Transport Processes I: Heat and Mass Transfer
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Lecture – 47
Mass and energy balance equations in spherical co-ordinates

We were deriving balance equations for concentration energy in the previous lecture for a general situation where there are variations in the concentration or the temperature field in all 3 spatial directions as well as in time. The basic law was that the change in mass in a time delta t is equal to mass in minus mass out plus acceleration.

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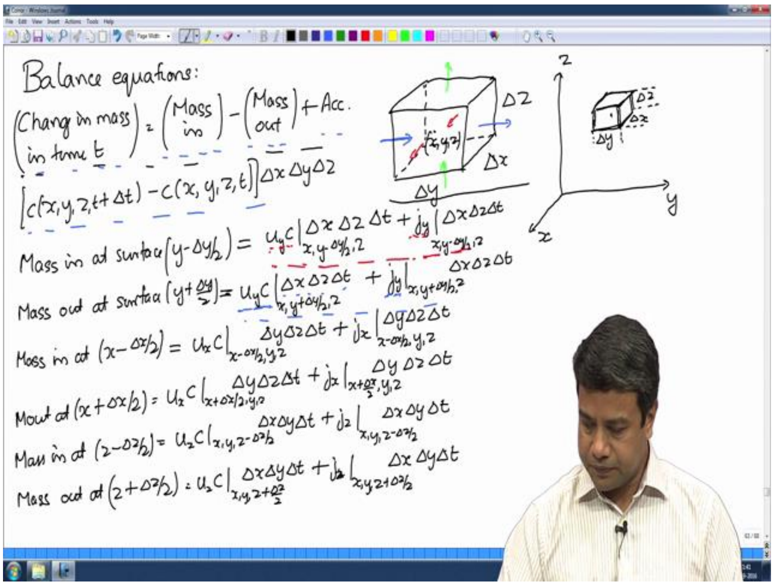

We had first done it for a Cartesian coordinate system where the x, y and z axis are perpendicular to each other, we had chosen a cuboidal volume whose surfaces are surfaces that are perpendicular to the coordinate axis. So, their surface is of constant coordinate and we had written a balanced equation for one such surface mass in minus mass out plus accumulation is equal to the change in mass in a time delta t. There are 6 phases, so there is mass in and mass out at 6 phases.

On 3 of these phases if the velocity is positive or the flux is positive, the mass is in and it increases the mass and on 3 others it is out and if they whole decreases the mass, mass in is going to be equal to the flux times the area times the time. Flux could be either due to convection or due to diffusion, the convective flux is the fluid velocity perpendicular to the surface times the concentration and the diffusion diffusive flux is given by Fick's law for diffusion or Fourier's law for heat conduction.

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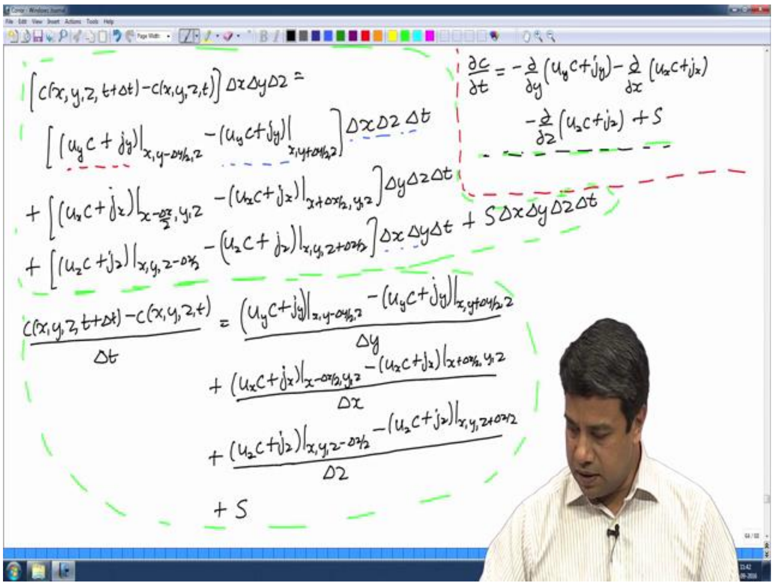

And on that basis, I had derived an equation of this kind for you, the conservation equation and written it more compactly as follows is. So, this is the conservation equation slightly rewritten.

