Transport Processes I: Heat and Mass Transfer
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Lecture – 44
Unidirectional transport: Spherical co-ordiantes. Heat conduction from a sphere

Welcome to the continuation of our lecture on singular perturbation solutions at high Reynolds number, we were looking at the oscillatory flow in a pipe in the limit of low Reynolds number I had told you that the viscous diffusion time is much smaller than the time period of the oscillation therefore, momentum diffuses instantaneously and what you get is a parabolic velocity profile, the solution for the steady flow except that the flow amplitude the maximum velocity corresponds to the pressure gradient at that particular instant in time, that is for the limit where the Reynolds number is small and we had seen how to do a regular perturbation expansion in order to look systematically correct for the effect of Reynolds number in the limit of low Reynolds number.

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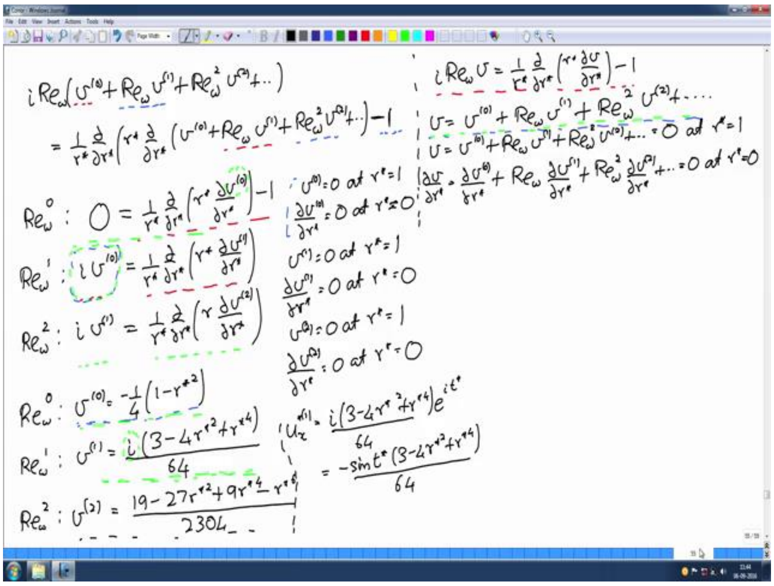

Then we were looking at high Reynolds number and as you recall if we had scaled the velocity by the inertial scales and based upon that we have got a conservation equation where the viscous term was multiplied by a small parameter.

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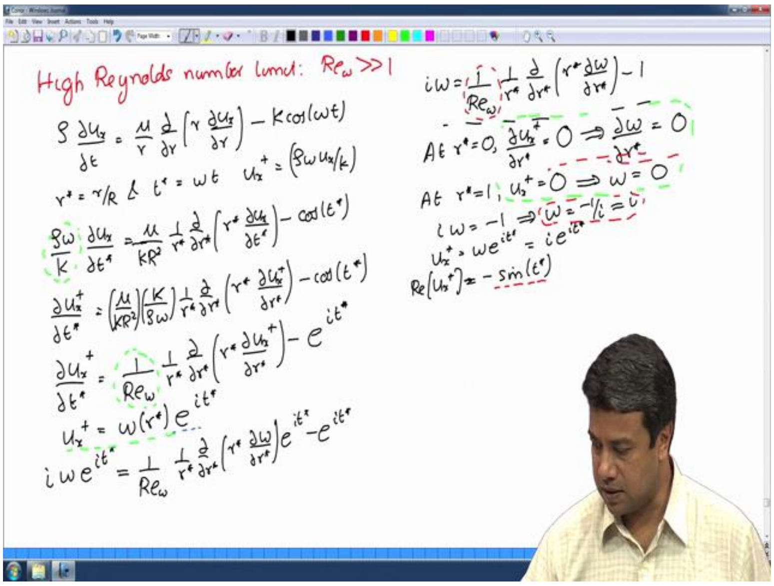

The viscous term was multiplied by a small parameter and when we try to solve this we ended up with an inconsistency because we were not able to satisfy both boundary conditions at the center as well as at the wall.

