Transport Processes I: Heat and Mass Transfer

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Lecture - 40
Unidirectional transport: Oscillatory flow in a pipe. Solution using complex variables

Welcome to our continuing discussion on the Flow in a Pipe. We had first solved the momentum balance equation for the steady flow to get a parabolic profile for the flow in a pipe, and I had shown you that we get the equation for the friction factor for a laminar flow in a pipe I had discussed qualitatively some features of transition to turbulence and some features of turbulent flows.

Turbulent flows are in general difficult to describe so the simple techniques that we have developed here we will in general not be applicable to a turbulent flow. Despite that there are some gross or some broad features of the turbulent flow that can be deduced just from simple analysis and dimensional arguments.

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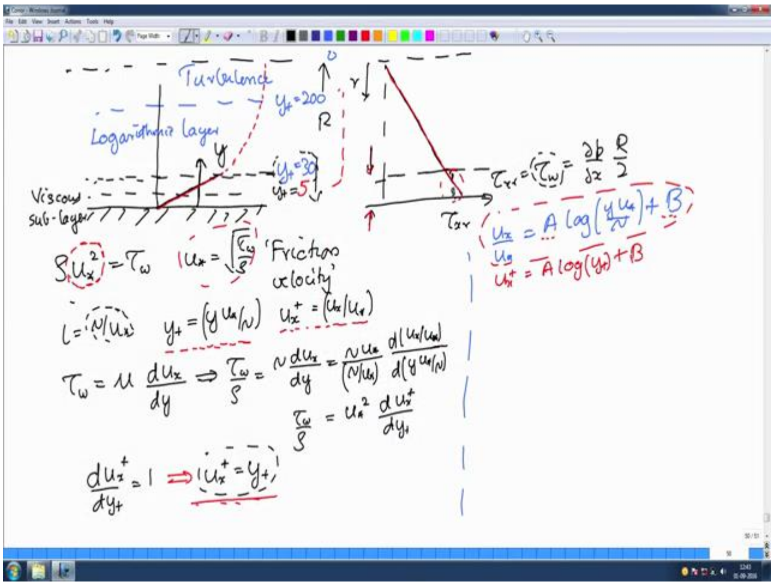

So, I had shown you that for a turbulent flow the bulk of the flow.

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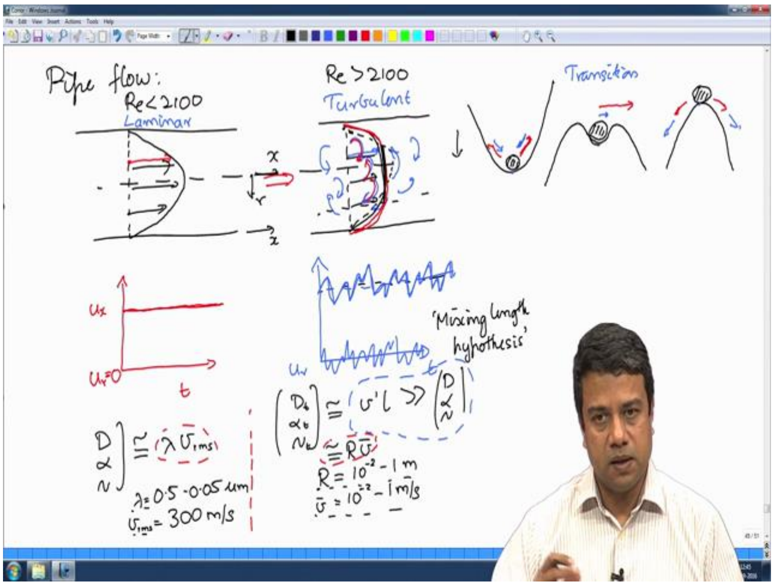

There are velocity fluctuations in all directions, there are eddy velocity fluctuations correlated parcels of fluid that are in correlated motion. So therefore, a turbulent flow is inherently unsteady and it is inherently three dimensional. The velocity fluctuations are not only in the flow direction they occur in all three directions. And these velocity fluctuations or these correlated parcels of fluid they transfer momentum far more efficiently than the transport due to molecular diffusion.

And for that reason, the transport coefficients in a turbulent flow turn out to be much higher than the transport coefficients in due to molecular diffusion. And for that reason the velocity profile is approximately flat here, because the transport takes place very fast. As you go close to the wall the turbulent velocity fluctuations have to decrease to 0, because the fluid velocity has to be equal to 0 at the wall. Therefore, as you go too close to the wall the turbulent velocity fluctuations have to decrease to 0. The length scale of these fluctuations also has to decrease to 0, because a regional correlated motion cannot have a size larger than the distance from the wall, because the velocity has to be 0 at the wall.

