Transport Processes I: Heat and Mass Transfer
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Lecture – 43
Unidirectional transport: Oscillatory flow in a pipe. Low and high Reynolds number solutions

We continue our series of lectures on fundamentals of transport processes. We were discussing transport in 1 dimension, we had first seen in a Cartesian coordinate system, we have looked at similarity solutions, separation of variables and we were now looking at a cylindrical coordinate system. Coordinate system suitable for example, the flow in a pipe or the heat transfers in an annular region or other such problems.

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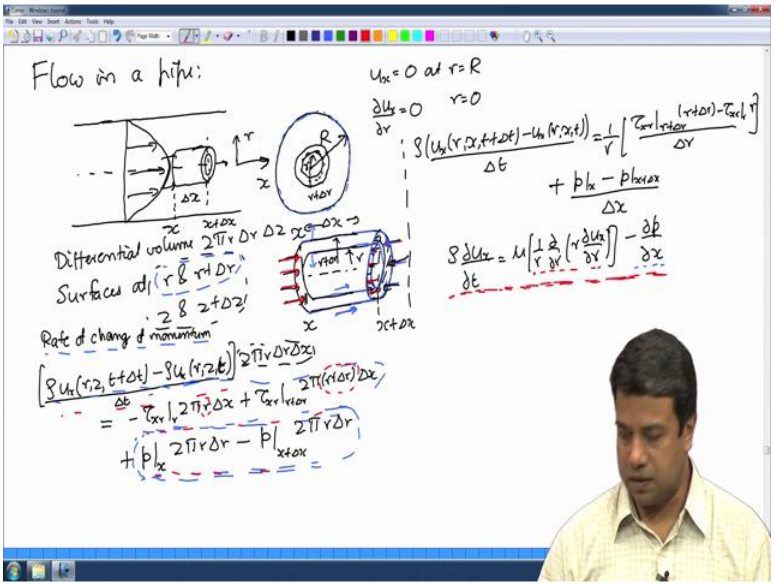

We had looked at in some detail at the flow in a pipe we had obtained the balance equations for the flow in a pipe due to a pressure difference between the inlet and the outlet, and the momentum conservation equation for the x momentum had this form, left side was the inertial term the right side the first term was the viscous term and the second was the pressure gradient and I had shown you that the pressure gradient had to be a constant for a pipe flow, provided the flow was fully developed that is there is no variation along the length of the pipe.

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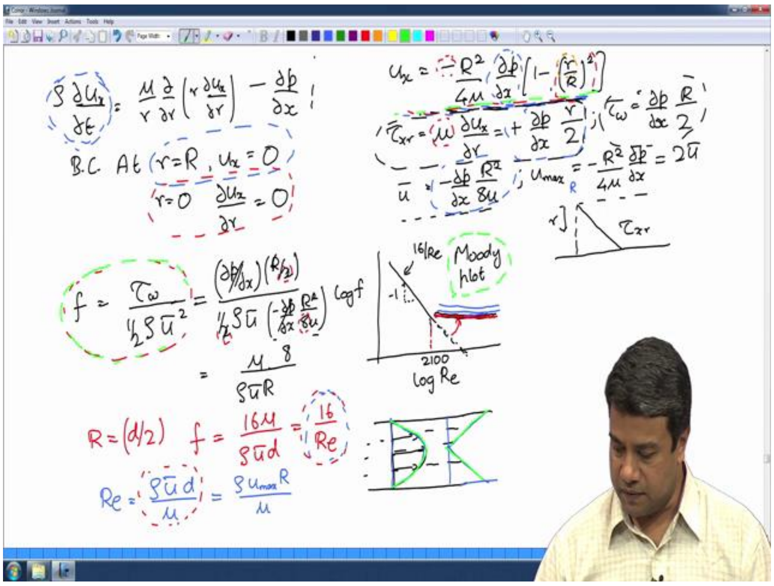

We had solved this to get the parabolic velocity profile for a pipe flow at steady state and from that we got the wall shear stress, the friction factor from that scaled form of the wall shear stress, the average velocity, the maximum velocity we found out that the velocity profile is a parabolic profile for a laminar flow, the shear stress is a linear function of distance from the center.

From this the wall shear stress we had got the friction factor as a function of Reynolds number; the moody plot that you see here friction factor goes a 16 by R e so long as the flow is laminar, at some point there is a transition, at that point the flow is not steady anymore you have viscous eddies which transports momentum in all directions; in particular the cross stream direction velocities are fluctuating in all 3 dimensions and the transport of momentum has also mass and energy is due to the fluid velocity fluctuations.