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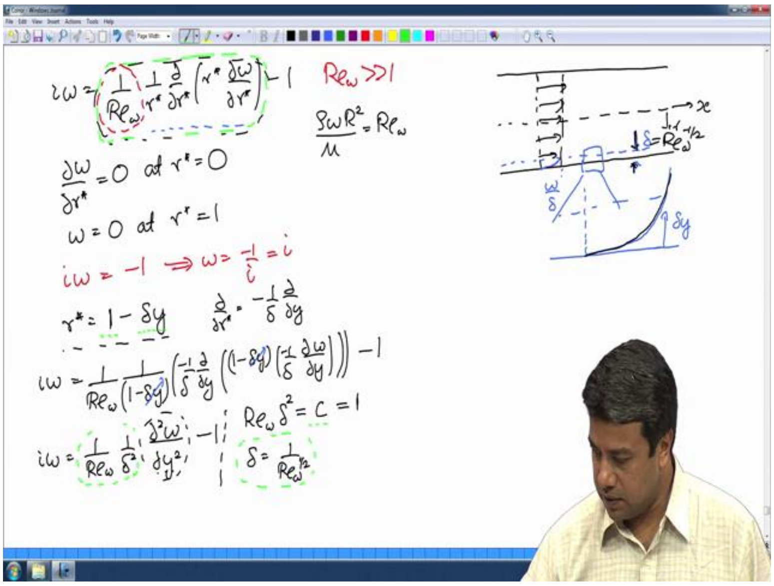

And we had identified the source of that inconsistency; mathematically speak us the viscous term is the highest derivative and therefore, if you set the viscous term equal to zero you do not get a differential equation anymore, there are no constants of integration and it is not possible to satisfy boundary conditions.

Physically it is because the zero velocity condition at the wall is because of the retardation of the flow close to the wall. The momentum diffusion from the wall or rather to the wall slows down the fluid close to the wall and reduces the velocity to zero this physical effect had been neglected when we neglected the viscous term in the conservation equation and therefore, we are not able to satisfy the boundary condition.

Even though the Reynolds number may be large that is the time required for momentum diffusion may be large compared to the time period for the flow reversal or the reversal of the pressure gradient. So, even though the time required for momentum diffusion may be large compared to the time required at the period of oscillation; when the Reynolds number is large what it means is that the time period for diffusion across a distance r is large compared to the time period for the pressure oscillation.

However diffusion still takes place and by the time the pressure diffuser reverses before the pressure reverses, the momentum has diffused to a small distance away from the wall that small distance is much smaller than the radius because the time period is very fast because the time required for the pressure to reverse is small compared to the time for diffusion to the center. Diffusion takes place only to a small distance close to the wall, within that distance the velocity increases from 0 to the bulk value, because the distance is small the gradient is large the gradient is the velocity change divided by the distance; when the gradient is large, the stresses are much larger than what you would anticipate on the basis of the length scale being the radius of the pipe and because of that you could have a balance between the viscous and the inertial forces even on the limit of very high Reynolds number.

This thickness over which this momentum diffuses from the wall before the flow has reversed of the pressure has reversed; this distance has to decrease as the Reynolds number keeps increasing. So, that is the scaling the length scale over which the velocity is increasing from 0 at the wall to the free stream velocity; that distance decreases in such a way that the inertial and the viscous terms are comparable even in the limit of high Reynolds number. To find that we had scaled the distance from the wall that the radius at the wall is one close to the wall is going to be equal to some small number,1 minus some small number that small number we can we had rewritten it as the scaling factor delta, which decreases as the Reynolds number increases, times this coordinate y the coordinate y. Continues to be a harder 1 in the limit as a Reynolds number becomes large, it is the distance from the wall the scaled distance, delta is the scale factor which depends upon the Reynolds number.

