Transport Processes I: Heat and Mass Transfer
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Lecture - 42
Unidirectional transport: Oscillatory flow in a pipe. Solution using complex variables contd

We continue our discussion of Time Dependent Oscillatory Flows in a Pipe. If you recall we had considered the pressure gradient in the pipe to be an oscillatory function in time.

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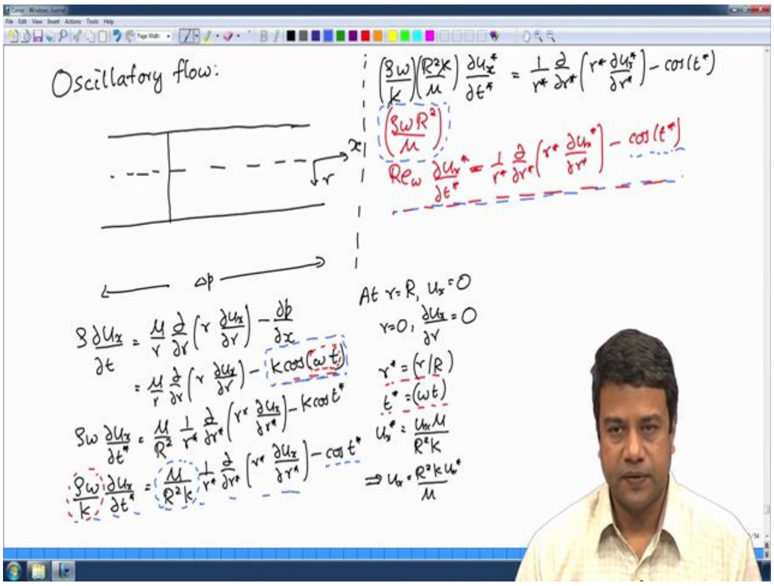

It scales as cos of omega t over here rather than having the steady pressure gradient we had considered an oscillatory pressure gradient. And the momentum balance equation of course was the same; this was for a fully developed flow. Assumption is that there is no variation in the velocity in the downstream direction, but the flow was not steady it was oscillatory. Can a flow be fully developed and unsteady? In this particular case the maximum velocity by which pressure is transmitted is at the speed of sound; the pressure transmission takes place through the fluid at the speed of sound. So, one can never have a fully developed flow because the pressure perturbations have to travel at the speed of sound so long as the distance is small. The time scale for pressure transport across the pipe will be much smaller than the time for the mean velocity.

So, in that limit we can consider the flow to be fully developed, but still unsteady. The assumption is that pressure is transmitted instantaneously infinitely faster than the fluid velocity. In that case we can consider the flow to be fully developed that is no variation in the stream wise direction but still unsteady, because any pressure differences that he put across the ends they travel at the speed of sound which is much faster than the fluid velocity. So, that is the assumption here that the pressure perturbations propagate instantaneously.

So, we had written down these equations for the oscillatory flow in a pipe and we had scaled them. In this particular case we had chosen the viscous scales for the scaling of the equations. And we had got an equation of this kind which contained the Reynolds number that Reynolds number was based upon the frequency of the oscillations and the radius of the pipe and the kinematic viscosity.