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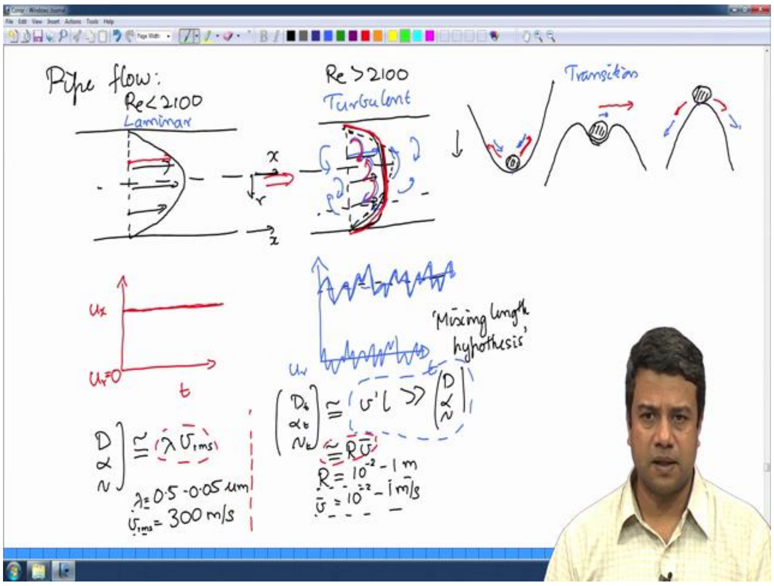

So, rather than the molecular velocity fluctuations and give you some idea of the mixing length hypothesis and how this gives rise to a much higher friction factor is compared to a laminar flow.

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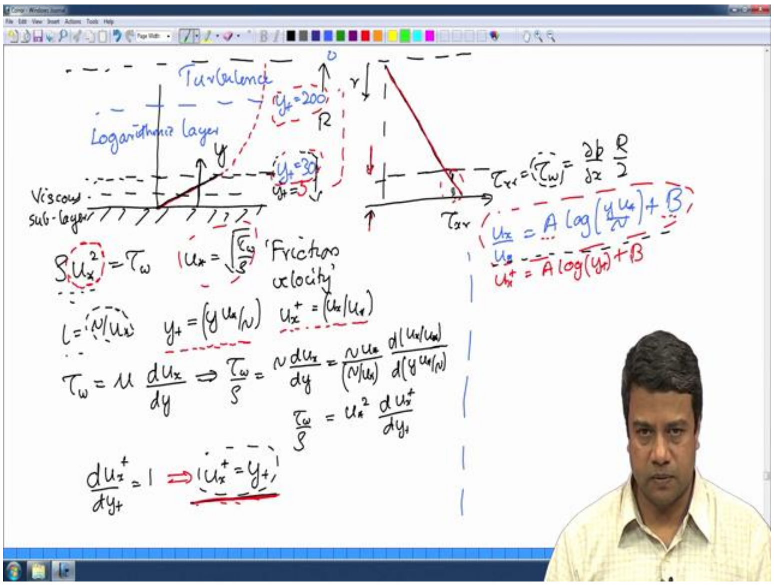

And we had looked at the structure of a turbulent flow qualitatively; near walls we have a linear velocity profile in the viscous sub layer is a logarithmic layer and then there is a bulk flow further away from the wall.

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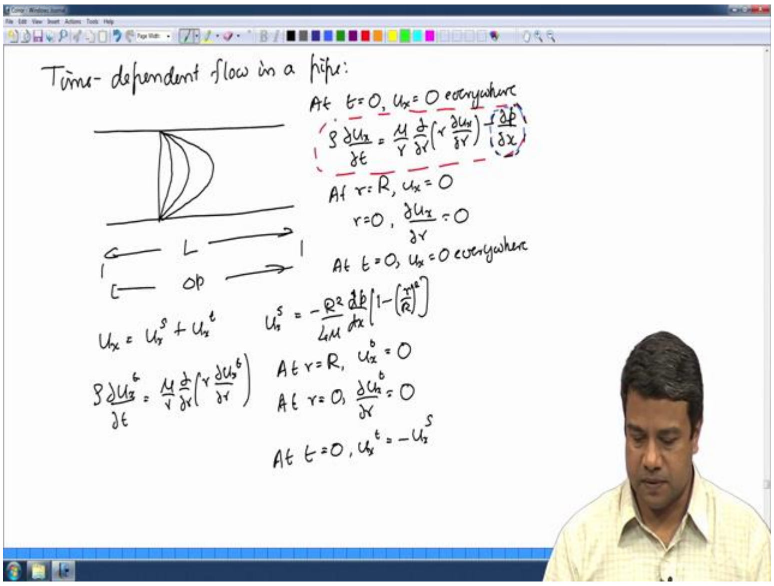

And then we were solving the problem of a time dependent flow in a pipe were in the pressure in the first instance we had looked at the separation of variables procedure, which had briefly sketched for you because it is very similar to what we had done for heat transfer and then we have looking at the case of an oscillatory flow.

