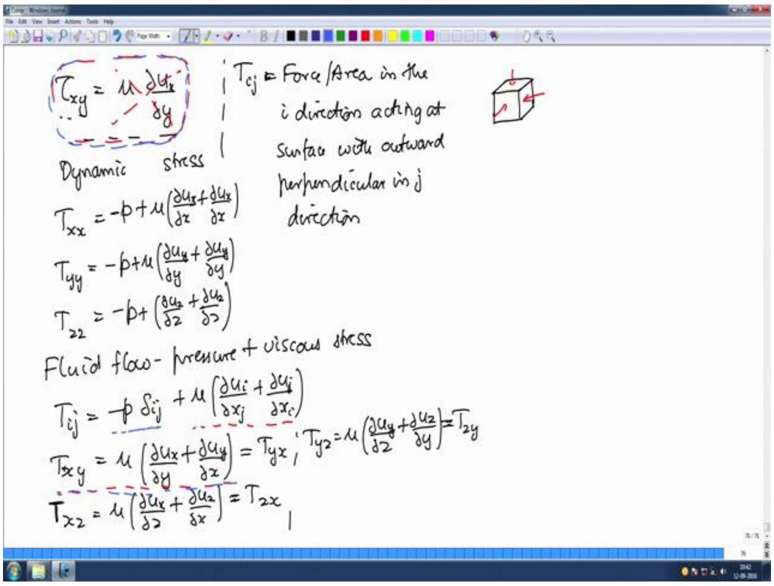

So far we had just been writing tau x y is equal to mu times partial u x by partial y, it turns out that this is valid only for unidirectional flows where the velocity is only in the x direction and the velocity variation is only in the cross stream y direction, this is not a general formula. If I take the differential volume, the force per unit area on the surfaces of this volume will actually consists of 2 parts, one is the component perpendicular to the surface due to the pressure as I said the pressure acts on all surfaces and it is compressive it acts perpendicular to the surface it acts along the inward unit normal to the surface it is compressive. So, it acts along the inward unit normal to the surface.

Therefore even if the fluid was stationary the pressure would still be present it would be acting along the inward unit normal to each surface; that means, that even for a system that is at rest the stress tensor hydrostatic stress you will find the t x, x is equal to p minus p. Note that I am defining here the stress t i j is equal to force per area in the i direction acting at surface with outward perpendicular in j direction that is my definition, the force per unit area acting at the surface force is in the i direction the unit normal is in the j direction.

For example, in this case, the stress for example, the momentum was in the x direction that is the force was exerted in the x direction the unit normal to the surface was in the y direction. So, the transport was in the direction it was across the surface and they are perpendicular to the surface was in the y direction. So, t x x force per unit area in the x direction acting at a surface whose unit perpendicular is in the x direction due to pressure that force actually acts in the minus x direction therefore, t x x is minus p because t x x is assumed to be positive. If unit normal is along the outward so, unit normal to the surface - t x y t x z and so on will be 0, t y y will also be equal to minus p and t z z will also be equal to minus p that is the hydrostatic stress.

What about when there is fluid flow when there is fluid flow you have pressure plus viscous stresses and for a Newtonian fluid the expression for the stress is t i j is equal to minus p delta i j plus mu into partial u i by partial x j where i and j could be x y or z where i and j could be x y or z. This first term here is the hydrostatic term you recall the delta i j is 1, if i is equal to j and it is 0, if i is not equal to j. So, if I had no fluid flow I would just get this term which means that t x x, t y y and t z z will be equal to minus p all other components are 0.

The second term here is the viscous stress, the second term here is the viscous stress and this depends upon the velocity gradient. So, in the presence of a velocity gradient if I have to write t x x for example, I would have a second term here. So, I will write this as the dynamic stress I would have a second term here which is plus mu in this case, i is x and j is x, partial u x by partial x plus partial u x by partial x. This one i is y and j is y, so I will get partial u y and then I will get additional terms where which are when x is t x y for example, the force per unit area in the x direction acting at a surface whose unit normal is in the y direction in that case there is no pressure contribution because delta I j is 0 when i is x and j is y, but I get a second term partial u x by partial y plus.

And you can see that this is also equal to t y x. So, it is symmetric where you take t x y or t y x you get the same results similarly t x z is equal to mu into partial u x by partial z plus partial u z by partial x t z x and similarly for y z. So, this is the actual expression for the stress tensor a generalization of this one this expression that I had here which I had used is valid only when u y is equal to 0 and u x is only a function of y whereas, this one is the more general expression for the stress tensor for a Newtonian fluid; incompressible Newtonian fluid.

I just briefly given you a description of the conservation equations, the momentum conservation equation, the mass conservation equation for an incompressible fluid, the mass conservation equation reduces to just a condition with the divergence of the velocity is equal to 0 that expression, we had derived earlier in different coordinate systems and the momentum conservation equation has this form rho into d u d t plus u dot grad u is equal to minus the gradient of the pressure plus mu del square u plus f where we have already discussed how these divergence gradient and laplacian operators are formulated in the different coordinate systems.

I should once again caution, this is valid only for incompressible fluids, density is a constant, viscosity is a constant, only then are these equations valid, why they are valid? They have a form that is remarkably similar to the concentration in temperature equations laplacian with the velocity on the right side and the velocity derivative plus u dot grad u on the left side. The additional piece here which is not present for mass and energy conservation is the gradient of the pressure and the other important distinction is that this term here is a non-linear function of the velocity for concentration and temperature equations this was a linear function of concentration temperature in this case, it is a non-linear function of the velocity that is actually what causes multiple solutions the transition to turbulence and so on and that is why fluid flows are more complicated than heat and mass transfer.

Heat and mass transfer, what is being transported is different from, what is doing the transporting? What is being transported is mass and energy, the transportation is carried out by the fluid flow, in the case of the convection by the fluid velocity field. In this case, the velocity the momentum that is being transported is the same as the velocity that is transporting it and that is why it is non-linear. So, we derived these equations and now, we will look at how to solve these equations in different limiting cases in particular in the cases where the diffusion is dominant first and then in cases where convection is dominant. We will continue this in the next lecture, I will see you then.