We had inserted that into this equation and just from a simple balance since you have two derivatives you get 1 over delta square in front of the highest derivative and for that to be of order 1 even in the limit of high Reynolds number, if you find the delta has to be equal to some constant by Reynolds number to the half.

In general this has to be a constant, but you can go through the derivation, even if you set any value of the constant the solution in terms of y will depend on the constant, the solution in terms of r star will not depend upon this constant you can go through that and verify that for yourself.

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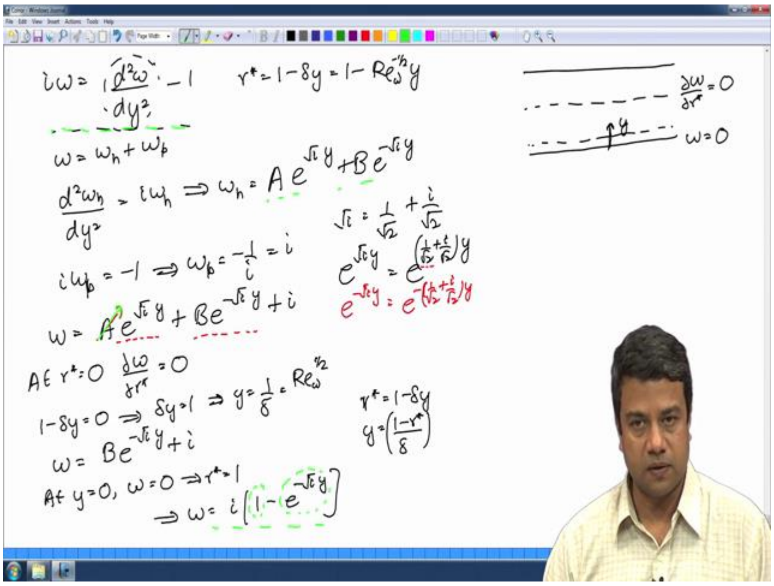

So, delta is equal to 1 over R e power half and on that basis we had actually managed to get a solution. Now we have a second order differential equation therefore, it is now possible to satisfy boundary conditions; in the homogeneous equation we get constants of integration, since the velocity disturbance due to the wall itself in the bulk the velocity is a constant very close to the wall the retardation effects slows down the velocity within the distance order delta from the wall, that velocity disturbance has to decrease to zero as y goes to infinity, that gave us that one of these constants is zero and the zero velocity boundary condition at the bottom gave us the other constant.

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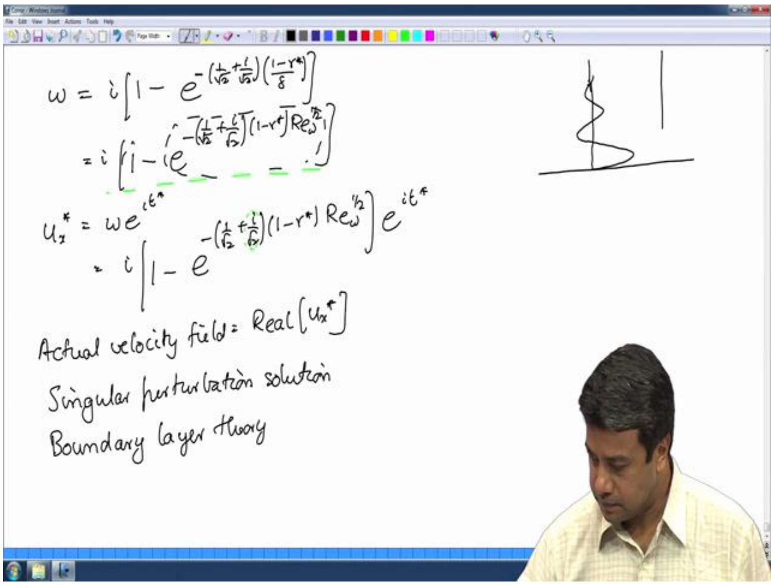

And therefore, if we go back and write the equation in terms of the original variables, we get an analytical solution in terms of the Reynolds number. So, it is not just a constant anymore, but it contains a term that is a function of the Reynolds number and then you can take the multiply that by e power i t and take the real part, in order to get the final solution for the velocity profile that will contain both real and imaginary parts; the real part will contain sin and cosine functions in time as well as because of this exponential this exponential of i by root 2 it will contain a function that oscillates in the boundary layer as well.