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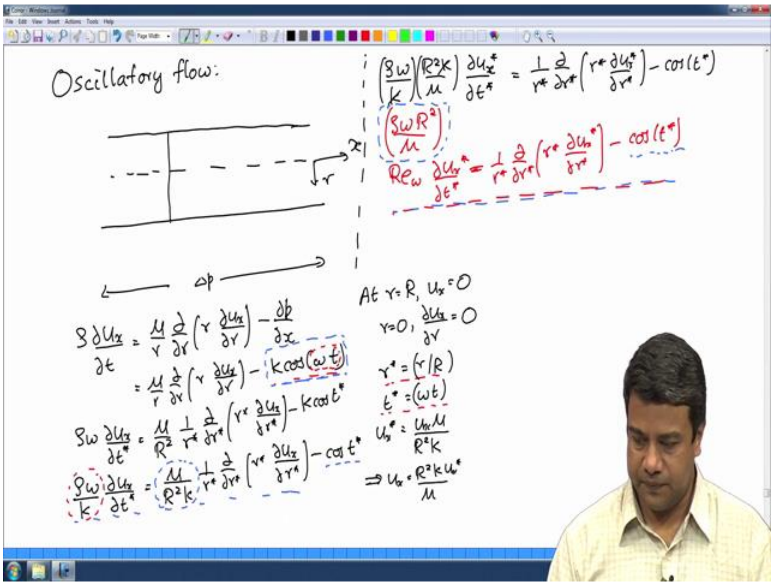

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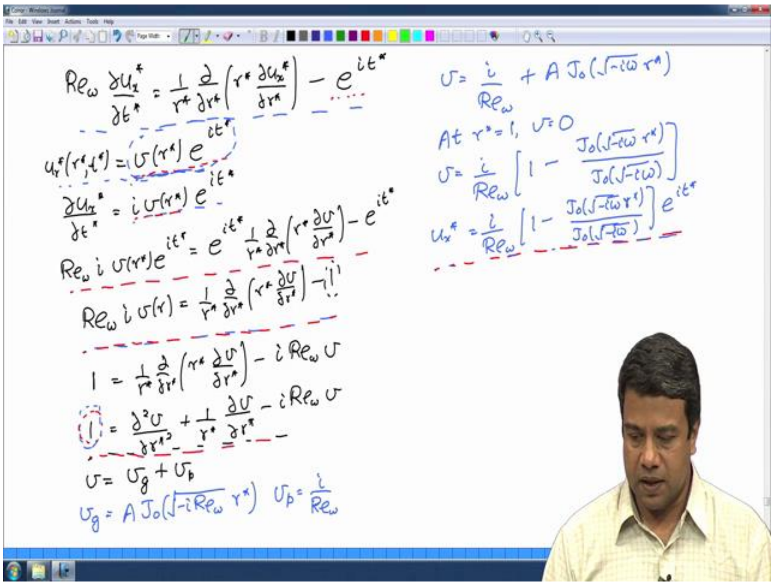

In an oscillatory flow we can use the substitution; the equation turns out to be a linear equation there Re omega is rho omega R square by mu, the ratio of inertial and viscous forces where we have use the frequency of oscillation as the inverse of the time scale and the radius as the length scale. If we scale by viscous scales the left hand side has a Reynolds number, if we scale the inertial scales as we saw a little later the right hand side has 1 over Reynolds number and the forcing function is of the form cos t and I had told you that we can simplify the equation by using the forcing function as e power i t instead, for this forcing the real part is cos t therefore, if I solve this equation this equation on top here the solution is going to be complex in general because the forcing is complex, if I take the real part of the solution I will get the solution for the forcing being cos t.

So, once I substitute e power it I can write the velocity as some function of r times e power i t and then I get differential equation only a function of r and I managed to get you an analytical solution in terms of Bessel functions, which satisfies all the boundary conditions we do not learn very much physically from this analytical solution. So, we went back and tried to look what are the solutions in the limiting cases when viscous forces are dominant and when inertial forces are dominant.