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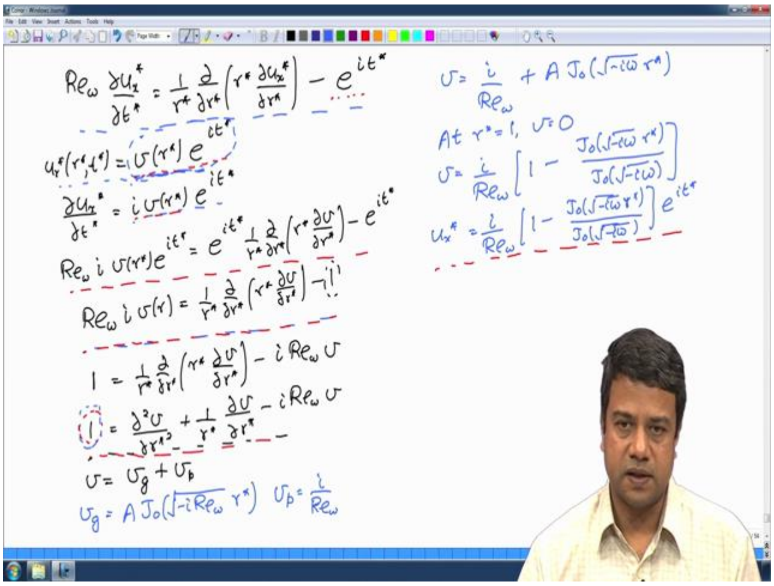

And we had actually got a analytical solution for this particular problem. In most cases when we consider transport phenomena of fluid mechanics in complicated geometries, we will not get an analytical solution. In this particular case we did get an analytical solution by first writing on the velocity as some function of the radius times e power i t. The real part of e power i t is the cos function, therefore the real part of the complex velocity field that I get by solving this equation should be the solution for the inhomogeneous term being the cos function.

Now, I can separate out the dependence on time and the dependence on position here, because of the equation is linear and the inhomogeneous forcing is of the form e power i t then one would expect the solution also to have that same frequency. There may be a phase shift, but for a linear equation the frequency has to be the same. So, on that basis we had done this decomposition of the velocity into a part that depends only on radius times and oscillatory function in time. And once you do that you get for the part that depends only on the radius that is v of r here you get an equation which is an ordinary differential equation in the radius which can be solved.

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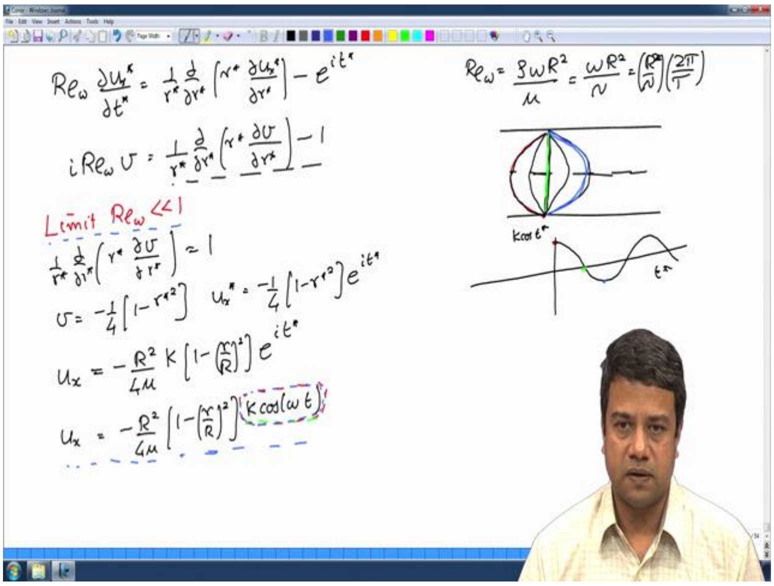

And as I told you that solution is a numerical solution, you can evaluate it, you can get numerical velocity profiles for that solution, but that solution in itself does not give you a physical understanding of what is going on in this flow. In order to do that we considered limiting cases: The first limiting case was when the Reynolds number is small at given you the physical interpretation of that. The time for momentum diffusion across the pipe is much smaller than the time period of variation of the pressure function, the time period for the oscillation of the pressure function.

In that case at any instant in time the flow looks like a steady flow, but with an oscillatory pressure function. And with that if I solved the equation I just get the velocity profile to be the parabolic velocity profile, but with the pressure gradient given by the instantaneous pressure gradient at that instant in time. So, that is what turns out to be the solution for this in this case.

