Transport Processes I: Heat and Mass Transfer
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Lecture - 39
Unidirectional transport: Laminar and turbulent flow in a pipe

In the last lecture we were looking at a pipe flow which is an important prototypical velocity profile that we encounter. If you recall we had done the momentum balance for a cylindrical shell in the pipe flow in the last lecture and we had got a balanced equation and that balance equation was of the following form.

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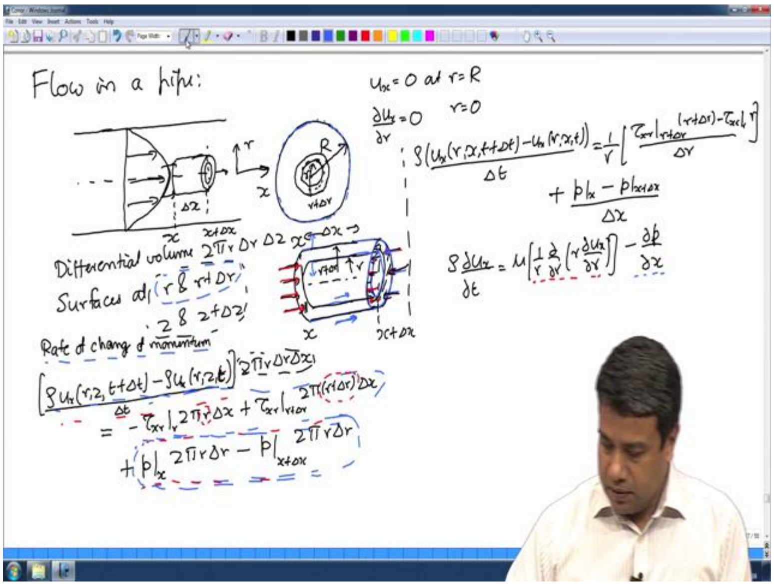

So just to write it once again; rho d u x by dt is equal to mu into 1 by r d by d r of r minus dp by dx. So, that was the balanced equation that we had got this slightly more complicated form of the operators because we considering the cylindrical volume. We got a similar equation for the concentration and the temperature fields, but without this additional term here without this additional term. This additional term is the pressure gradient the pressure difference between the two ends of the differential volume divided by the distance.

And when we solved momentum transport problems we had got an equation of the kind rho partial u x by partial t is equal to mu 1 by r d by d r. Without a pressure but with body forces we had got plus f x, where f x is the body force density force per unit volume along the x direction. So, the point I am trying to make here is that the pressure gradient acts exactly the same as a force density. The negative of the pressure gradient in the x direction is equal to the force density. The negative of the pressure gradient is the excess force that is exerted in the plus x direction; that is analogous to a force density.

I could have got the same velocity profile if the pipe had been vertical and I had a body force it was rho times g the gravitational acceleration. So, the velocity profile that you get from a pressure gradient is the same that you would have got from a body force, the pressure gradient is just equal to the body force density. So, that is primarily the role that pressure gradient plays in momentum transfer it attacks it exerts an additional body force density per unit volume.

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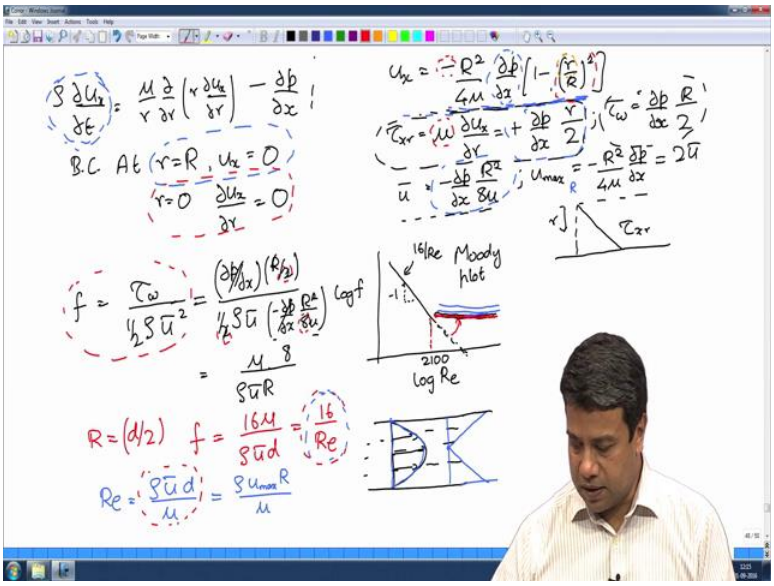

