Transport processes I: Heat and Transfer

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Lecture - 41
Unidirectional transport: Oscillatory flow in a pipe. Solution using complex variables

Welcome to our continuing discussion of Flow in a Pipe as a part of our course on Fundamentals of Transport Processes.

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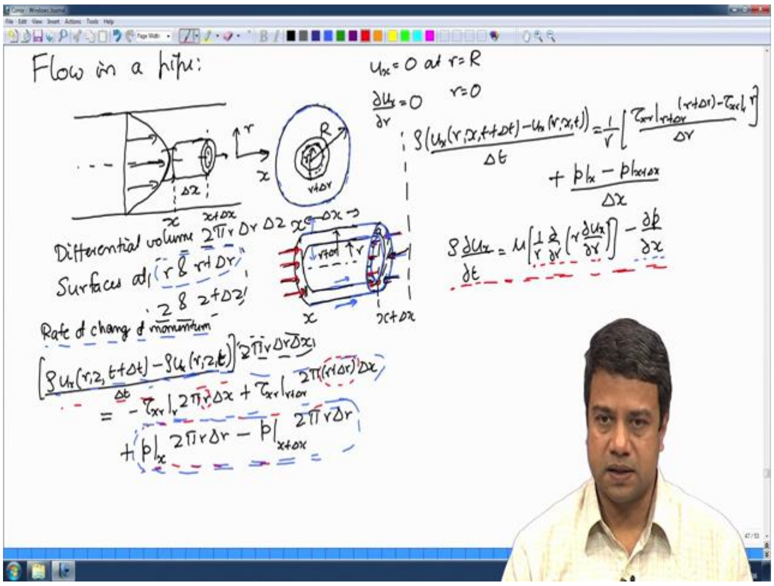

I had in the previous lecture derived for you the momentum balance equation for the flow in a pipe. It contains the inertial term on the left, a viscous term on the right, not quite the second derivative but slightly more complicated because our surfaces of constant coordinate our curved surfaces. And there is also a pressure gradient on the right. The pressure gradient arises because the flow is being driven by a pressure difference across the ends and that pressure acts along the inward perpendicular to each differential volume. So, for the x momentum equation that pressure acts on the annular surfaces which are perpendicular to the x direction or the axial direction, whereas the shear stress acts at the curved surfaces where the unit normal is to the radial direction; this stress acts tangential to the surface.

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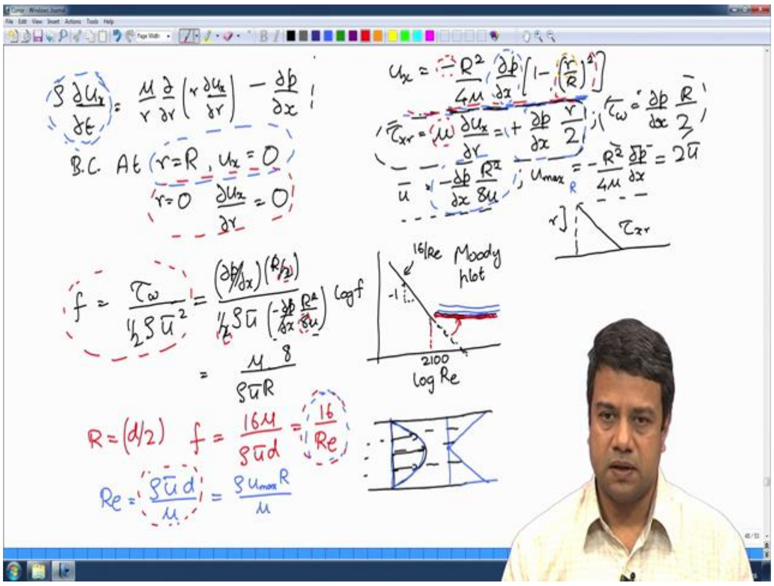

We solve this first for a steady flow, and we had obtained the parabolic velocity profile and calculated the shear stress the average velocity the maximum velocity and so on. And we had got this expression for the friction factor versus Reynolds number. Friction factor is 16 by Re. And as I told you in the last lecture this is only for what is called a laminar flow, as the Reynolds number is increased at some point the flow undergoes a spontaneous transition to what is called a turbulent flow.