So, the solution for this velocity profile it has to be zero at the wall, you will contain a term that oscillates slowly away from the wall ok and if this part of the velocity field and then superposed on that is the constant velocity, you have to add up that two to get the total velocity.

So, physically as I said even though viscous effects even though the Reynolds number is large, viscous effects are still important within regions close to the wall because momentum diffusion perpendicular to the surface required to slow down the fluid can take place only due to viscosity therefore, you have to have viscous effects being important in the thin region close to the wall, the extent of the layer is the extent to which viscosity will diffuse within a time scale comparable to the time period of oscillations and once I do that I can get a solution for this problem. In all cases where the viscous effects are small compared to inertial effects, Reynolds number is large or peclet number is large for mass and heat transfer these problems, will all reduce to singular perturbation problems, where the disturbance due to the flux from the surface will be limited to a thin region near the surface and that thickness of that region is determined from the condition that inertial and viscous forces continue to be of equal magnitude within that thin region even in the limit as the Reynolds number goes to infinity of the peclet number goes to infinity.

The boundary layer thickness delta will be some power of the Reynolds number or the peclet number. It will not in general be minus half as in this case it could be a different power, but once I scale it that way within this region the inertial and viscous effects are equal to magnitude and therefore, I can solve the boundary conditions and from that calculate the flux at the surface by taking the derivatives of the temperature field or the concentration field. So, this serves as an illustration of what I have been emphasizing throughout that when inertial effects or convective effects are dominant, diffusive effects still have to be incorporated within a thin region near the surface.

So, this will complete our treatment of unidirectional flows, it transport in one dimension; the only other thing that I need to do for used to derive this in what is called a spherical coordinate system before we go on to deriving general expressions for transfer.

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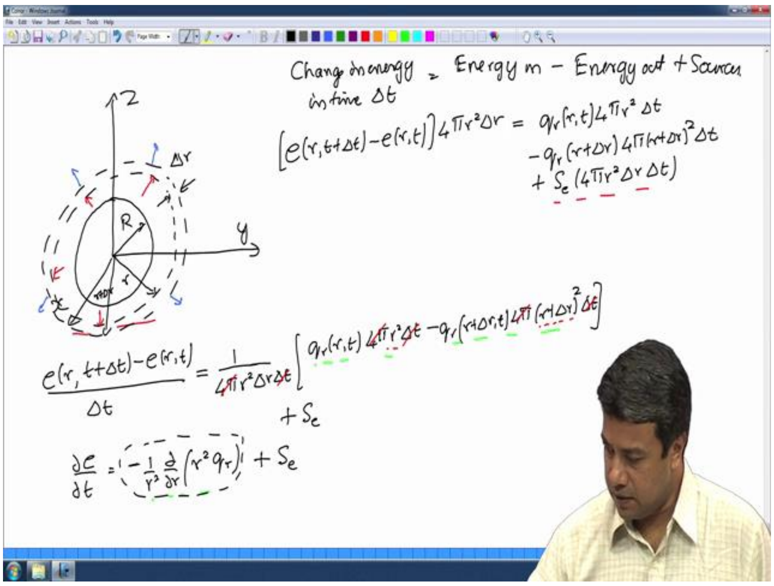

In a spherical coordinate system the for example, if you wanted to look at the transport around the spherical particle, you would like the surface of the particle to be a surface of constant coordinate.