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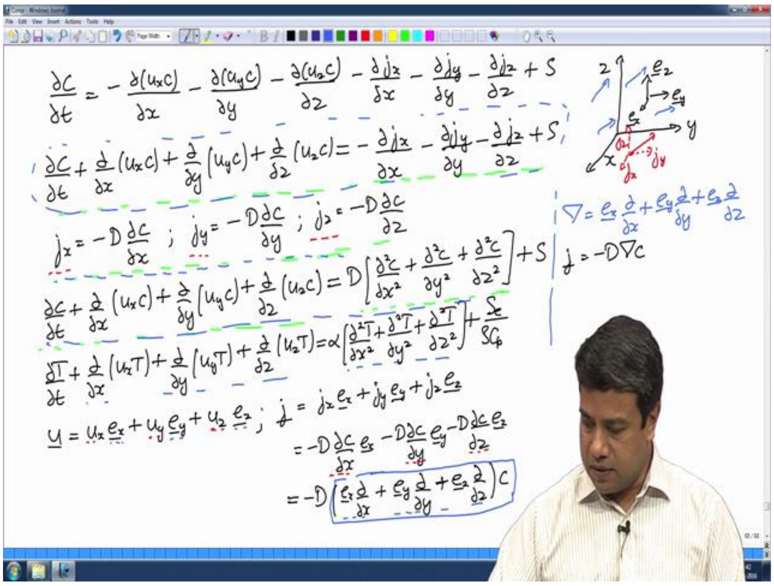

So that the rate of change of concentration and the convective transport appears on the left side and the diffusive transport appears on the right side and then I write the flux in terms of the grid the derivatives of the concentration field and get an equation for the for the concentration field that looks something like this.

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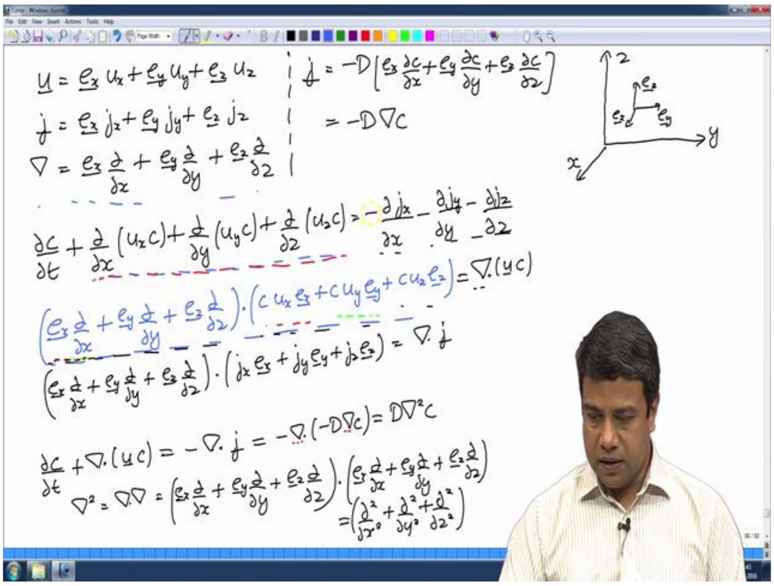

I would also derive for you rewritten this a little bit by defining vectors velocity vector of course, we know the velocity is a vector it has a direction and a magnitude at any point. The flux can also be written as a vector the flux as 3 components in the x, y and z directions which depend upon the derivatives in the x, y and z directions therefore, I can write the vector flux as the unit vector in the 3 directions times the component that gives a flux vector the resultant flux vector at a given location and I had also derived for you a vector derivative and in terms of this the conservation equation was quite easy.

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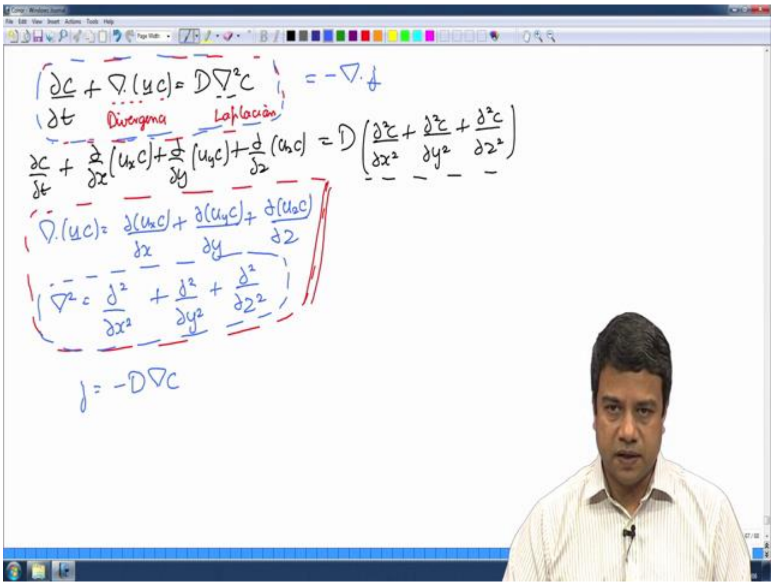

The conservation equation just contains the time derivative the divergence of the velocity times concentration, there is the convective part and the diffusive part which contains the laplacian of the concentration field.