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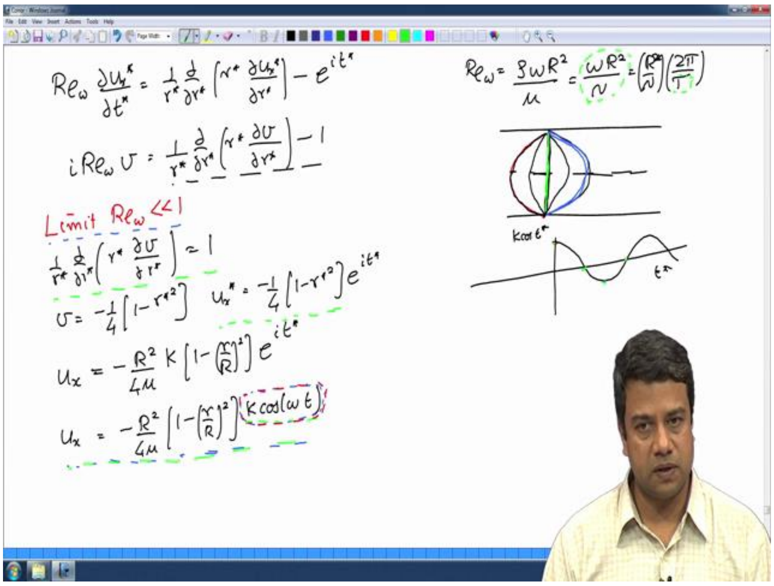

First when viscous forces are dominant, you can just solve this equation if you recall this equation has exactly the same form as the steady fully developed equation, solution is also there for a parabolic solution, in the limit of low Reynolds number. I told you the physical reason for this, when the Reynolds number is low the Reynolds number can also be written as the ratio of 2 timescales, the frequency of oscillation is 1 over the time period of this oscillation, the frequency of oscillation is 1 over the time period of the oscillation and this factor R square by mu can be written as the time the diffusion time over a distance capital r over a distance r the diffusion time is R square nu.

So, this is the ratio of the diffusion time over a distance r divided by the period of oscillation; when the Reynolds number is small the diffusion time is small compared to the period of oscillation therefore, the diffusion of momentum is instantaneous in comparison that to the time period of variation of the pressure. So, the diffusion is instantaneous, we basically get back a parabolic velocity profile with the maximum velocity related to the value of the pressure gradient at that particular instant in time, because the velocity field is responding instantaneously to variations in the time variations in the pressure therefore, we got a simple relation for the velocity itself.

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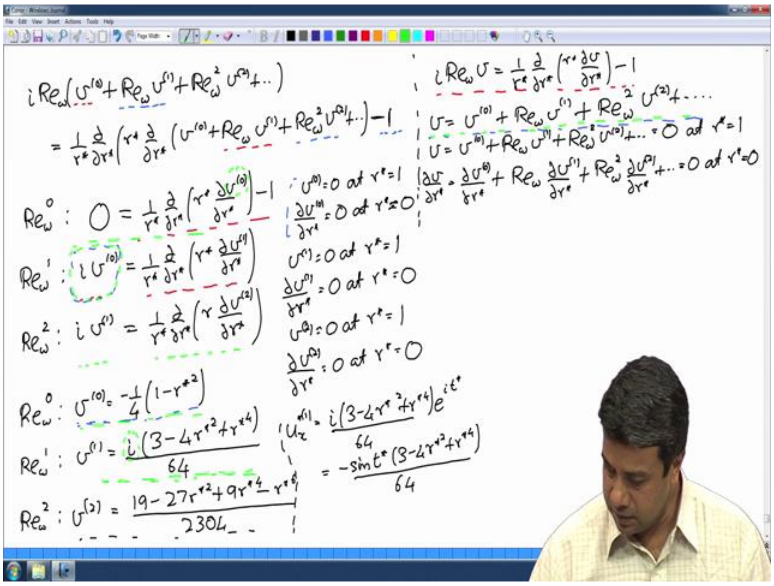

This equation is not exact; however, the Reynolds number is small, but in general it will not be zero and in that case we can systematically improve our solution by writing the velocity as a cds in the Reynolds number; since the Reynolds number is small the leading order term will be much larger than the next higher term that first order term will be much higher than the second order term and so on.

So, I can substitute that into the equations and into the boundary conditions and identify a set of equations at each order in this expansion, solve each of those independently for the leading order equation we get back the 0 Reynolds number limit, we can solve that for this solution v 0 we get a solution of this kind. That can be inserted because that appears as an inhomogeneous term in the next equation of the series, you can insert that get the solution, that solution appears as an inhomogeneous term in the next equation in the series and so on. So, you can systematically get solutions up to whatever order that you want by using this procedure it is called a regular perturbation expansion and this uses two advantage the fact that the Reynolds number is small in the limit that the Reynolds number goes to 0 this series is always convergent. The first term is always much smaller than the zeroth term, second term is much smaller than the first term and so on. So, that is how we can do a regular expansion in this small parameter in order to get velocity fields, improved approximation for the velocity fields which include inertial effects to whatever extent that you would like.