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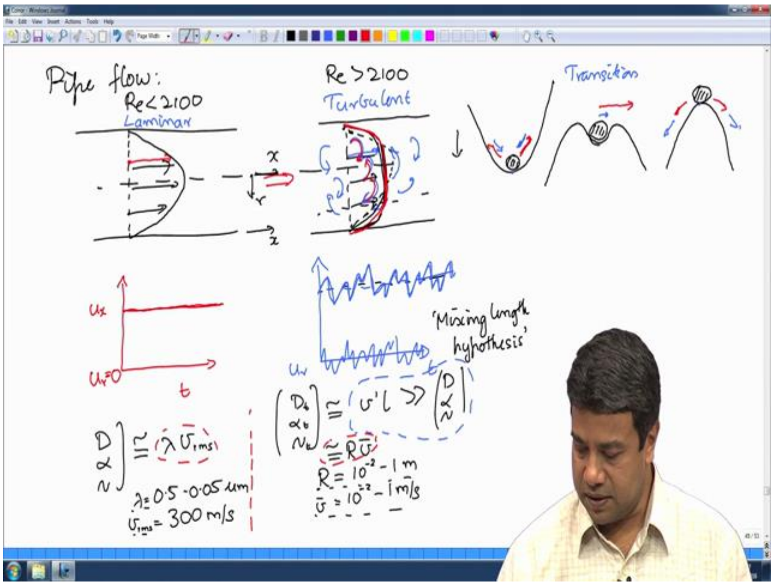

In the turbulent flow the fluctuating velocities are not 0, the flow is inherently unsteady. There are velocity fluctuations in all directions not just in the stream wise direction. And it is these fluid velocity fluctuations rather than the molecular velocity fluctuations that actually transport mass momentum and energy in a turbulent flow. And the diffusivities due to these fluid velocity fluctuations are actually much larger than those due to the molecular velocity fluctuations.

And therefore, in the centre of the pipe the velocity that the transport takes place primarily due to fluid velocity fluctuations. The transport rates are much faster and that is why the gradient is much smaller than what you would expect for a laminar flow. The parabolic profile for a laminar flow near the wall of course these fluctuations have to decrease to 0.

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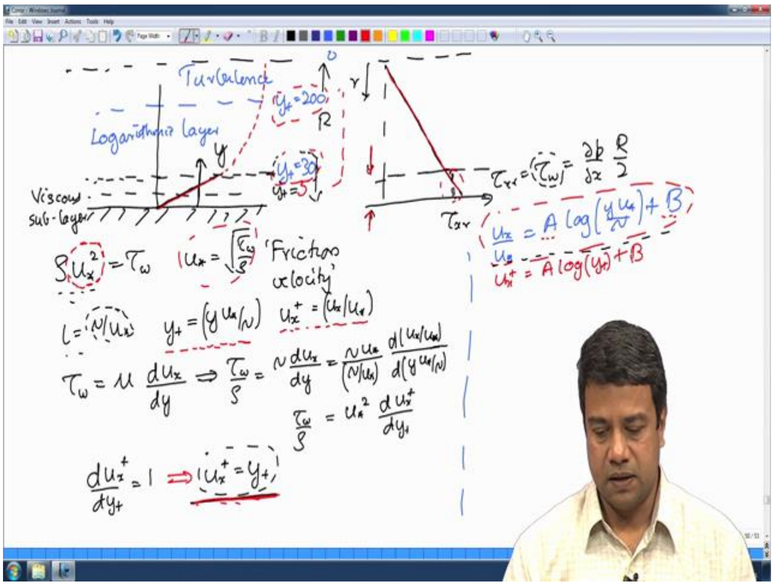

And we had discussed some characteristics of the near wall variation of the velocity profiles. While the flows highly fluctuating has large fluctuating turbulent fluctuating velocities near the center these velocities have to decrease to 0 on the other wall, because the fluid velocity has to be 0 at the wall. Similarly, the size of those eddies has to decrease to 0 at the wall. And therefore, you have a thin region close to the wall called the viscous sub layer where the flow is viscous, laminar; the transport takes place due to molecular diffusion rather than due to turbulent discussion.