The diffusive part is basically minus the divergence of the vector flux and that vector flux can be written terms of the diffusion coefficient using Fick's law for diffusion and based upon that we have got an equation which contains the divergence of the velocity times concentration on the left side and the laplacian of the concentration del dot del on the right side. So, those are the conservation equation. This is for a Cartesian coordinate system where the laplacian is at the second derivative in the 3 directions added up in general that is not the case and to illustrate that I had taken for you the example of a spherical coordinate system.

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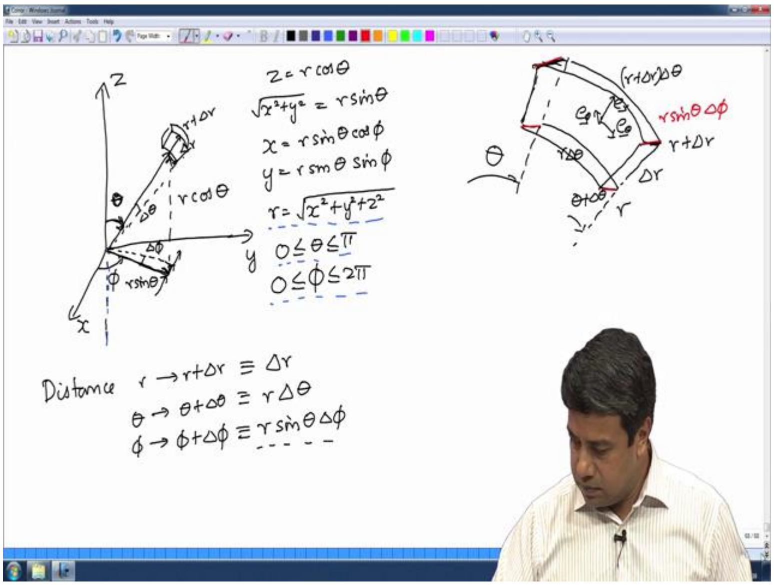

I just defined the coordinate for you in the previous lecture. Spherical coordinate system is useful whenever you are dealing with a configuration where the bounding surfaces have a spherical symmetry. I will show you a little later that the interpretation is a little more general there are many things that you can do in a spherical coordinate system which are not that easy to do in a Cartesian coordinate system.

But in general, it is useful whenever you have surfaces of spherical symmetry in that case you would prefer the surface to be a surface of constant coordinate; that means, that the distance from the center from the origin has to be one coordinate the center of the spherical geometry. So, that distance is the radial coordinate r. However, you need 2 other coordinates in order to completely specify a position in 3 dimensional space and those 2 other coordinates are expressed as angles.

The first one is the azimuthal angle theta which is the angle from the plus z axis theta is equal to 0 is the plus z axis and theta is equal to pi is the minus z axis theta is equal to phi is for minus z axis and; obviously, once you go all the way to the minus z axis you can go no further therefore, theta goes from 0 to pi. The projection of the radius vector onto the x y plane the angle that that makes with the x axis is phi it is called the meridional angle this goes around the z axis it goes around the z axis therefore, when you start along the x axis phi is equal to 0, you go all the way around the z axis and come back, you will find that phi is equal to 2 pi. Those are the axis.

As said this is also used for locating positions on spheres the earth for example, is sphere and if you want to locate a position on this sphere you need 2 angles in this particular case the z axis is the north south axis, the angle from the z axis can go from 0 to pi that is 0 to 180 conventionally that actually corresponds to the latitudes, but conventionally the latitudes go from 90 degrees on the north pole to minus 90 on the south pole. So, that is by convention.

If you want to define the angle from the north pole itself when theta would go from 0 to pi and the angle around the north south angle around the z axis is the meridional angle and that gives you the longitudes, the longitudes go from 0 to 3 6ty they start at 0 and then they go all the way around to 360 that is the angle phi, in this case, the 0 longitude is the x axis. So, that is the coordinate system.