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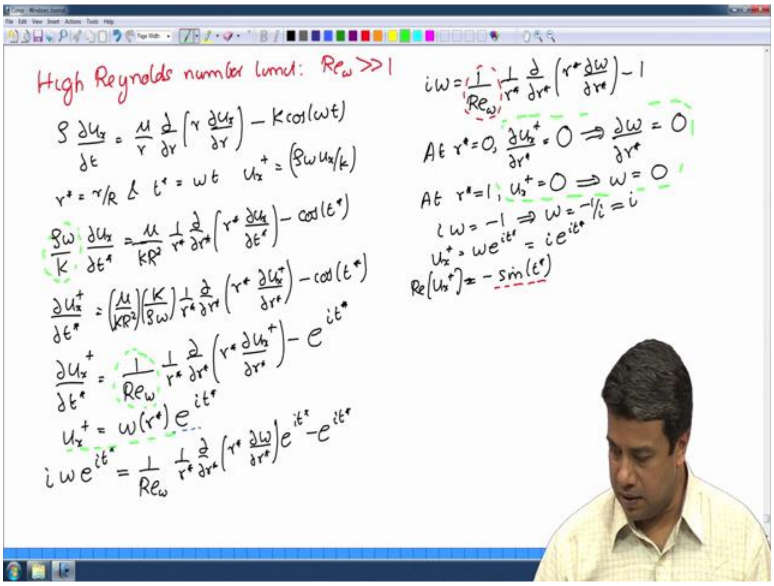

Last time we had looked at the opposite limit where the Reynolds number is large, we have done the scaling as usual, we had scaled the velocity by the inertial scale rather than the viscous scale; we had scaled the velocity by the inertial scale rather than the viscous scale and for that reason we had got the Reynolds number appearing the inverse of the Reynolds numbers, appearing in front of the viscous term in this particular case; where U x plus is the velocity that is scaled by the inertial scale and not the viscous scale that you x plus satisfies the same set of boundary conditions as the original U x and we use the same substitution a function of r times e power I t substitute into the equation and then try to solve it that was where we were in the last lecture.

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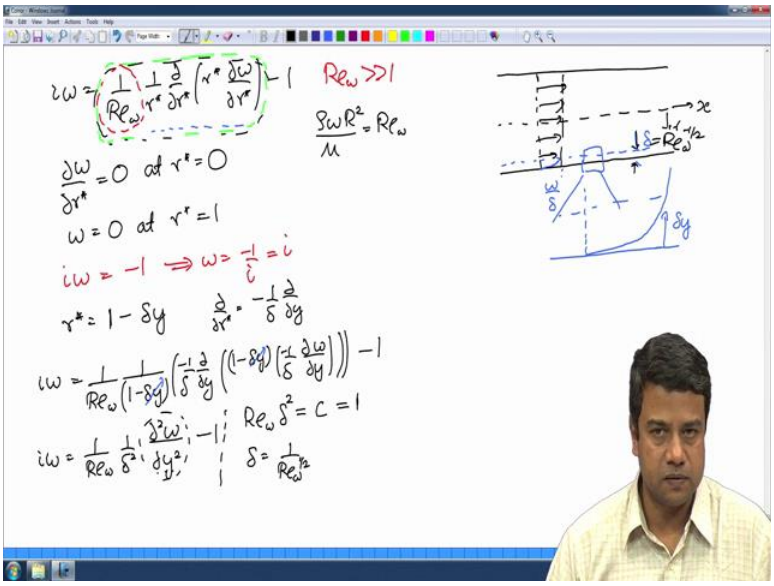

Now, when we try to solve this equation we found the equation of the form i w is equal to 1 by Re omega, 1 by r star d w by d r star, minus 1 that was the equation for the function w if you recall; the boundary conditions were that partial w by partial r star is equal to 0 at r star is equal 0; the symmetry condition along the axis of the pipe in order to ensure that the velocity field is continuous and differentiable at that point, if the derivative does not go to 0 the derivative non 0 it will get different derivatives from different directions.

So, therefore, this is there is no physical boundary there, there is only a boundary in the coordinate, at that point the function has to be continuous and differentiable and for that we require that the derivative has to be equal to 0. The second condition was that w is equal to 0 at r star is equal to 1. So, that was the second condition that we have; now from these 2 conditions if we consider the limit Re omega much greater than 1, this term goes to 0 because it is 1 over a large man sorry 1 over a large number therefore, you would think that you can approximate i w is equal to minus 1, which implies that w equal to minus 1 over i, it is just equal to i itself; that was the equation that we had got earlier he with neglected this term and we got i w is equal to minus 1 which means that w is equal to I itself and then we had tried to enforce the boundary conditions, w is equal to a constant.