Within this region the relevant parameter is the shear stress and from that we get the velocity scale as a friction velocity. And if I scale the velocity by the friction velocity the velocity gradient is just equal to 1 when the coordinate is skilled in wall units. The wall unit is of course the ratio of the kinematic viscosity and the friction velocity because these are the only two parameters that are important in this region. And I told you that there is also a logarithmic layer where you have a logarithmic law for the velocity profile. And then the logarithmic layer varies from about y plus is equal to 30 to about y plus is equal 200. That is the classical picture though I should warn you that this picture keeps getting refined in time. The viscous sub layer is about y plus b equal to 5.

So, that was what we had seen in the last few lectures.

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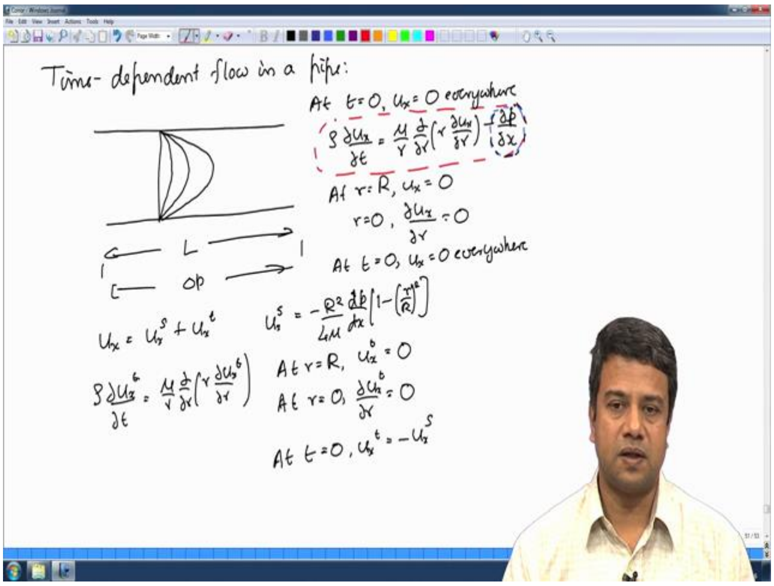

I had told you briefly how to solve the time dependent problem for the flow in a pipe. It was very similar to the flow down an inclined plane that we had done previously.

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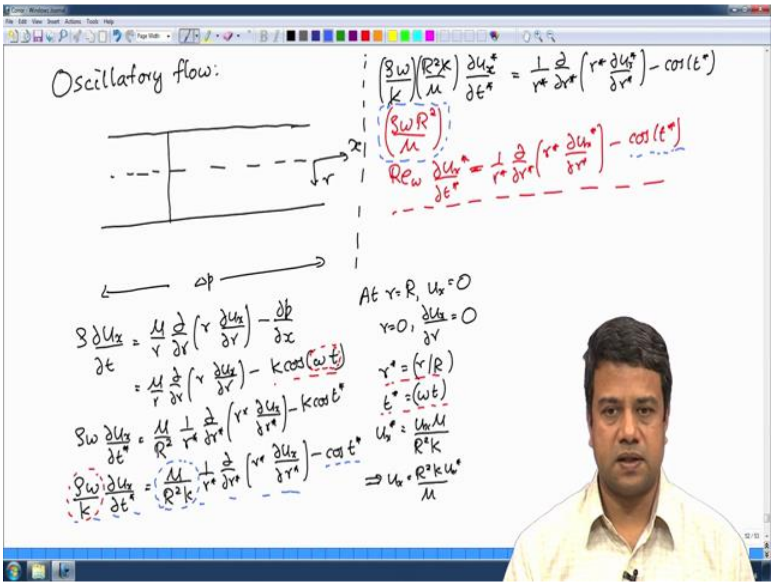