Now we have to construct a differential volume in this coordinate system, the differential volume are bounded by surfaces of constant coordinate, but first you have to first define what is the distance moved when these angles changed by a little bit because in the Cartesian coordinate system delta x delta y and delta z was the distances traveled, if you change x to x plus delta x, the distance traveled is delta x it has units of length all 3 coordinates have units of length.

In a spherical coordinate system, all 3 coordinates do not have units of length 2 of these are angles therefore, I need to define the distances carefully, when I travel from r to r plus delta r in the radial direction that distance is delta r itself. So, in the radial direction, if I go a small distance delta r in the radial direction to r plus delta r that angle is delta r itself now, how about if I change the theta coordinate from theta to theta plus delta theta by change the theta coordinate from theta to theta plus delta theta, what is the distance traveled? That distance traveled is going to be the radius times the angle it is going to be the radius times the angle in radiance. So, this has to be equal to r times delta theta is the distance traveled when you go theta to theta plus delta theta the distance traveled is r times delta theta and when I go phi goes from phi to phi plus delta phi and I go a small distance delta phi along the phi coordinate when I increment phi by a small distance delta phi this distance is; obviously, the radius times delta phi the radius in this case is r sin theta. So, those are the distances that are traveled and; obviously, the volume of this any differential volume that you construct the 3 distances of that volume will be these 3.

Let me expand that hopefully for you, this surface here I take a surface at r and the surface at r plus delta r at theta and theta plus delta theta and then in the phi direction that surface will go from phi to phi plus delta phi. So, let me construct that surface let me enlarge that surface for you I have a surface at r and the surface at r plus delta r. So, this as the location r this is r plus delta r this distance is delta r. So, my differential volume is bounded by surfaces of constant coordinate surfaces of constant coordinate in the r direction r at r and r plus delta r surfaces of constant coordinate in the theta direction this angle is theta here this angle is theta here. So, this angle is theta and this angle is theta plus delta theta.

It is bounded by 2 surfaces of constant theta at theta and theta plus delta theta, the distance traveled in this is going to be r delta theta and over here it will be r plus delta r delta theta. So, those are the phases that are perpendicular to the r direction and perpendicular to the theta direction note that if I write some kind of a coordinate system, here the r direction will be in the direction of increasing r e r the theta direction will be the direction of increasing theta e theta the phi direction is along the x y plane. So, it is perpendicular to both of these if phi direction is perpendicular to both of these it goes along the x y plane. So, in the phi direction if you increment by a small angles phi right you get something that goes like this and these phases on the sides r at phi and phi plus delta phi these phases which are perpendicular to these 2 r at phi and phi plus delta phi and this distance that you travel in the phi direction this distance are you travel in the phi direction is going to be equal to r sin theta times delta phi that is this distance in the phi direction so that is going to be r sin theta delta phi.

This is the differential volume that I will construct 2 of the coordinates are perpendicular to the 2 of the surfaces are perpendicular to the r direction, 2 of the surfaces are perpendicular to the theta direction and 2 of the surfaces are perpendicular to the phi direction, 2 surface is perpendicular to the r direction that is the in this case the bottom and the top surfaces, 2 of the surface that are perpendicular to the theta direction that is the left and the right phases and 2 of the surfaces that are perpendicular to the phi direction and that is the in this case it will be the front and the rear 2 other surfaces that are perpendicular to the phi direction. That is a differential volume that I will construct.

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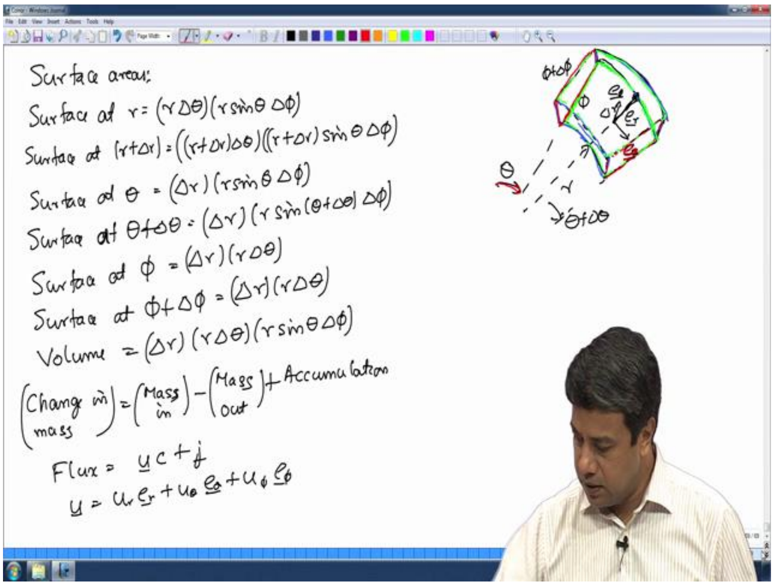