Now, this is of course only when the Reynolds number is exactly 0. If the Reynolds number is small, but is nonzero can we get an improved solution for the velocity profile? In other words can be systematically improve this velocity profile to take into account the effects of inertia in the limit of low Reynolds number.

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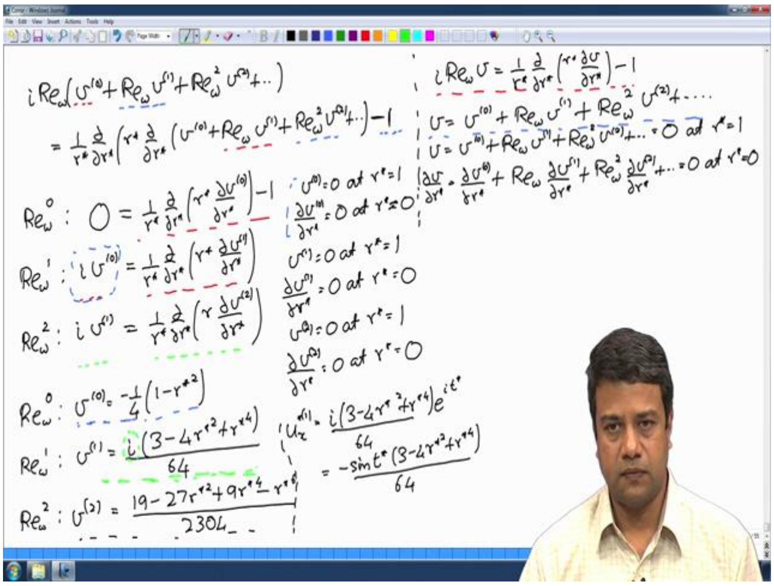

So, back to my equation for v I have i Re omega v is equal to plus, sorry minus 1. Now, since Re omega is a small number and we already have the solution when Re omega is 0, what one can do is to write v as the sum p naught plus. So, in the limit as Re omega goes to 0 the second term in the series will be much smaller than the first one, the third term will be much smaller than the second one and so on.

So, this expansion I can insert back into the equation for v. So, if I do that what I get is i Re omega into v naught plus is equal to; and then had a inhomogeneous term minus 1. So, that is the expansion that I get. Now I can equalize terms which are of equal powers of Re omega on the left and the right side, I can equalize, I can write equations for term set of equal power of Re omega on the left and the right side. If I take Re omega is going to 0 the power of Re omega power 0 terms which have 0th power in Re omega these are terms which are not multiplied any power of Re omega in this equation. In this particular case on the left side I get 0, because even the left side the first term itself is multiplied by Re omega. So, on the left side I get 0, on the right side I get 1 by r d by dr of r d b 0 by dr minus 1.

So, that is all terms in this equation which are have the multiplied by the 0th power of Re omega. What about terms that are multiplied by the first power of Re omega, power 1. On the left side I have i times v 0, on the right side the term that multiplies Re omega power 1 is this one is equal to note that there is no inhomogeneous term because this term is only multiplied by Re omega power 0. Then if I take terms that are multiplied by Re omega power 2 I will get i v 1, this 1 is multiplied by the square of Re omega here on the left side. And on the right side I will get 1 by r d by dr of r dv 2 that is this one and so on. You can get higher and higher terms.

We have to also get the boundary conditions at 0 1 2 order. In this particular case at each order we require that v 0 is could be equal to 0 at r star is equal to 1 and dv 0 by dr. The condition at r is equal to 0 is the symmetry boundary condition, that the velocity gradient has to go to 0. And at r is equal to 1 is the wall of the pipe and at that location the velocity has to be 0 at all instants of time. Same thing applies for v 0, v 1 etcetera because I can expand the boundary conditions as well. So, the boundary condition is that v is equal to v 0 plus v 2 0 at r star is equal to 1 and partial v by partial r which is; so that is the second condition.