Obviously, it does satisfy the boundary condition at r is equal to 0. The derivative has to be 0 if the function is a constant the derivative is equal to zero, but we were not able to satisfied the other boundary condition that the velocity has to be 0 at r is equal to 1; velocity is 0 implies that w has to be equal to 0 at r is equal to 1, this solution does not satisfy that boundary condition why is that?

Mathematically the reason is that when I have reduced this equation even I have neglected the highest derivative here, I have reduced this equation from a second order differential equation to zeroth order differential equation there are no derivatives anymore and for original equation was the second order differential equation therefore, I was able to satisfy 2 boundary conditions and I have reduced the equation, the equation is reduced to a zeroth order equation and there are no boundary conditions that can be satisfied anymore. Mathematically because I have neglected the highest derivative, I do not have constants of integration, it not have constants of integration in the solution to be able to satisfy the boundary conditions that is the mathematical problem; physically what is the problem?

So, let us think about it physically ok what the solution is predicting is that the velocity is just a constant at any instant of time the velocity is just the constant independent of radiance. So, what the solution is very thing is that the velocity is a constant; this constant is not able to satisfy the no slip condition at the wall, why does the no slip condition arise at the wall that is because the wall is exerting a frictional force on the fluid and retarding the fluid so that the velocity comes to 0 at the wall, that frictional force is basically viscous it is the diffusion of momentum from the wall, the diffusion of momentum from the wall is slowing down the fluid close to the wall and therefore, reducing the velocity to 0.

In my momentum conservation equation simplified momentum conservation equation, I have neglected this diffusion terms from the wall therefore, in the solution there is no diffusion therefore, there is no way for the fluid to be retarded close to the wall which means that the velocity cannot decrease to 0 at the wall itself, that is the physical problem that is the physical reason why we are not able to satisfy the no slip boundary condition; as I have emphasized many times in this course, convection after all the inertial forces are convective they are along the fluid velocity. Convection only transports momentum along the flow direction; momentum perpendicular to the flow direction has to be transported by diffusion. In the limit of high Reynolds number I have naively neglected diffusion therefore, there is no momentum transport to the wall and therefore, there is no way to satisfy the 0 velocity boundary condition at the wall.

So, what is the solution? Obviously, near the center the relevant length scale is the pipe radius rho omega R square by mu is a Reynolds number based upon the pipe radius. So, this is a ratio inertial and viscous effects assuming that the relevant length scale is the pipe radius; however, as one goes close to the wall if there is the region close to the wall whose thickness delta is much smaller than the pipe radius it is a region close to the wall if there is a region of small thickness, whose thickness is much smaller than the pipe radius over this distance is the velocity increases, that means, that the velocity gradient in this region will go as this w by delta; the velocity gradient will go as w by delta; this could be much larger than what I have assumed here that the length scale is r and therefore, I have scaled the velocity gradient all by r.

So, in order to incorporate the effect of momentum diffusion close to the surface, close to the surface the distance of the region is small therefore, the gradients are large therefore, as you approach close to the surface, viscous effects could become comparable to inertial effects because viscous effects have 2 gradients in them, they have 2 derivatives if the length scale is smaller the derivatives are larger and therefore, the viscous effects could be much larger than what they were assumed to be which is to scale with one over the pipe radius.

So, how do I analyze this problem? So, what I need to do is to rescale my length scale in the region close to the wall, rescale the length scale in the region close to the wall by delta the small thickness rather than by the pipe radius assume the length scale to be delta rather than the pipe radius; how will I calculate delta? Delta will be calculated from the condition that inertial and viscous effects continue to be of equal magnitude within this region delta, even as the Reynolds numbers goes to infinity. So, to repeat as the Reynolds number becomes larger and larger, this delta will become smaller and smaller the thickness will become smaller and smaller, the gradients will become larger and larger in such a way that the inertial and the viscous effects continued to be of equal magnitude in the limit as the Reynolds number goes to infinity.

So, this is the region over which I expect the velocity to increase. So, what I will do is I will define another coordinate here. I will call it as delta times y; delta is the scaling factor the scaling that I have to use for the length scale in order to get the inertial and viscous effects of equal magnitude, y is a dimensionless distance from the wall. So, in the limit as Reynolds number becomes large delta will become smaller and smaller y will remain the same ok. So, therefore, I will define r star is equal to, at the pipe wall r star is 1, close to the pipe wall r star will be equal to 1 minus delta times y; where delta as I said is a scale factor which depends upon the Reynolds number, it should get smaller and smaller as the Reynolds number becomes larger and larger; y is the coordinate which is order 1, it continues to be order 1 in the limit as a Reynolds number goes to infinity.