Now for this differential volume, I have to write a balanced equation and in that balance equation, I need to be able to identify all the surfaces first. So, the surfaces, this is at r and this distance is delta r ok this is at theta and this is at theta plus delta theta I moved the small distance theta and the delta theta and then I have this third phase which is perpendicular to the phi direction. So, this will be at the angle phi and this will be at the angle phi plus delta phi and I have to write a flux balance for these mass balance.

But first I have to identify the surface area of these surfaces, what is the surface area? Surface at r that is this surface the surface at r is this surface that is the surface at the location r perpendicular to the radial coordinate the surface area of that surface has to be equal to the length in the theta direction times the length in the phi direction that is the surface area the length in the theta direction is going to be equal to r delta theta the length in the phi direction is r sin theta delta phi.

It is going to be the surface at the radial location r perpendicular to the r direction; the next is the surface in perpendicular to the radial coordinate at r plus delta r, what is the surface at r plus delta? r is equal to r plus delta r delta theta into so, those are the 2 surfaces that are perpendicular to the r coordinate the blue ones are perpendicular to be r coordinate and then there are 2 surface that are perpendicular theta coordinate, there are 2 surfaces that are perpendicular to the theta coordinate because theta is increasing along this direction therefore, there are 2 surfaces that are perpendicular to the theta coordinate.

So, the surface at theta it is going to be equal to this distance delta r this distance delta r times the distance in the phi direction r sin theta delta phi; if surface perpendicular to the theta direction is the distance in the r direction times the distance in the phi direction and similarly the surface at theta plus delta theta is equal to delta r into r sin of theta plus delta theta delta phi. So, as theta is changing the surface area is also changing.

Those are the surfaces in the r direction, the surface is in the theta direction, now there are 2 surfaces in the phi direction the surface is in the phi direction r the surface in front and similarly there is a surface at the back those are the surfaces in which you increment the meridional angle phi. Therefore, the surface at phi surface at phi is going to be equal to delta r into r delta theta that is going to be equal to the distance traveled in the r direction times the distance traveled in the theta direction, that is going to be the surface area of the surface at phi and similarly you have surface at phi plus delta phi is equal to delta r times r delta theta.

Those are the different surface areas and the total volume of the surface is going to be equal to delta r into the distance moved in the r direction times the distance moved in the theta direction times the distance moved in the phi direction so that is going to be equal to the total volume of this differential volume. And for these I have to write the balanced equation that is change in mass is equal to mass in minus mass out plus any accumulation and as I said the mass in and the mass out are equal to the flux times the area times time the flux consists of 2 parts flux, it consists of 2 parts, u times c plus j, the convective flux plus the diffusive flux where the velocity vector u is u r e r plus u theta u theta plus u phi e phi the velocity is decomposed in terms of the components along the r direction.

At the local location, I can insert the coordinate system the e r is along the direction of varying r e theta is along the direction of varying theta and e phi is along the direction of varying phi perpendicular to the plane e r is along the direction where r changes it is a along with the direction of r theta changes because it is angled from the z axis. So, this is e theta and if you have to decompose these into different components and then write the fluxes times the surface area at each of these.

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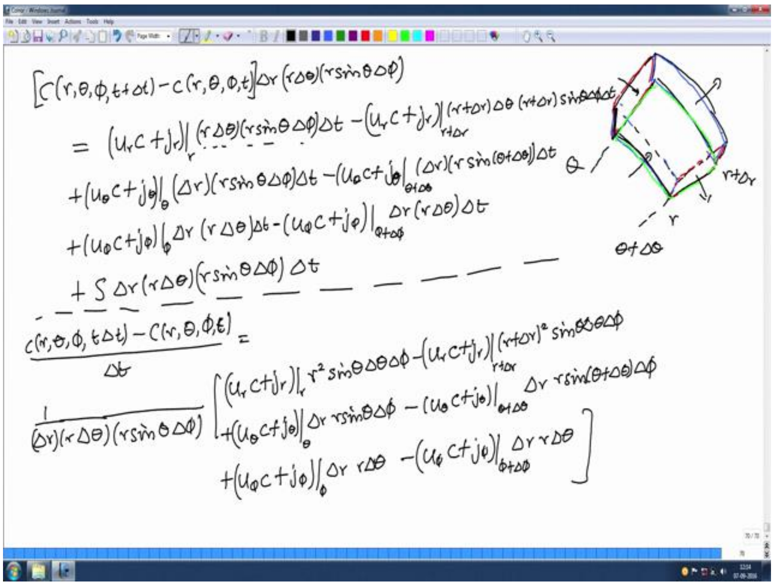