And each of these boundary conditions you can take all terms of coefficients Re omega power 0 power 1 or 2 and so on, and get boundary conditions individually for each of those, and each of these will be exactly of the same form; v 1 is equal to 0 at r star is equal to 1 and partial v 1 by partial r star; and similarly v 2. So, those are the boundary conditions at each order. And you can sort of see a way in which we can solve the series of equations. For the first equation it contains only v 0 and the inhomogeneous term, this was exactly the same equation that we had in the limit of 0 Reynolds number which we had just analyzed. Therefore, we have a solution for that; we have a solution for the first equation. That solution is v 0 is equal to minus 1 by 4 1 minus r star square.

Now, the equation at Re omega is order one here. This v 0 is appearing as an inhomogeneous term on the left hand side. The boundary conditions are homogeneous the boundary conditions are just at the velocity 0 at 1, the gradient is 0 at 0; boundary conditions are homogeneous. This inhomogeneous term the equation that we had already solved for is appearing as an inhomogeneous term, therefore I can integrate it and get the solution; I can integrate it straight away and get the solution. Therefore, at Re omega for 1 the solution is of the form i into 3 minus 4 r square plus r power 4 by 64.

And if you look at the next equation, I already have an equation for v 2 in which v 1 appears as an inhomogeneous term. However, v 1 is already evaluated here. So, I can just put in that inhomogeneous term and get the solution. The solution is a little bit complicated, but it is straightforward you can just integrate out this equation with the boundary conditions to get the solution; 2304. So, as you can see you get a series of solutions of higher and higher order. That was obtained by doing this expansion for the velocity field; expanding the velocity field in terms of this function Re omega.

So, you can systematically improve, you can get the order Re correction the order Re square correction Re cube correction and so on. As I said before the order are e correction was just real, so if I take multiply this by e power i t and take the real part I will just get cos t; that is exactly in phase with the pressure variation as I showed you in the last lecture; this function the velocity is exactly in phase with the pressure variation. The first correction here is multiplied by i. So, if I take i times e power i t and take the real part of that I will get something that is proportional to the sin function, and therefore that is out of phase. So, the presence of inertia gives you a solution that is out of phase. So, it creates a phase difference between the pressure gradient in the velocity field, because I have an i here therefore I will get u x star 1 will be equal to i into 3 minus 4 r square plus r power 4 divided by 64 e power i t. If I take the real part of that, I will get minus sin t star into r square; then I am getting something that is exactly out of phase.

The next higher order term is once again in phase it is real; next one will be out of phase and so on. So, you can systematically correct the solution to get higher and higher order terms. These are called asymptotic expansions in the limit of Reynolds number going to 0. At Reynolds number is equal to 0 the flows purely viscous, but if the Reynolds number is small but nonzero you can bring in the effects of inertia on this flow like actually calculating the corrections to the velocity profile due to inertial effects.

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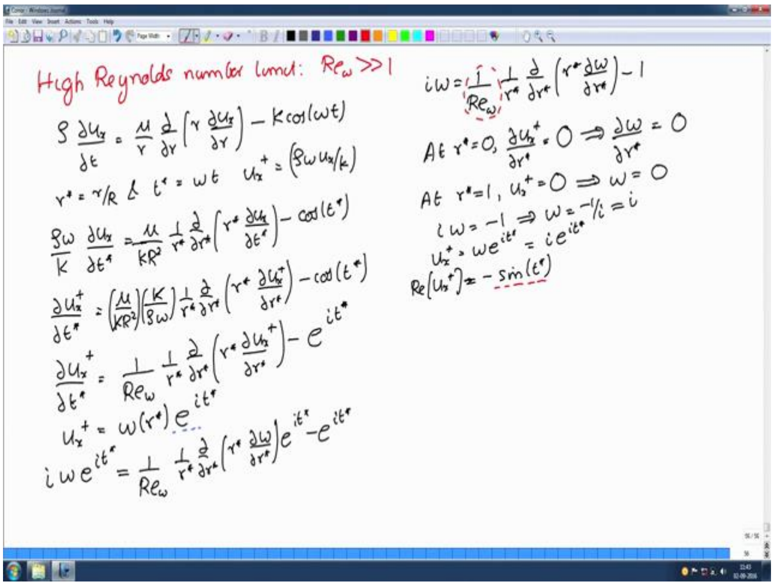