So, rather than writing the surface as it is called this is x y and z: as x square plus is equal to r square I am sorry R; you writing it this way, it is the preferable to consider this surface itself as a surface of constant coordinate; that means, that the distance from the origin has to be a constant value r, rather than having this as the boundary condition it is the preferable that we change our coordinate system, so that this spherical surface becomes a surface of constant coordinate r is equal to capital R.

So, in this coordinate system the surfaces of our constant coordinate are surfaces of constant R and if I wanted to do a differential balance in this coordinate system, it has to choose a differential volume which is bounded by surfaces of constant coordinate; the differential volume in which one surface is z r and the other surface is z r plus delta r and write a balance for this kind of a surface this is called as spherical coordinate system, you will see a spherical coordinate system a little further detail as we go through the course, here I will just derive you expressions for the spherically symmetric case and for a spherical coordinate system.

So, if you are solving the heat transfer problem for example, I say that the change in energy in time delta t is equal to energy in minus energy out plus any sources. So, that is going to be the balance condition. Change in energy is going to be equal to the difference; energy density at time t e at r t plus delta t minus e at r, t times the volume; what is the volume of this spherical shell is equal to the surface area times this distance delta r, volume is equal to the surface area times the distance delta r. So, this is going to be equal to 4 pi r squared delta r that is the volume.

Energy in is at the surface at r is going to be equal to the flux times the surface area times time energy in is equal to the flux q r at r t times the surface area, a surface area is 4 pi r square times delta t, the time interval. Fluxes transfer per unit area per unit time so flux times the area 4 pi r square times delta t; minus the energy out the energy out is at the location r plus delta r.

So, the energy out is going to be equal to minus q r at r, plus delta r into 4 pi into r plus delta r the whole square into delta t; delta t plus sources source in general is equal to source per unit volume per unit time times of volume r square delta r times delta t.

So, that is the balanced equation; now I divide throughout by volume divide throughout by time in order to get a differential equation. So, therefore, I will get the energy at r will be written here by delta t. Now for these flux terms I have to be a little careful, have to divide throughout by volume and by time. Volume is 1 by 4 pi r square delta r times delta t, into q r at r t times 4 pi r square delta t, minus q r at 2 r plus delta r the whole square delta t then when I divide this last term by the volume and time I just get plus S e and if I take the limit as delta t goes to 0 delta r goes to 0, you can see that the 4 pis r will cancel out over here they are just constants, delta t will cancel out; however, that r square will not cancel out in general because the surface area is different at these two surfaces at the inner and the outer surfaces the radius radial location is different therefore, the surface area is different. If I take the limit as delta t goes to 0 and delta r goes to 0 I will get partial e by partial t is equal to 1 by r square, d by d r of r square q r this term one by r square d by d r of r square q r will be equal to q r times r square at r plus delta r minus q r times r square at r.

Here we have the opposite of that we have q r times r square at r minus q r at r square times at r plus delta r therefore, there's going to be a negative sign here plus the source. So, this is the differential operator in a spherical coordinate system; 1 by r square d by d r of r square d q r.

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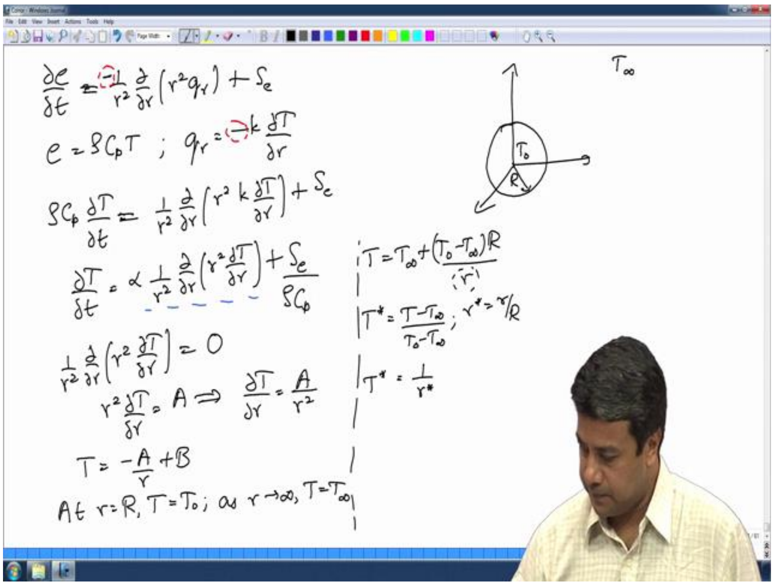