And where we ever had last lecture was to discuss the oscillatory flow in a pipe. This of course has applications, and physiological flows for example the heart is constantly pumping in an oscillatory manner. The pressure signals from the heart of course have a complicated form. In this particular case we considered a very simple form for the pressure variation just a sine function. However, as I told you in the last lecture this is not a severe loss of generality because, any complicated wave form can be written as the sum of sine functions; the sine and cosine functions.

So, what we had done last class was to first scale the equations. We had used the frequency itself for the time scale and the radius for the length scale, and when we scaled these equations we got an equation of this kind where the inertial term on the left is multiplied by a Reynolds number. And that Reynolds number is based upon the frequency of oscillations and the radius of the pipe. And then we had a viscous term on the right. In this particular case we had chosen to scale the velocity by the viscous scale. So, that the coefficient of this velocity u is equal to 1. And then we had this forcing term. So, it is a linear differential equation with an inhomogeneous forcing term.

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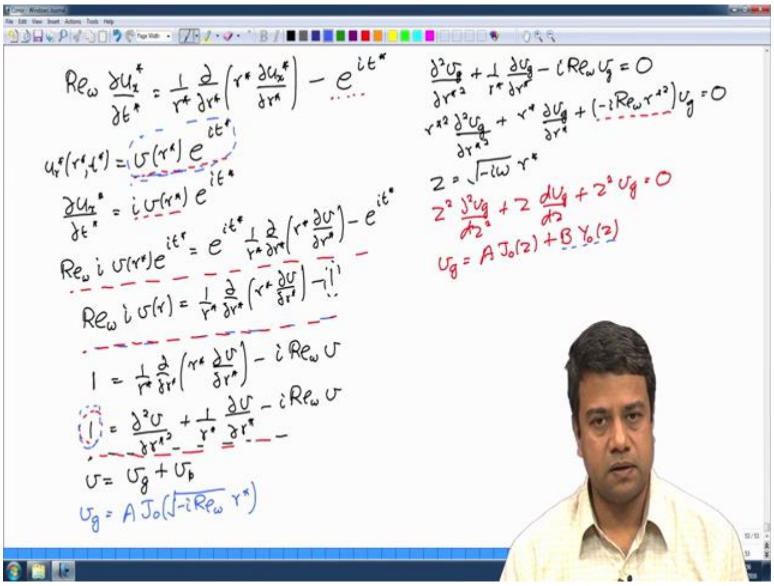

And we had started to solve that; the equation itself is a partial differential equation which is being forced by an inhomogeneous term which is sinusoidal in time. Since this is a linear equation it is linear in the velocity if the forcing is sinusoidal in time one would expect the response as well to be sinusoidal in time. Based upon that we had postulated that the velocity has got to be equal to some function v times e power i t.

Recall that in this particular case we are solving for the complex velocity, because we have a complex forcing function. The actual physical velocity is just going to be the real part of this complex velocity, because the physical forcing is the real part of this complex forcing function. We have written cos of t as e power i t. Cos of t is actually only the real part of a variety. So, we are solving this equation and we had postulated that the response for the velocity should also be sinusoidal in time and we had inserted it into the conservation equation. And of course, the factor of e power i t cancels on all the terms. I just get an inhomogeneous equation which is only a function of the radial coordinate; inhomogeneous but now a ordinary differential equation which we can solve.

So, as I had rewritten it in the last lecture it contains an inhomogeneous term on the left and a term which is a function of velocity on the right. So, how do we solve an inhomogeneous differential equation, a linear differential equation? We separate it out into a general solution and a particular solution. And the general solution is one that follows the same differential equation, but with the inhomogeneous term set equal to 0.

So, that is the general solution. This particular solution we had already seen in the context of cylindrical coordinates. The way you solve it is to multiply it both sides by r square. So that is going to be the differential equation just, rewritten by multiplied throughout by r square.