So, once I have done this rescaling, I insert that into the equation, I will get d by d r star equal to minus d by d y. So, I can write the equation as i w is equal to 1 by Re omega, 1 by 1 minus delta y into minus 1 by delta d by d y acting on r star which is 1 minus delta y into minus 1 by delta partial w by partial y minus 1 and I told you delta is a small number it is much smaller than 1. So, in this case I can neglect delta y in comparison to 1 you can neglect delta y in comparison to 1 now we will get I w is equal to 1 by Re omega, 1 by delta square d square w by d y square minus 1. So, therefore, I get back the original equation that I had for a Cartesian coordinate system in terms of y and that is because in this wall region, when this distance delta is much smaller than the radius, the layer close to the wall a basically seems like a plane layer, it basically seems like a flat layer because the radius the thickness is much smaller than the radius of coverage therefore, the layer looks like a flat layer therefore, you get the differential operator that I had for a simple Cartesian coordinate system.

So, therefore, now this I said the delta is a function of Re omega and it has to be it is not a function of r it is a scale factor, it has to be chosen in such a way that the inertial and the viscous terms continue to be of equal magnitude in the limit as the Reynolds number goes to infinity therefore, Re omega times delta has to be equal to a constant value in that I am sorry delta square, in the limit as Re goes to infinity. So, this gives us the value of delta, the constant can be any value any constant of order 1, only thing is there to be fixed it should not go to either 0 or infinity as the Reynolds number goes to infinity; turns out without loss of generality, you can actually choose this constant to be just 1, that will change the solution in terms of y, but it will not change the solution in terms of r. So, therefore, delta is equal to 1 by Re omega power half.

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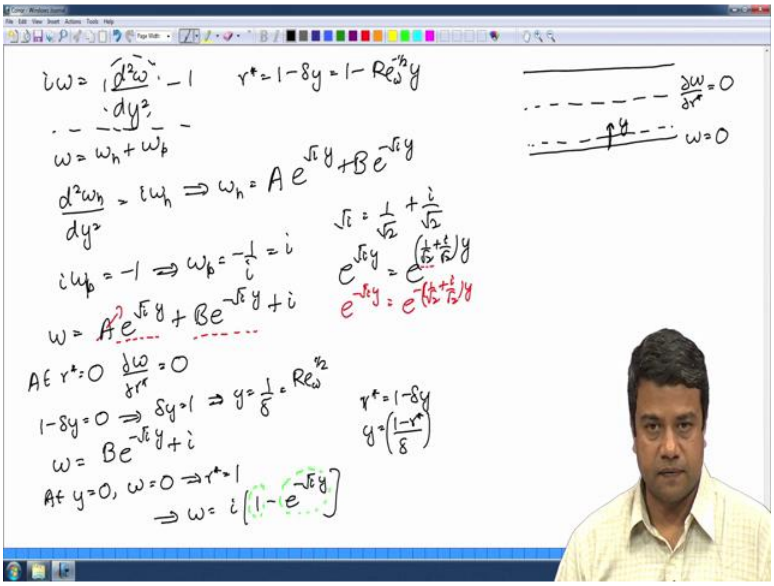

So, therefore, within the small layer they have to solve the differential equation - i w is equal to d square w by d y square minus 1; where y r star is equal to 1 minus delta y is 1 minus Re omega bar minus half. So, what this balance has basically told me is that the thickness of this layer is Reynolds number power minus half, as I said it goes to 0 as an ensemble goes to infinity in such a way that within this region inertial and viscous effects are continued to be of equal magnitude within this region along, in the outer region of course, inertial effects are much more dominant in comparison to viscous effects.

So, now we can solve this equation w can be written as this is an inhomogeneous equation therefore, w has 2 parts: 1 is the homogenous part plus a particular part; the homogeneous part satisfies the homogeneous differential equation d square w homogeneous by d y square is equal to i w, which implies that w homogeneous is equal to e power root of i into y.

Particular solution we had already derived earlier, the particular solution is any 1 solution that satisfies this equation and the simplest solution that satisfies this equation is a constant; i w particular is equal to minus 1, constant because if I take the derivative of this I will get 0 therefore, this term for the particular solution ends up being 0, which implies that w particular is equal to minus 1 over i which is equal to plus y therefore, the total solution for w is equal to A e power root of i y plus B e power minus root of i y plus i. So, there is the total solution for w how do we find the constants? We have to enforce the boundary conditions; the boundary conditions were that partial w by partial r star is equal to 0 at the center and w is equal to 0 at the wall. Now by this rescaling that I have done, I do have constants of integration which can be used in order to enforce the boundary condition.