And because of that the diffusivity due to the turbulent fluctuations decreases to 0 as you go close to the wall, because both the magnitude of the velocity is decrease and the length scale decreases. And therefore, close to the wall you have a region called the viscous sub layer within which the flow is dominated by viscosity. You do not have turbulent diffusion. In this region, the characteristic velocity scale is obtained from the wall shear stress, because you have the region is a very thin region and you can consider the shear stress to be approximately a constant across that region that constant is equal to the wall shear stress.

From that we managed to get a friction velocity and a length scale, the wall unit. Wall unit based upon the kinematic viscosity and the friction velocity of the wall. So, close to the wall when you express the velocity in terms of the friction velocity and the distance from the wall in wall units you just get a linear profile u x plus is equal to y plus. Since the wall shear stress is much higher than what it would have been for a laminar flow, I said that the turbulent diffusivity is much higher, so the rate of transport to the wall is much higher than what it would have been for a laminar flow, Therefore, the slope near the wall is higher then what it will have been for a laminar flow.

That is the reason that close to the wall the gradient of the velocity is higher than what you would get for any current laminar flow. And therefore, the shear stress is also higher because the shear stress is the viscosity times the velocity gradient. And beyond this viscous sub layer you have a region called the logarithmic layer where the velocity profile is logarithmic in the distance from the wall. Then beyond that there is the bulk of the flow.

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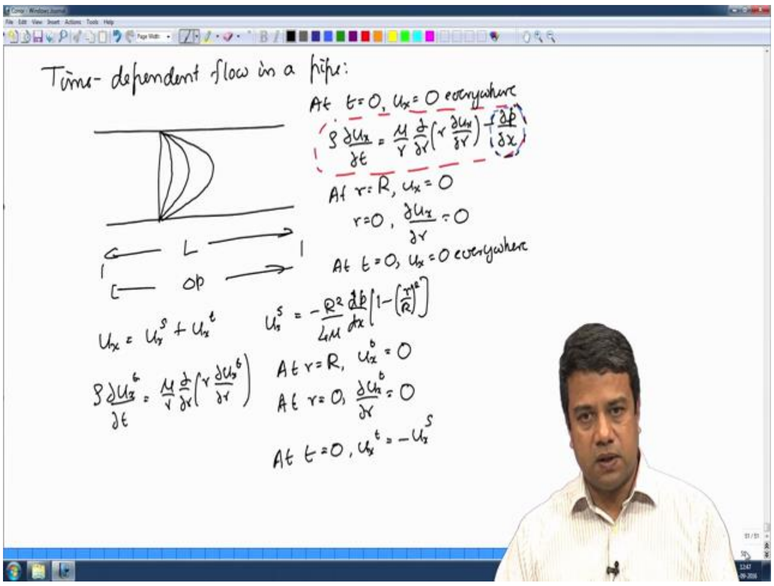

I said we will start a discussion on oscillatory flows in this lecture. I briefly told you how to get the solution for the startup of the flow in a pipe.

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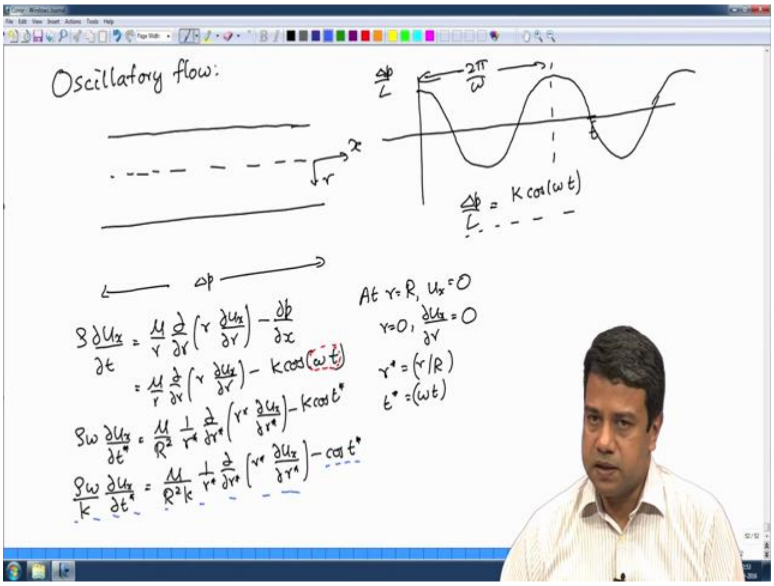

And in this lecture we will talk about an Oscillatory flow. So, have a pipe and I have a pressure difference across this pipe across this length L and this pressure difference is an oscillatory function; delta p by L as a function of time if I plot that if plot the special difference is a function of time it is an oscillating function. So, I will approximate delta p by L as some function cos some constant k times cos of omega t; where this time period is 2 pi by omega, omega is the frequency of the oscillation.