So, in this case I have solved the equations got a parabolic velocity profile, if you recall here the parabolic velocity profile for that we had calculated the mean velocity, the maximum velocity, the shear stress. The velocity profile itself was parabolic as shown here. The shear stress turned out to be linear, the shear stress was 0 at the center, the shear stress is proportional to dp by dx times r by 2, so it was 0 at the center increased linearly towards the wall the slope was the same on both sides.

And from that we had got the expression for the friction factor there is a function of a Reynolds number. Laminar flow we saw that it was 16 by Re. At a Reynolds number of about 2100 there is a transition to turbulent. And the laminar flow is no longer observed, and I would explain to you what that transition means the previous lecture.

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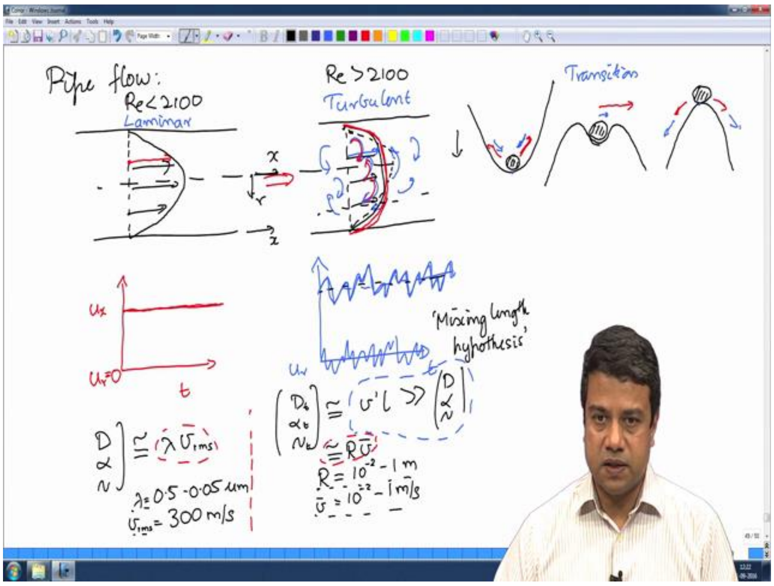

Even though the laminar flow is a solution of the momentum equation at all Reynolds numbers it is not a stable solution. Below a Reynolds number of about 2100 it is stable, so that if you put a small perturbation the system actually comes back to its original state. Above 2100 it put the perturbation and if that perturbation is sufficiently large in amplitude the system does not come back to its original state. And that is what causes the transition to a turbulent flow.

The turbulent flow is not steady, there our velocity fluctuations at every location in all three directions. In a laminar flow if I plot the velocity as a function of time it will be only along the stream wise direction and it will be equal to a constant value, there is no cost cross stream velocity. And therefore, in a laminar flow the flow is steady; the fluid fluctuating velocity is 0 even though there may be molecular fluctuating velocities. It is these molecular fluctuating velocities that give rise to diffusion.

In a turbulent flow on the other hand, there is an average velocity when you average over time but instantaneously the velocity is fluctuating in time. It fluctuates in all directions; in all three directions you get a fluctuating velocity in time. So, you get a velocity a fluctuation superpose on the stream wise velocity and you also get fluctuations perpendicular to the plane of the flow. And this is because you have parcels of fluids called eddies which are in correlated motion moving along and across the flow so the flow is not steady. And therefore, one cannot write a balanced equation where one considers cross stream transport due to momentum diffusion alone due to the sheer stress alone, because there is a fluid velocity in that direction as well. And that fluid velocity is what transports momentum across the flow primarily.