So, therefore, the boundary condition is that at r star is equal to 0, partial w by partial r star is equal to 0; note that root of i is equal to square root of i is equal to 1 by root 2 plus or minus i by root 2 plus i by root 2 therefore, e power root of i y is equal to e power 1 by root 2 plus i by root 2. Now at r is equal to 0 it is the center, that corresponds to the coordinate y going outside this viscous layer at the wall, this corresponds to the coordinate y going outside the viscous layer; at this location you would expect that the velocity correction due to the viscous layer at the wall has to decrease to 0 the velocity correction due to this viscous layer has to decrease to 0, because r star is equal to 0 corresponds to 1 minus delta y is equal to 0 which is delta y is equal to 1, which means that y is equal to 1 by delta which is equal to Re omega power half.

So, this corresponds to y going to infinity in the limit as Reynolds number is large, delta is small therefore, this corresponds to y going to infinity, in that limit you require that the derivative has to be 0. Look at the two terms here, the first term is exponentially increasing because it has the real part to a positive power; the second term is exponentially decreasing because e power minus root i y is equal to e power minus 1 by root 2 plus i by root 2 y, this is exponentially decreasing. The exponentially increasing term does not have 0 slopes at the origin therefore; I have to set this constant to be equal to zero.

So, basically the requirement that the perturbation due to the presence of the wall has to decrease into the fluid fixes, that you can have only the exponentially decreasing term in the solution. So, therefore, my solution becomes of the form which satisfies the boundary condition at r is equal to 0 minus root i y plus i and then I have at y is equal to 0, w has to equal to 0 because this corresponds to r star is equal to 1 at the wall w has to be equal to 0 and therefore, this gives me the solution for w is equal to i into 1 minus e power minus root i. So, that is the solution for this w. Note that we recover in the limit as y going to infinity as in the bulk of the flow when you go outside of this boundary layer, we recover the solution that we had originally got when we neglected the viscous terms, we get this additional correction to that only very close to the surface.

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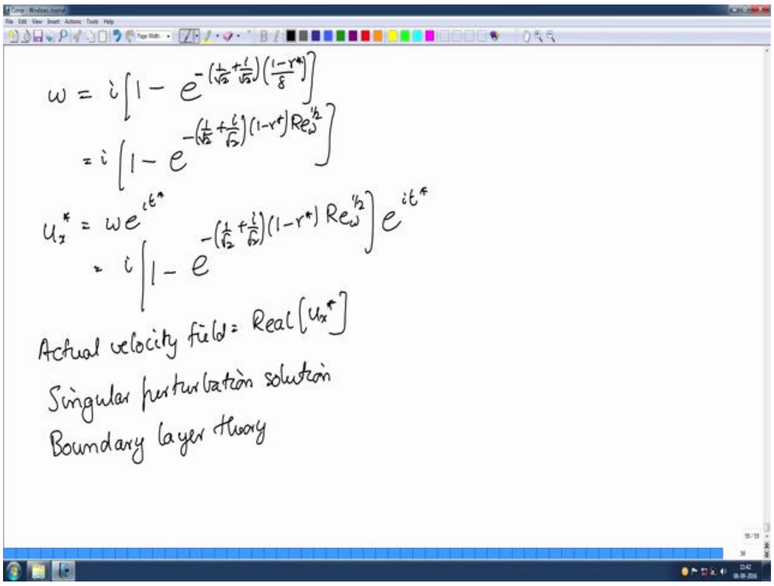

So, if I go back and I write my function w in terms of substitute for y, we know that r star is equal to 1 minus delta y which means that y is equal to 1 minus r star by delta and delta was Re power minus half. So, solution I will get is 1 minus e power minus 1 by root 2 plus i by root 2 into 1 minus r star by delta; which I can also write it as 1 minus e power minus into 1 minus r star; and delta we know is Re omega power minus half. So, that is w. U x star is equal to w e power i t star is equal to i into 1 minus e power minus; I am sorry this should be Re power plus half because e power i t star and the actual velocity field is just equal to going to be equal to the real part of U x star. So, that is going to be actual solution for the velocity field.

So, I will briefly continue this in the next lecture to explain the physics once again and then we will come back and we will solve problems in the next coordinate system which is a spherical coordinate system. So, I will briefly continue this in the next lecture to explain the physical principles behind this kind of a solution, this is called a singular perturbation solution and this is often used in what is called boundary layer theory and convective effects are large compared to diffusive effects. I will go through this a little bit in the next lecture and then we will proceed to looking at this spherical coordinate system I will see you then.