So, I am approximating this pressure gradient as a sinusoidal pressure gradient with one frequency. It turns out that you can solve the problem for any gentle shape because any waveform we know can be expressed as the sum sine functions. So, if you have any arbitrary periodic waveform; this periodic waveform can always be written as the sum of sine functions of different frequencies. So, if I solve it for one particular sine function I can get a solution for arbitrary periodic wave forms by just adding up the solutions due to the different sine functions that comprise this periodic waveform by just adding up the sine functions that comprise this periodic waveform. I will come back and discuss that a little later.

So, the point I am making is that considering a sinusoidal profile is not a loss of generality, any waveform can be expressed in terms of sinusoidal profiles. So, for now we will consider a sinusoidal profile. Momentum conservation equation for this case, so this is x this is r rho d u x by dt is equal to mu by r t by dr minus dp by dx minus k cos omega t. So, that is the momentum conservation equation. And I have the boundary conditions at r is equal to capital R u x is equal to 0 and at r is equal to 0 d u x by dr is equal to 0. The symmetry condition there because there is no physical boundary at the location r is equal to 0.

So, how do we solve this equation? It helps first to scale the equation; I can scale the equation the radial distance r star equal to r by capital R. And what about the frequent that the time scale; I already know that omega t is dimensionless, therefore I can scale define a scaled time t star is equal to omega t. What do I scale the velocity by; it is not yet clear at the present juncture what would I scale the velocity by. So, if I do it in this fashion if at scale omega t s t star the equation that I will get is rho omega partial u x by partial t is equal to mu by r square. I have two derivatives with respect to r in this equation, so I will get mu by r square 1 by r star t by dr star of minus k cos t star, because omega t was equal to t star so you just get k cos t star over here.

Now, I can divide this entire equation by this pressure gradient k, because delta p by L the amplitude of that pressure gradient is k. So, I can divide throughout by k. So, I will get rho omega by k partial u x the t star that is equal to mu by R square k minus cos of t star. Once I have divided throughout by k, this term here is dimensionless that means, that all the other terms in the equation are also dimensionless, because in an equation all terms have to have the same dimension. So, one term is dimensionless it means that all the other terms are also dimensionless. So, I can scale the velocity now in one of two ways.

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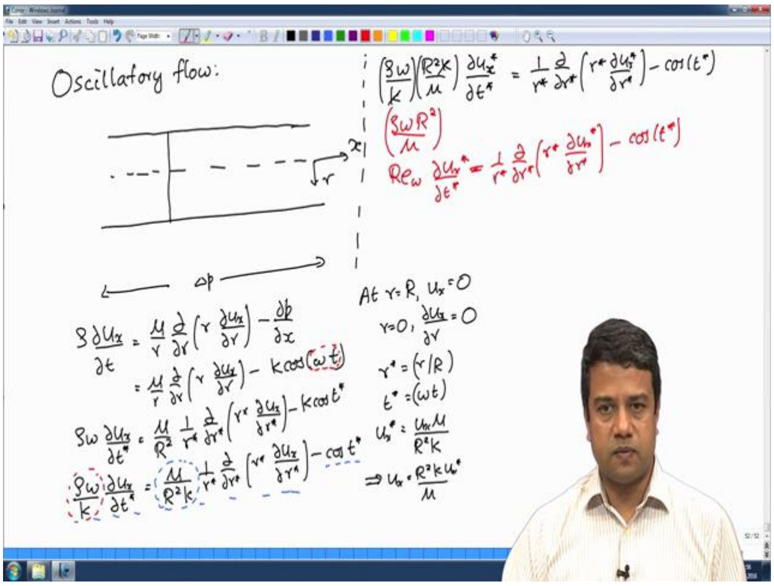

The first way is to scale it by the viscous scales. The other way is to scale it by the inertial scale. Which one do we choose? That of course depends upon whether in the problem the ratio of inertia to viscosity is large or small, and we do not yet know what is the ratio of inertia to viscosity so for the moment we will just scale it by a viscous scale. So, we will define u x star is equal to u x mu by R square k. I will define u x star is equal to u x mu by r square k which means that q x is equal to R square k u x star put, that into the conservation equation. So, the first term I get rho omega by k into r square k by mu into partial u x star by partial t star.