The velocity profile if you plot it looks flatter at the center and steeper close to the walls. At the walls itself the velocity has to be 0. All components of the velocity have to be 0; the mean velocity has to be 0 but the velocity fluctuations also have to be 0, because the absolute fluid velocity is 0 at the wall from the no slip condition. So, at the wall itself the velocity of the turbulent eddies has to decrease to 0, the velocity fluctuations have to decrease to 0. What that means is that if I go a small distance away from the wall the maximum length of the correlated turbulent fluctuation has to be equal to the distance from the wall itself, because you cannot have a correlated fluctuation over a larger distance, the velocity has to decrease to 0 at the wall.

Now, analogous to molecular diffusion where the diffusion coefficients were equal to the mean free path times the root mean square molecular fluctuating velocities. In turbulent fluids as well one can make an analogy and in exact analogy that the diffusivities are proportional to the mean root mean square the turbulent fluctuating velocities. Recall the fluid velocity can be decomposed into a fluid mean velocity and the fluid fluctuating velocity in a turbulent flow. This is not the molecular fluctuating velocity; this is the fluid fluctuating velocity because the fluid velocity itself fluctuates in time.

In that case one can make a loose analogy that the diffusivities will be a magnitude of the fluctuating velocity times a size of eddies. This goes by the name of the mixing length hypothesis; l in this case is the mixing length, the distance of which eddies mix the fluid, and v prime is a characteristic velocity of the eddies. And therefore, the turbulent diffusivity will scale as v prime times l. And that I told you showed you for typical systems the fluid fluctuating velocity is much smaller than the molecular fluctuating velocity; the molecular velocity fluctuations k less the speed of sound of the order of hundreds of meters per second in gases and about thousand meters per second in liquids.

The fluid fluctuating velocity on the other hand is only of the order of meters a second or less. So, there is a difference of about two orders of magnitude, the fluid, the molecular velocities are two orders of magnitude larger than the fluid velocities in practical applications. However, the molecular length scales the mean free path is of the order of 10 power minus 7 to 10 power minus 8 meters, so it is smaller than the system size by over 5 orders of magnitude. Because of that the molecular diffusion coefficients are much smaller than the turbulent diffusion coefficients if you define it by the mixing length hypothesis.

And therefore, the transport of momentum is primarily by turbulent diffusion across the center of the pipe. This turbulent diffusion process is much faster than the molecular diffusion process, therefore the transport rates are much higher and consequently the friction factor is higher because from the transport rates are higher the shear stress is much higher. And the friction factor is much higher than that for a laminar flow where the transport is due to molecular diffusion.

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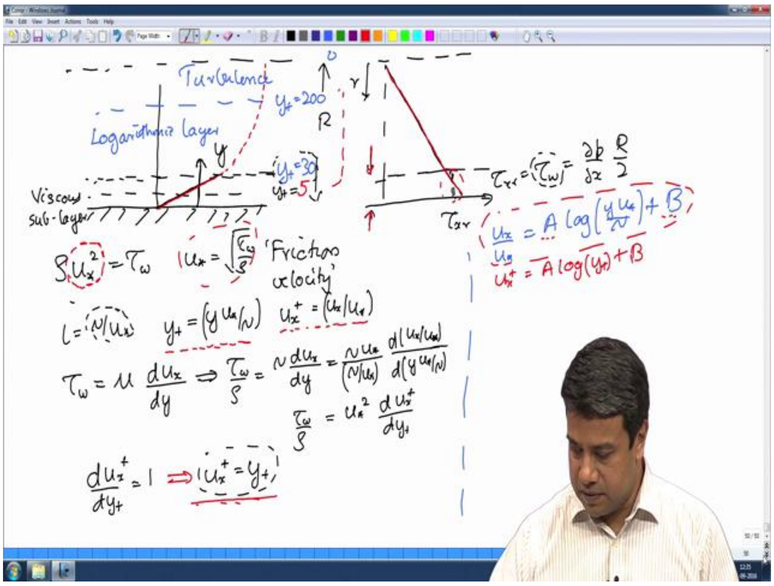