Now you can see that on the left side the first two terms are 0 dimensions in r. So, if I scale r by any value is first two terms do not change. So, I can for example scale, define a new coordinate z is equal to root of minus i omega times r. That way, so this is just a new coordinate; is if I do that the first two terms do not change that last term just becomes z square. So, the equation now becomes z square d square v general by d z square plus z dv general by d z plus z square v general is equal to 0. Just by rewriting the independent coordinate and this is an equation we have already seen in the context of heat transfer and cylindrical coordinate. The solution is just of the form A J naught of z plus B Y naught of z; the Bessel functions. This is a Bessel equation of 0th order. Recall that the Bessel equation was z square d square by d z square plus z dv by d z plus z square minus n square v is equal to 0 In this particular equation the index n is equal to 0 and therefore the general solution is just A times J naught of z plus B times Y naught of z.

Of course I had explained in my discussion on heat transfer this function Y naught goes to infinity as z goes to 0. Therefore, the coefficient has to be equal to 0 and I get the general solution as just A times J naught of z which is root of minus i Re omega times r star. So, that is the general solution for this differential equation. What about the particular solution? The particular solution is any solution that satisfies this equation; the inhomogeneous equation.

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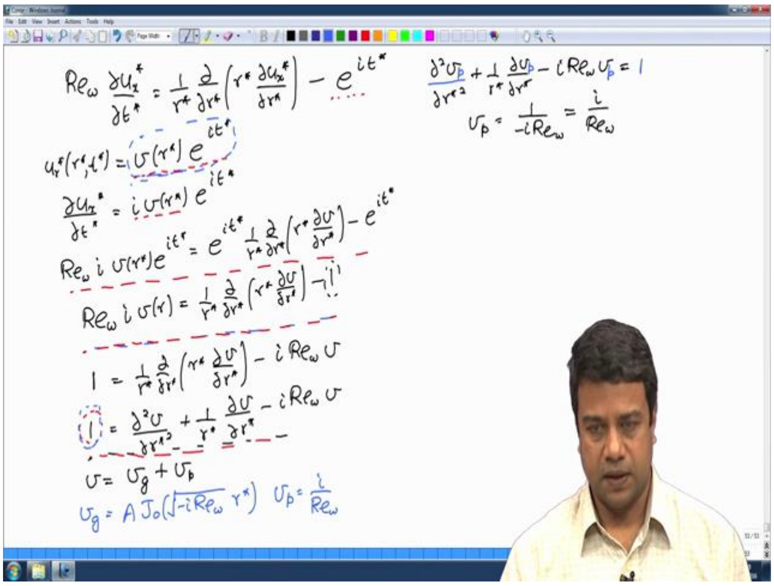

That satisfies the inhomogeneous equation, and I just need one solution so I can just choose the simplest one. And the way you find out the solution is to first test there is a constant satisfy this equation. In this particular case a constant just does satisfy this equation, because if the particular solution were just equal to a constant these derivatives both will be 0 and I will get that v particular is equal to 1 by minus i Re.

Note that i is the square root of minus 1. So, I can also write this as i by Re omega. So, this is the particular solution. The particular solution is v p is just equal to i by Re omega. And the total solution is the sum of these two.