And then if I use the constitutive relation if the balance equation that I had was partial e by partial t is equal to plus the source, e I can write it as rho C p times T if it is at constant pressure and Fourier's law for heat conduction q r is equal to minus k times d t by d r and if rho and C p r constants I will just get rho C p partial T by partial t is equal to. I have two negative signs here one in the equation and the other in the expression for the heat flux. So, these two will give me a positive, I will get 1 by r square d by d r of r square d T by d r plus I am sorry I written k there plus the source. If k is independent of position once again if you assume that the thermal conductivity is in depend of position the temperature just becomes d T by d t is equal to alpha 1 by r square plus the source by rho C p.

So, there is a diffusion equation in the spherical coordinate system; once again this operator is different because as we saw in the previous slide the surface area is changing as the radius changes, the total transport is equal to the flux times the surface area. So, both the flux and the surface area are changing therefore; you get a complete more complicated expression for the temperature.

This can be solved in simple cases; if I wanted to solve for example, the temperature field around, a spherical particle; let us say the particle was at temperature T naught and far away the temperatures T infinity. The surface of the particle the temperature is T naught there are no sources within the fluid around the particle and we will assume that it is at steady state. In that case the equation just reduces to 1 by r square d by d r of r square d T by d r is equal to 0 or if I integrate once r square d T by d r is equal to some constant A therefore, T will be equal to minus A by r plus B, minus A by r plus B is going to be the temperature.

Now, at r is equal to R, we know that T is equal to T naught and as r goes to infinity, T is equal to T infinity; from this we can calculate the two constants A and B and the final temperature that you will get is T infinity, plus T naught minus T infinity, R by r alternatively if I done my scaling T star is equal to T minus T infinity, by T naught minus T infinity and r star is equal to r by R, in this equation could well have been written as T star is equal to 1 over r star, that is the temperature field.

So, the temperature basically decreases as 1 over r, the temperature minus the temperature at infinity decreases as 1 over r as r becomes large the flux goes as 1 over r square because temperature goes as 1 over r, the flux goes as 1 over r square and the surface area increases proportional to r square therefore, the flux times the surface area is a constant as it should be there are no sources or sinks.

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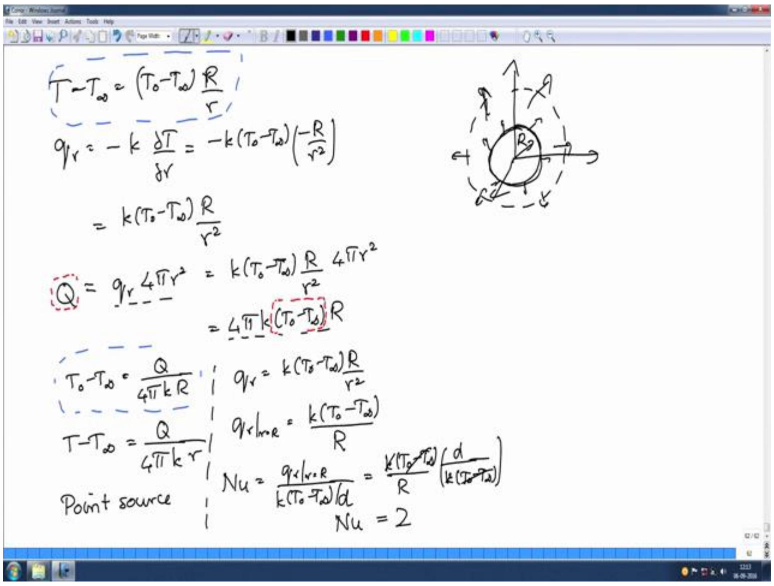