However, as you compose close to the wall the turbulent fluctuating velocities have to decrease to 0. And as I discussed in the previous lecture very close to the wall you have a layer of fluid where the flow is primarily viscous because the turbulence fluctuations have to all decrease to 0 by the time you would come very close to the surface, the turbulent velocities at p 0. And the hole very close to the wall the turbulent fluctuations are damped out you have a region called the viscous sub layer where the flow is primarily viscous. What is the length scale of this viscous sub layer? It is not the macroscopic size the pipe diameter r because the viscous sub layer is actually confined to a thin layer near the wall.

What determines the thickness of the viscous sub layer? The only quantity of relevance here is the shear stress in the viscous sub layer, because the shear stress as I showed you is a linear function of distance from the center of the pipe. However, as you approach close to the wall if this thickness is much smaller than the radius of the pipe the length scale for the shear stress is the radius. Therefore, if this thickness is much smaller than the radius then within this region you can consider the shear stress to be approximately a constant, and approximately equal to the shear stress at the wall itself.

From that I can actually get a velocity scale; from the shear stress you can get a velocity scale from the shear stress it is called the friction velocity, it is based upon the wall shear stress; it is called the friction velocity. I get that from dimensional arguments by equating rho u square to the shear stress and that gives me the friction velocity as u star which is equal to root of tau w by rho. So, that is the relevant parameter, the relevant velocity scale within the viscous sub layer. Now how does one get a length scale? We know that within the viscous sub layer viscous effects are important, and therefore the only other dimensional parameter that is of importance is the kinematic viscosity. And from that you can get only one length scale L is equal to u by u star; where u stars the friction velocity.

Therefore, I can scale the distance from the wall by this length scale. If y is the distance from the wall I can define a scaled distance y plus it is called the distance in wall units, this length scale is what is called one wall unit because it is based upon the shear stress at the wall and the kinematic viscosity so it is called one wall unit. And y plus is defined as y u star by nu.

So, for the flow very close to the wall y plus must be the relevant length scale the total radius of the pipe is not. Similarly, I can define the velocity u x plus is equal to u x by u star, because that is the only velocity scale very close to the wall. And close to the wall you would expect this velocity scaled by the friction velocity to be a function of this distance from the wall in wall units. So that is the fundamental relation close to the wall. If the system is completely viscous at the wall itself there is no cross stream diffusion due to turbulent velocity fluctuations because they have already got damped out. And if the flow is purely viscous then you require that the wall shear stress has got to be equal to the viscosity times d u x by dy at the wall, because the flow is purely viscous.

Or I can define by the density this is equal to nu d u x by dy close to the wall. And if I express this now in terms of the scaled velocities this can be written as nu u star divided by nu by u star d of u x by u star by d of y u start by nu. Just expressing it in terms of the scaled velocities and this is equal to u star square du x plus by dy plus. This is equal to tau w by rho. Just expressing in terms of the distance expressed in wall units and the velocity scaled by the friction; velocity the friction velocity is u star. But we know that u star square is just equal to tau w by rho.

Therefore, very close to the wall the velocity scaled velocity u x plus by dy plus should be equal to 1, because I have used the wall shear stress to scale the velocity therefore d u x plus by dy plus has to be equal to 1. And this implies that very close to the wall u x plus is just equal to y plus the distance in wall units in the viscous sub layer. Experimental studies show that this law is applicable approximately up to about y plus of about 30; y plus less than 10 you see a linear velocity profile. So, y plus less than 10 you will see a linear velocity profile.

Within this viscous sub layer this linear law applies the only thing is that I have to define the velocity in terms of the friction velocity rather than the mean flow velocity. Now the turbulent flows of course when the distance is comparable to the radius of the pipe in between the two in experiments it is found that there is what is called a logarithmic layer. This logarithmic layer can be justified on the basis of matched asymptotic analysis which I will not have time to go through in this course; this can be justified on the basis of matched asymptotic analysis.