Let us proceed, first of all change in mass within the time in interval delta t, it is going to be equal to the concentration at r theta phi t plus delta t minus r, the concentration times the volume delta r, r delta theta into r sin theta delta phi so that is the change in mass within an time interval delta t is equal to mass in minus mass out mass in at the process at r. So, have this coordinate system once again so I will have a mass in at the surface at r because if the velocity is positive in the r direction the mass increases within in this volume and I have a mass out at r plus delta r. So, let us destroy the differential volume once again this is r, r plus delta r this is theta this is theta plus delta theta and the phi coordinate is perpendicular to both of these.

Mass in at the surface r it is going to be equal to u r times c plus j r at r times the surface area surface area perpendicular to the r direction I said was r delta theta into r sin theta delta phi. So, r delta theta is the angle is the length in the theta direction and r sin theta delta phi and length in the phi direction minus the mass out at the location r plus delta r minus I am sorry, I should multiply it by delta t as well because this is a flux times an area times time and this is r plus delta r delta theta times sin theta delta phi delta t.

Those are mass in and mass out at r and r plus delta r; plus mass in at theta and theta plus delta theta u theta c plus j theta at theta times the surface area. Surface area perpendicular to the theta direction surface area perpendicular to the theta direction that is the left and the right phases here: note that the theta direction were these 2, the phases that were perpendicular to the theta direction the surfaces that are perpendicular to the r direction were these 2 and the phi direction were the front and the rear phases, the front and the rear phases.

Perpendicular to the theta direction what are the surfaces that are perpendicular to the theta direction? The surface area is delta r times r sin theta delta phi times delta t. So, that is the flux times the surface area and what is going out at the surface at theta plus theta delta theta. So, what I just use the flux in here, the flux out at the location theta plus delta theta is going to be equal to minus u theta c plus j theta at theta plus delta theta into delta r into r sin into delta t and then I have the surfaces the front and the rear at phi and phi plus delta phi.

I have to take the flux which is u phi c plus j phi at phi multiplied by the surface area the length in the r direction times the length in the theta direction r delta theta minus u phi c plus j phi at phi plus delta phi into delta r into r delta theta. This the length, the surface area perpendicular to the phi direction is the length in the r direction times length in the theta direction perpendicular to the r direction is the length in the theta direction times the length in the phi direction and so on. And I have to multiply it by delta t here and now I have to divide throughout by volume and by time I have to divide by throughout by volume and by time, when I do that on the left side I will just get see at I should add the accumulation term here the accumulation term is just of the form total source times the volume delta r r delta theta r sin theta delta phi times delta t.

Now, I have to divide throughout by volume and by time, on the left side the term will just be equal to c at r theta phi t plus delta t minus c at r theta phi t by delta t. On the right side I will have 1 over delta r into r sin theta delta theta, I am sorry; r delta theta, r delta theta into r sin theta delta phi into u r c plus j r at we will expand this at r into r plus delta r the whole square plus u theta c plus j theta at theta times delta r into r sin theta delta phi minus u theta c plus j theta at theta plus delta theta into delta r into r sin theta plus delta theta delta phi plus u phi c plus j phi at phi into delta r into r delta theta into just dividing throughout by volume and by time.

You can see that for each of these terms certain intervals will cancel out for example, in the first term sin theta delta theta and delta phi will cancel out, but I will have a 1 over delta r in the denominator. Similarly for the second term delta r and delta phi r delta phi will cancel out, in the third term delta r and r delta theta will cancel out. So, this is the difference equation now we have to reduce it to a differential equation.

So, I will continue the reduction of this now to a differential equation we will continue that in the next lecture. Please go through this and familiarize yourself with the way the differential volume is constructed, what are the lengths of all the size of the differential volume, but on the surface areas once you have done that correctly then you know what is the mass in and the mass out by just multiplying the flux times the surface area. We will continue this in the next lecture, I will see you then.