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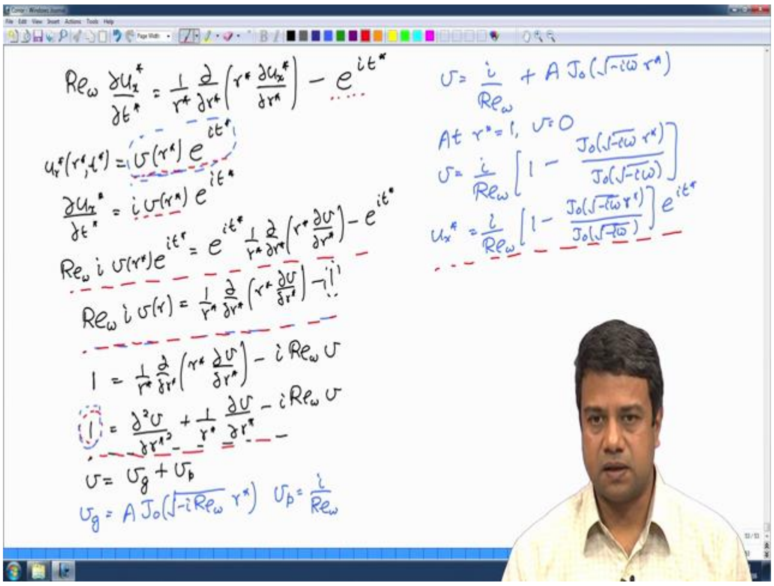

The total solution v is equal to i by Re omega plus A J naught of root of minus i omega r star. We have one constant left, we had already set one constant equal to 0 in the general solution because we had required that the velocity has to be finite at r is equal to 0. And the second Bessel function y 0 did not satisfy that condition. Now we have one more constant and that has to be determined from the condition that at r star is equal to 1 v is equal to 0 the most left condition that the velocity has to be equal to 0 at r star is equal to 1. And using that you can quite easily determine what is the solution v is equal to i by Re omega into 1 minus. So, that is the final solution for v.

And the final solution for u z u x star is going to be equal to this times e power i t star. That is the complex velocity field, and the actual velocity field that you will see in practice is the real part of this complex velocity field. So, we have managed to get solutions for this velocity field. It is in terms of Bessel functions, nevertheless it can be evaluated and you can actually plot the velocity field as a function of distance and the radial coordinate at different points in time.

But as you can see an expression like this does not give you very much physical insight. And in order to get physical insight we have to go back end and look at the equation consider different limiting cases and try to get solutions which will give us physical insight into the structure of the flow in different limiting cases. In our initial discussion I had said that the fundamental balance here is between viscous and inertial forces. In the limit when the inertial forces are small compared to the viscous forces, you get one type of solutions. You will (Refer Time: 17:33) where the inertial forces are large compared to the viscous forces you get another type of solution.

Can we get those two solutions types of solutions in this case? So, let us go back to the equation and try to use some physical understanding, in order to get solutions which are valid in different limiting situations.

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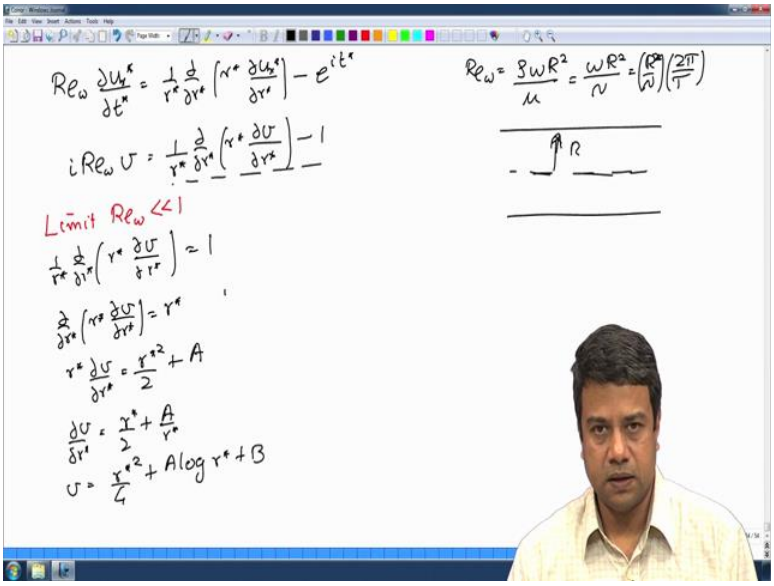

So, our equation was Re omega u x star by d t is equal to minus e power i t star. Or if I had written in terms of the function v I get only an our ordinary differential equation, I am sorry I should not be there; i Re omega v is equal to minus 1, if I take the limit where the inertial forces are small compared to the viscous forces. Note that Re omega is equal to rho omega square, I am sorry. This can also be written as omega R square by the kinematic viscosity. R square by mu is a viscous time scale, so you can write this as R square by mu which is a viscous time scale; time it takes for momentum diffusion across the radius of the pipe, times omega which is 2 pi divided by the time period of the oscillation.