But, within the logarithmic layer this goes has a logarithm of the velocity; this goes as a log of the velocity. So, if I express u x plus is equal to A log y plus plus B. There is a logarithmic region where the velocity profile basically goes logarithmicly. And then near the center of the pipe it becomes nearly flat. This logarithmic layer varies from y plus about 30 to 200 is the classical conventional wisdom, this is called the one common law for the logarithmic velocity profile in this logarithmic layer.

I should note that this is an area where there is a lot of current research, and these limits are currently being questioned as to whether they are actually applicable or not based upon experiments in very large you know systems. However, this law is just a dimensional necessity; the viscous sub layer does exist beyond that is a logarithmic layer. The logarithmic layer can be justified on the basis of matched asymptotic expansions, where these constants are available by fitting experimental data to these logarithmic profiles. And these constants there are questions whether they are universal or whether they depend on geometry whether they tend to the same limit as the pipe diameter goes to becomes larger and larger whether it is a pipe or a channel so on. But, conventionally this is approximately between y plus of about 30 and 200 you have a logarithmic layer beyond that you have the parabolic velocity profile.

So, that is the structure of the flow near the wall of the pipe, the flow is entirely viscous very close logarithmic. And then you have the central region where the transport is primarily due to turbulent diffusion due to the costume mixing due to eddies. And within the region since it is very well mixed the velocity is approximately a constant. Similarly, for momentum and the mass and heat transfer the concentration or the temperature will be approximately a constant in that central region. And in that central therefore most of the transport takes place across this logarithmic layer and the viscous sub layer close to the wall of the pipe. It is well mixed in the bulk of the flow the turbulent mixing is a dominant mechanism for momentum transfer.

So, this is a brief summary of turbulent flows; as I said we are not able to do this analytically because the flow is not three dimensional. We have so far looked only at transport in one direction; we have not yet looked at three dimensional flows. We have also not looked at unsteady flows; we focus primarily on steady flows in this case. That does give us what is the laminar velocity profile, it does not give us the turbulent velocity profile. But there are some broad features of the turbulent velocity profile that can be deduced just on simple reasoning.

The largest eddies have to be comparable to the size of the pipe. The velocity fluctuations are comparable, they are smaller but they are comparable to the mean flow velocity. And this turbulent eddies that are primarily responsible for the cross stream transport. As you come close to the wall the velocity has to be equal to 0 at the wall itself so the velocity has to be damped out, the fluctuations have to be damped out by the time you come close to the wall. Very close to the wall there is a layer or which viscous forces are dominant. And across this layer the wall shear stress is approximately a constant;

from that wall shear stress I got only the friction velocity as a dimensional parameter and based upon that friction velocity I can define a velocity profile close to the wall.

Note that this friction velocity is not the mean velocity; this comes out from the wall shear stress and the density. However, if I define the velocity in terms the friction velocity and the distance from the wall in terms of wall units in terms of wall units then the velocity profile close to the wall is just a linear profile with a slope equal to 1 within this viscous sub layer. As I said this viscous sub layer it is valid only when this y plus is much less than about 30 or so gives the limits are you variously given as 5 10 and so on. A conservative limit would be to consider this as just equal to 5 but I said these limits are still under active research.

So, y plus less than about 5 is a good conservative limit when the velocity profile is linear. Y plus going from about 30 to 200 the velocity profile obeys another law a logarithmic law. In this particular case the distance from the wall is still measured in wall units. The velocity is expressed in terms of the friction velocity; you get a logarithmic function with constants. These are called the one carbon constants. And depending upon the configuration these turn out to be universal constants. In particular the constant A is approximately about 2.5, and this is for a smooth pipe and then it will change the depending upon the roughness of the pipe these constants could change as well.

So, that is the broad summary of the features of the turbulent flow in a pipe. So, we looked at steady flows, laminar flow this steady; turbulent flow is not steady, but on average it was a steady flow with an average velocity profile which has these features.