So, let us look at the solution once again T minus T infinity is equal to T naught minus T infinity into R by r, what is the flux; q r is equal to minus k partial T by partial r, this will be equal to minus k into T naught minus T infinity into R divided by. So, you should write this as the derivative of 1 over r is going to be equal to minus R by r square; this is going to be equal to k into T naught minus T infinity into R by r square. So, that is the flux. The total heat generated the flux times the surface area the integral the total heat transported per unit time is going to be equal to q r into 4 pi r square, which will be equal to k into is equal to 4 pi k into T naught minus T infinity into R.

So, this thing if I had a particle of radius R, if I looked at the total amount of heat coming out per unit time at any location, the total amount of heat coming out of any surface at any radius, at any radius we have put a surface and look at the total amount of heat coming out that has to be equal to the flux times the surface area at that location and that is equal to a constant it is independent of position; obviously, the total amount of heat coming out has to be a constant it should be independent position provided there are no sources or sinks; what comes out of here at steady state has to go out of the next surface.

Now, rather than expressing the temperature in terms of this temperature difference I could as well express it in terms of this total heat coming out per unit time. How do I do that? I just substitute T naught minus T infinity is equal to Q by 4 pi k into R from this equation, and I substitute that into this expression here and if I substitute for T naught in this manner, what I will get is that T minus T infinity is equal to Q by 4 pi k into r. Expressed in this manner the temperature field does not depend upon the radius of the sphere, it depends only upon the total heat that is coming out from the sphere. So, it does not depend anymore on the radius of the sphere even if the radius goes to zero so long as you know what is the total amount of heat coming out you know what is the temperature field, this is what is called the temperature field due to a point source in a spherical coordinate system, we will see more of that as we go through this course.

Now, that we know the heat flux we can now calculate, we know that q r is equal to k into T naught minus T infinity R by r square. I can calculate the flux at r is equal to capital R at the surface and the surface of the particles I can calculate the heat that the heat flux q r at R is equal to r is equal to k into T naught minus T infinity, at r is equal to r this just becomes one over R and from this now we can get back our Nusselt number. The Nusselt number is defined as the flux at r is equal to R divided by k delta t by the diameter of the particle; Nusselt number is usually defined with respect to the diameter of the particle.

So, this is k into T naught minus T infinity by R into the diameters of the particle divided by k into T naught minus T to infinity and you can see everything cancels out, the ratio of diameter and radius is just two. So, this gives us the expression for the Nusselt number when the transport is dominated by thermal diffusion and the transport is dominated by thermal diffusion, the Nusselt number is identically equal to two because transport is only due to diffusion for a spherical particle.

In the similar manner if I had solved the mass transfer problem, the Sherwood number would have been 2 in that case and that limiting case for diffusion dominated flows, Nusselt number is equal to 2, Sherwood number is equal to 2 comes out of a very simple balance and this corresponds to a point source if I specify the total amount of heat coming out rather than the temperatures regardless of the particle size, the temperature field will be exactly the same.

So, this is a brief introduction to a spherical coordinate system, here I have considered that there are variations only in the radial direction. In general there will be variations in other directions as well, how do we derive conservation equations when there are variations in multiple directions. Those conservation equations we will start in the next lecture general conservation equations for 3 dimensions. First conservation equations that include both convection and diffusion first I will do that for a Cartesian coordinate system and then I will do that for a spherical coordinate system after that we will come back and see how to solve problems where the flow is either diffusion dominated or convection dominated. So, we will continue with general conservation equations in the next lecture I will see you then.