So, everything we have discussed so far is in the low Reynolds number limit. Let us go to the high Reynolds number limit. In this particular case if you recall previously we had scaled the velocity by the viscous scales. In this particular case at high Reynolds number since inertia is dominant compared to viscosity we have to scale the velocity by the inertial scales. Let us go back to the original equation: rho times d u x by dt is equal to mu minus k cos of omega t, we had scaled r star is equal to r by r and t star is equal to omega t. With that I had got rho omega times partial u x by partial t star equal to mu by r square minus k cos t star. And I had divided this equation throughout by k so that the inhomogeneous term became dimensionless.

So, I divided this equation throughout by k; they have a k in the denominator here. And this k here disappears. Now I have to scale the velocity by the inertial scales. So, I should define u x star is equal to rho omega u x by k. If I do that on the left side I get partial u x star by partial t star will be equal to mu by k r square into k by rho omega into; in order to not confuse with the viscous scaling that I had used previously I will define this with a plus sign, then a star I will define it with a plus sign. So if I do that, understood that u plus is scaled by inertial scales. And then my equation for u plus becomes partial u x plus by partial t star. You can easily see that the k cancels out here and I just get the factor of 1 over Re omega on the right side minus cos t star.

As before instead of using cos as the inhomogeneous term I prefer to use e power i t; if I use e power i t the only differences that I will get a complex solution for the velocity, when I take the real part of that I will get the actual physical velocity profile. So, now when using cos I will use e power i t star. And then I express u x plus is equal to some w of r star some other function e power i t star and insert that into the balance equation, I will get i times w times e power i t star. Because if I take the derivative of this e power i t with respect to time I just get a factor of i coming out, and w is only a function of r so when I take the derivative of that it comes out of the differentiation.

This becomes 1 by Re omega into 1 by r d by dr of r d w by dr e power i t minus e power i t. And I can divide throughout by e power i t to get an equation which is an equation for w alone. That equation for w will be i times w is equal to minus 1. So, that is the final equation for w. And w has to satisfy the boundary conditions at r is equal to 0 partial u x plus by partial r is equal to 0; the symmetry condition because of the cylindrical coordinate system. And that implies that partial w by partial r is equal to 0. And at r star is equal to 1 u x plus has to be equal to 0, because you have a no slip condition at the wall at r is equal to 1 and that implies that w is equal to 0, good.

So, the only difference was that I have a 1 over Re in the viscous term instead of an Re in the inertial term. In the limit of Re going to 0 the inertial term goes to 0 because it is multiplied by Re. In the limit of Re becoming large the viscous term can be neglected because it is multiplied by 1 over Re. And if I neglect the viscous term I just get an equation i w is equal to minus 1 which implies that w is equal to minus 1 over I, it is just equal to i itself. So, that is a solution for w. The solution for u x plus will be equal to w e power i t is equal to i times e power i t.

And the real part of that will be u x plus will be the real part of i e power i t which is equal to minus sin of t star. That is the solution that we have and we have to now enforce the boundary conditions. Boundary condition at r is equal to 0 d w by dr is equal to 0 this equation does satisfy the boundary condition. The second boundary condition at r is equal to 1 w has to be equal to 0. Does this equation satisfy that boundary condition? We cannot satisfy the boundary condition, because we did not have any constants of integration in this equation. Why is that? In the limit of high Reynolds number when we had neglected the viscous term we are not able to satisfy the boundary condition, think about it.

We will continue this in the next lecture. Why is it that we are not able to satisfy the boundary condition in this case? I will give you the physical reason as we go along in the next lecture. We will see you then.