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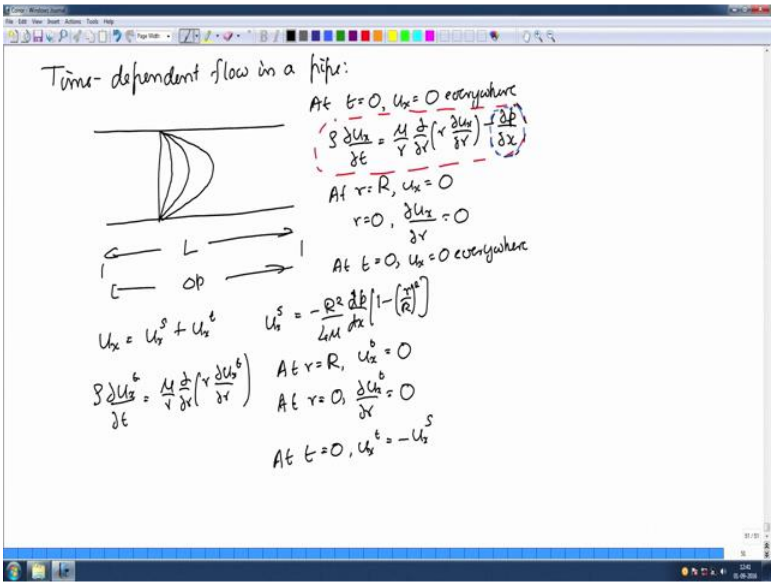

Next we will look at time dependent flows. One can look at two different kinds of situations: one is the unsteady flow, the flow that the time required for the flow to become steady. So, typical configuration in that case would be: this is the steady parabolic velocity profile; however I could consider a case where initially the fluid is stationary, so initially I do not have any pressure difference across the walls of the pipe. Initially the pressure difference is equal to 0. So that initially at t is equal to 0 u x is equal to 0 everywhere in the pipe.

And then at time t is equal to 0 I apply a pressure gradient, I apply a pressure difference delta p across this length at time t is equal to 0. And then I watch how the velocity profile develops into this steady velocity profile, for the velocity increases until it reaches the final steady value with progression in time. So, that is the unsteady problem. The equation for that is of course, rho d u x by dt is equal to mu minus dp by dx. The boundary conditions are that at r is equal to 0, I am sorry; at the wall of the pipe at r is equal to 0 we have the symmetric condition and the initial condition is that at t is equal to 0 u x is equal to 0 everywhere.

How do you solve this problem? Separation of variables; so I need to get in this particular case the equation is inhomogeneous. Therefore, I need to separate out the velocity into a steady plus a transient velocity. You had already done that for the flow down inclined plane and looked at what kinds of solutions result, the exact same procedure has to be used in this case.

Since I have already done it two times I will not go through it in this case again. What we need to do is to express the velocity as a function of a steady plus a transient part. And the transient part is what I already have; minus r square by 4 mu dp by dx into 1 minus r by r the r whole square. That is the steady part. For the transient part, the total equation contains is inhomogeneous term; the steady equation contains the same inhomogeneous term. Therefore, the transient equation will be of the form rho the u x transient by dt is equal to mu, when I subtract out the steady equation from the total equation this pressure gradient will also gets subtracted out, just as the body forced at in that case.

So, this is the homogenous equation subject to boundary conditions which are also homogeneous, sorry it r is equal to homogeneous boundary conditions. The in homogeneity will be at t is equal to 0 u x transient is equal to minus of the u x steady; at t is equal to 0 the transient part will be equal to the negative of the steady part because the total velocity has to be equal to 0. And therefore, you can solve this by separation of variables; the solutions will be in the form of Bessel functions in the radial direction and in the form of exponentials in time and you can get the constants using the orthogonality relations. So, there is a simple extension to a pipe flow of the same unsteady problem we had solved for the flow down an inclined plane.

Next lecture I will start another kind of flow which is an oscillatory flow. Oscillatory flows are known in the physiological system for example, flows in the blood vessels are oscillatory in nature. How does one get a solution for an oscillatory flow? So that is something that I will start in the next lecture, that flow will for us illustrate how a competition between convection and diffusion plays an important role in momentum transport. So, I will start an oscillatory flow in the next lecture. I will see you then.