So, this is the physical interpretation of this Reynolds number. The viscous time scale divided by the period of the oscillation, the viscous time scale for momentum diffusion across the radius. In the low Reynolds number limit this viscous time scale is much smaller than the time period of the oscillation. The time it takes for momentum to diffuse across the radius of the pipe is much smaller than the period of the oscillation. So, at each instant in time the momentum is diffusing instantaneously across the entire pipe. So, that is the physical interpretation of the limit of low Reynolds number in this case.

So, at low Reynolds number I have to solve just the viscous part of the equation; 1 by r d by dr of r d u x by dr is equal to 1; I am sorry I should put v here please cut me this correction. This is just equal to 1. And I can solve this, I will get v is equal to 1 by 4 r star square plus C 1. So, let us integrate this. I will get d by dr of r dv by dr is equal to r which means that r dv by dr is equal to r square by 2 plus some constant A dv by dr is equal to r by 2 plus A by r. Therefore, v is equal to r square by 4 plus A log r plus B. So, that is the solution.

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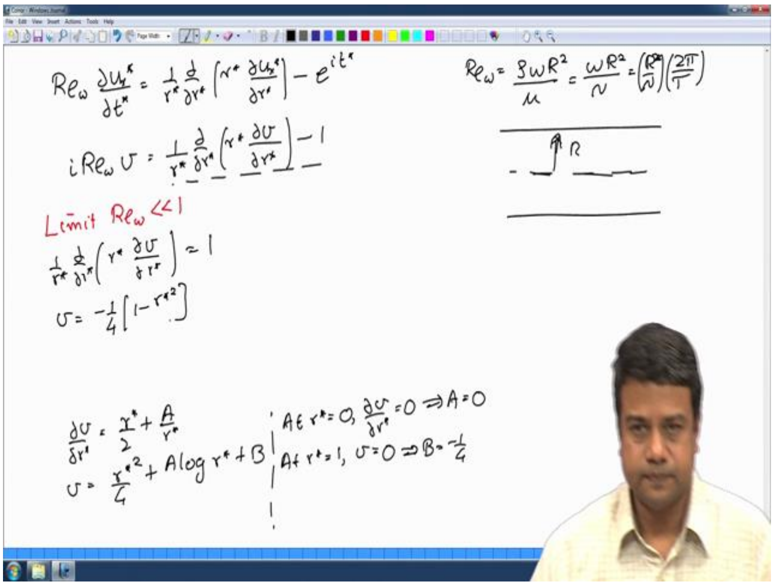

And the requirement that the velocity has to be finite at r is equal to 0 means that a has to be equal to 0; v is equal to required that the slope of the velocity at r is equal to 0 dv by dr is equal to 0 implies that A equals 0, because A goes has a logarithmic functions which take the derivative it goes to infinity at r is equal to 0. And at r star is equal to 1 v has to be 0 the no slip condition; at r is equal to 1. This would imply that B is equal to minus 1 by 4.

Therefore, my solution for v is equal to minus 1 by 4 into 1 minus r star square. You might recognize that this solution procedure is exactly the same procedure that we had used for the steady flow in a pipe.