Since, I scaled by the viscous scale on the right side I will just get 1 by r d by dr of r the d u x by dr minus cos of t. And in this first term here k cancels out; therefore this becomes rho omega r square by mu that is the coefficient. This is a Reynolds number based upon the frequency of oscillations and the radius of the pipe. So, this is the conservation equation which contains just one dimensionless parameter, Reynolds number based upon the frequency of oscillations and the radius of the pipe.

Now, how do we solve this equation? So, the equation is will formed Re omega partial u x by partial t is equal to 1 by r p by dr of r partial u x by partial r minus cos of t. This is a partial differential equation it contains functions of both its function of both time and radius, it is an inhomogeneous partial differential equation, it is an oscillatory so the velocity exactly repeats itself after each cycle.

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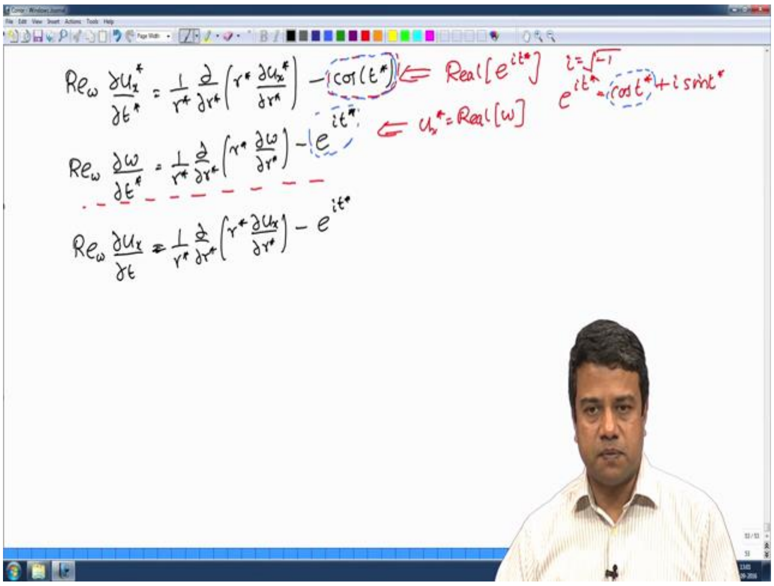

Now, how do we solve this equation? The simplest way to solve this equation if you recall I had my equation was of the form Re omega partial u x by partial t is equal to minus cos of t star. This cost t it is the real part of e power i t. The real part of this function e power i t, because e power i t is cost t plus i sine t so this is the real part of e power i t.

So, what I can do instead is to solve another equation. Since the inhomogeneous term in this equation is the real part of the inhomogeneous term in this equation, this is the inhomogeneous term in the momentum equation is the real part of the inhomogeneous term of this modified equation and the equation is linear; what that means is that the solution u x star will be equal to the real part of this modified equation w; will be the real part of the solution for this modified equation. Note that this is a linear equation, the real part of the inhomogeneous term for this modified equation; the real part of this inhomogeneous term is the inhomogeneous term for the momentum equation. So, the real part of the solution of this w for this modified equation will be the solution for u x. So, that is always guaranteed in the theory of complex variables. I should not hear that i is equal to square root of minus 1 and e power i t can be written as cos t plus i sine t.

Therefore the real part of e power i t is just the cos function which was the inhomogeneous term here. So, the strategy for solving is as follows: I will solve this equation for w, I will get a function the solution for w which will be some function of r and t because the inhomogeneous term is complex the solution w will also be complex. Once I have found the solution for w that complex function. If I take the real part of that complex function I will get the solution for u x so that is the idea if the real part of the solution for w will turn out to be the solution for u x.

And this is always guaranteed if you have a linear equation. So, rather than writing in terms of w I will just write it in terms of u x itself; I will just write the equation in terms of u x itself partial u x by partial t is equal to 1 by r star minus e power i t star. With the implicit understanding that because e power i t is a complex function the solution for u x will also be a complex function. The real physical velocity profile will be the real part of this complex function with that understanding.

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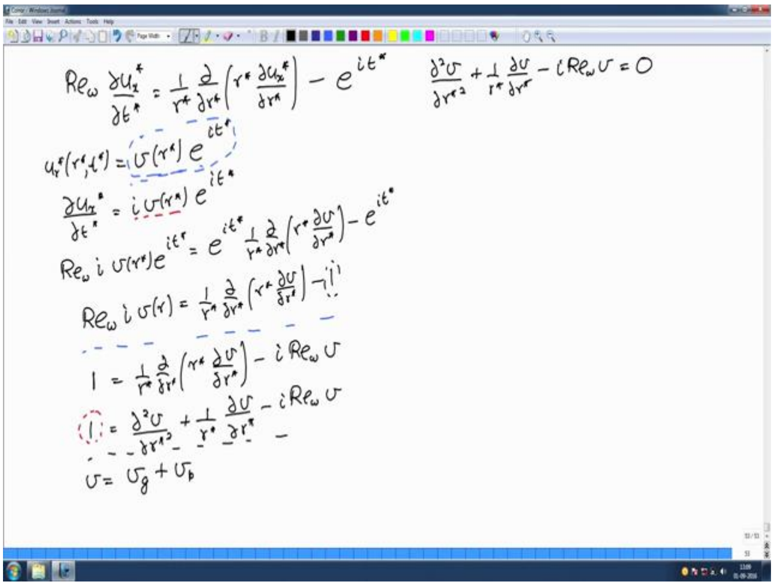

So, on this basis this, this is the equation we have to solve. Let us solve the equation with the inhomogeneous term becoming e power i t. How do I solve this? Since the inhomogeneous term is an exponential I can write the velocity u x also in terms of exponentials; I can write u x star is equal to some function v of r star e power i t star. And if I put that in to this equation you will find that this factor of e power i t star is common in all terms, and therefore it will cancel out. So therefore, d u x by dt is equal to i v of r e power i t.

Therefore, I will get Re omega into i v of Re power i t is equal to, now when I take the derivative with respect to r I get only derivatives of v; therefore I will get e power i t coming out 1 by r d by dr of r d v by dr minus e power i t. And this factor of e power i t cancels out on both sides, and I will get Re omega i v of r is equal to 1 by r d by dr of r d v by dr minus 1.

So, the four singles of the form a power i p and the equation is linear. If the forcing is of the form e power i t and the equation is linear then the response also had to have the same frequency. There may be a phase shift but the frequency has to be the same, because the forcing is at one particular frequency and the equation is linear equation with that forcing then the response also has to have that same frequency and for that reason we were able to write down the velocity in this form here.

And now this equation that results it is only a function r there is no time dependence in this equation, and that is the basic simplification here. I had said originally that we are solving partial differential equations they are functions of space and functions of time. And there is no standard method for solving partial differential equations; we have to use physical insight into to reduce them to ordinary differential equations. We had seen two such procedures: one was the similarity solution where there was a deficit of dimensions and therefore we were able to reduce the equation to an ordinary differential equation, by n defying the similarity variable as z by root alpha t or r by root alpha t against the case may be.

The second was separation of variables, where we wrote it in terms of the product of a function of position and a function of time. And use separations of variables to get those solutions individually add them up and then determine the coefficients from the inhomogeneous term. This is the third such method. In the case of oscillatory flows you know that if the forcing is at one particular frequency the response also has to be at that same frequency. And therefore, if I express the time dependence, in this particular case the velocity is a function of r and t the velocity is a function of position and time. If I express that time dependence as sinusoidal time dependence and I put it back into the equation I get an equation that is only in the spatial coordinate; it is an ordinary differential equation.

So, I can simplify this equation a little bit what I get is that d square v by, let me do it in steps; I can take the inhomogeneous term to the left side I will get 1 is equal to 1 by r d by dr of r d v by dr minus i Re omega v. Or my equation becomes 1 is equal to d squared v by t r square plus 1 by r d v by dr minus i Re omega v. This is an inhomogeneous equation it is an inhomogeneous equation for the solution v. So therefore, I can write the solution v for v as the sum of two parts; the general solution plus the particular solution. The general solution is the solution of the homogenous equation. The homogenous equation in this case is the equation with the inhomogeneous term z equal to 0. So, the homogenous equation is d square v minus i Re omega p is equal to 0. And the particular solution is any one solution of this inhomogeneous equation; it can be any one solution of this inhomogeneous equation.

We will continue this solving this equation in the next lecture. Kindly go through the solution of inhomogeneous linear differential equations, ordinary differential equations, how do you separate it out into the homogenous solution and the particular integral, and how do you add up the two in order to get the final solution. So, I will continue the solution of this equation in the next lecture.

So, far what we have done for this oscillatory flow in a partial differential equation and based upon some physical insight we had reduced it to an ordinary differential equation; a function of the radius r times e power i t. And we have to find out this complex function of r. So, once I find out the solution for r and multiplied by e power i t take the real part that is going to be the solution for my velocity profile. This I will continue in the next lecture. I will see you then.