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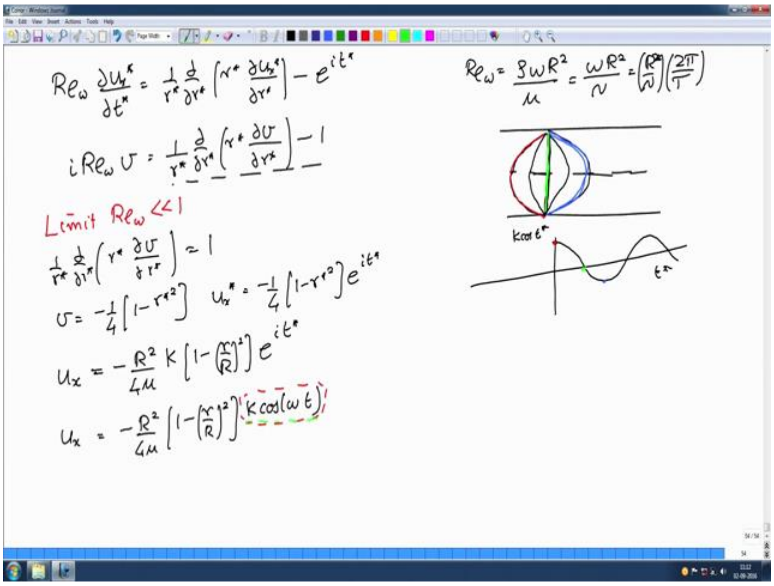

If I get u x star from this, I will get minus 1 by 4 into 1 minus r star square into e power i t star. And if I express the velocity back in terms of dimensional variables, if you recall we had scaled the velocity by k by mu r square. So, if I express the velocity back in terms of dimensionless variables I will get minus r square by 4 mu times k into 1 minus r by r the whole square times e power i t star. And if I take the real part of this I will get u x will be equal to minus r square by 4 mu into 1 minus r by r the whole square into k cos of omega t.

So, we get back the parabolic velocity profile except that instead of a steady pressure gradient we now have this oscillatory pressure gradient. So, at each instant in time the velocity profile is a parabolic profile, except that it has a pressure gradient which corresponds to the value of the pressure at that particular point in that oscillatory function. It has a negative sign because the flow is in the direction of decreasing pressure, and as you change the pressure the direction when cos t is 1 you will get the maximum velocity is parabolic then you will get slower velocity, when cos t is equal to 0 at half way to the cycle it is 0 and it exactly reverses for the other side of the cycle. This is and the pressure gradient is going through an oscillatory function, this is a cos function so should be careful here; k cos t star is on oscillatory function.

So, when the pressure is maximum in the positive direction the velocity has to be negative. At this point the velocity has to be negative because the pressure gradient is positive, so the velocity has to be negative. Pressure increase of downstream distance, whereas when it goes through the maximum negative you get this one is opposite parabolic velocity profile. And when it goes through 0 the velocity is also 0 throughout the pipe. So, you get the same solution that you got for a parabolic profile except that in this case the value of the pressure gradient is the instantaneous value, as it is oscillating.

And that is not too surprising physically; the low Reynolds number limit corresponds to the case where the momentum diffusion time across the pipe is much smaller than the period of oscillation. Therefore, at each instant of time the momentum is diffusing instantaneously and you get the time scale for variation of pressure is much longer. So, it looks like at each instant of time as far as momentum diffusion is concerned you just have a steady pressure gradient. And that gives you a parabolic velocity profile. The amplitude of that velocity profile changes as the amplitude of the pressure gradient changes.

So, we get back the exact same parabolic velocity profile except that it is the instantaneous, pressure gradient that has to be put in to this velocity profile. So that is the limit of low Reynolds number. However, even in the limit of low Reynolds number inertia should be important; even in the limit of low Reynolds number inertia could be important. If a Reynolds number is low, but not 0; in that case can we get an improved solution that includes the effects of inertia in this case.

So, how to get an improved solution that includes the effects of inertia in the low Reynolds number limit? There is something that I will start my discussion in the next lecture. And then we will proceed to looking at obtaining a physical understanding in the limit of high Reynolds number in this case. So, we look first at the limit of low Reynolds number next lecture and then we will go on to the limit of high Reynolds number. That will be the program for the rest of this analysis of oscillatory pipe flows. And I will use that as a way of illustrating the balance between convection and diffusion.

We will continue this lecture and then we will continue this discussion in the next lecture. We will see you then.