Transport Processes I: Heat and Mass Transfer
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Lecture - 37
Unidirectional transport: Friction factor for flow in a pipe

Welcome back to our discussion of the Flow in a Pipe, our discussion of momentum transport as part of the course on Fundamentals of Transport Processes. The last class we were looking at the flow in a pipe, this is a flow that is used often. And in this case the surface is a cylindrical surface, a curved surface. And therefore, it is appropriate to use a cylindrical coordinate system for analyzing the flow in a pipe.

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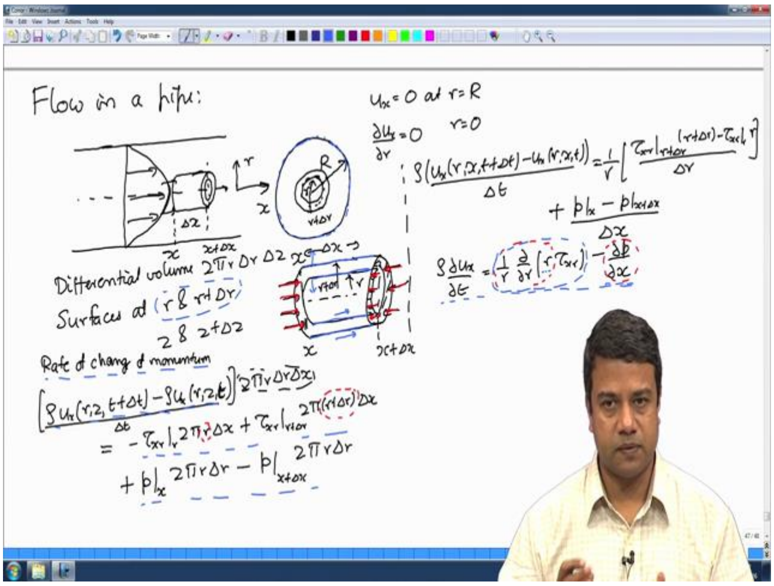

So, we considered a pipe of radius r across which there was a pressure difference that was a plane, and that difference in pressure was responsible for generating a flow in this pipe. This pressure difference of course, has no real analog in mass and heat transfer so it is specific to momentum transfer. We have done the balance equations between a surface at r and the surface at r plus delta r. We had written that the rate of change of momentum is equal to the sum of the applied forces. This momentum balance equation was done for the x direction; the x momentum, the stream wise momentum.

So, in that case we have written that the rate of change of momentum between two instance of times separated by delta t which is basically the difference in momentum density times the volume is equal to the forces; there are forces exerted at this outer surface at r plus delta r, the inner surface at r. As well as the end caps the surfaces at x and x plus delta x at these surfaces, there is of course a flux of momentum coming in and a flux of momentum going out gives this momentum coming in through the surface because the fluid velocity is along the x direction. There is momentum leaving the surface as well on the right side because the velocities in the x direction.

In addition to that there is a pressure force pressure. Pressure as I have told you in the last class is compressive; it always x inward to the surface, if the pressure is always acting inward to any surface, so it x along the inward unit normal to the surface, the inward perpendicular to the surface. Therefore, the pressure at these two end surfaces is what enters into the x momentum balance equation. The pressure at the curved surfaces is not in the x direction, so therefore this pressure does not enter into the momentum balance equation. What enters into the momentum balance equation is the shear stress in the x direction which actually x parallel to the surface. So, this is what it does not to the x momentum balance equation. However, at the two end surfaces the pressure is perpendicular to the surface and it is in the x direction, so that pressure x interns in to the x momentum balance equation.

So, these were the two forces exerted: the first set at the surface at r and r plus delta r, the second set at the surface is at x and x plus delta x the two end surfaces. So, once we have written these momentum balance equations correctly it is only a matter of dividing by volume and taking the limit, and we end up with this momentum balance equation. This contains the shear stress term; this is slightly complicated because for the curved surfaces the surface area changes as the radius changes. And this factor of our actually takes that into account, because the surface area is proportional to the rate parameter which is proportional to the radius.

So, that is the additional term due to the momentum due to the curvilinear coordinate system, and then I have a pressure gradient here a derivative of pressure with respect to the x direction. This appears because the pressure at the two end caps is different in the pressure at x and x plus delta x have to be different, because that is what is driving the flow.

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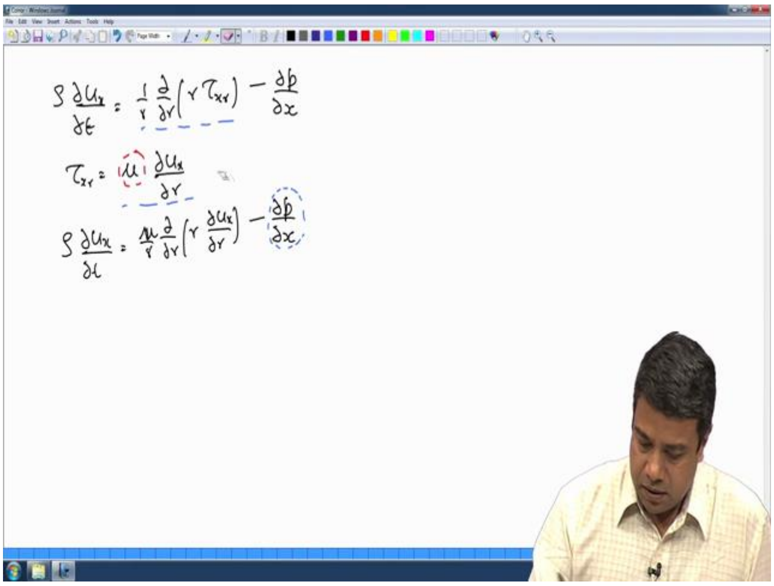

So, I have written that momentum balance equation here and then written down Newton's law of viscosity and got a balanced equation for the x momentum.

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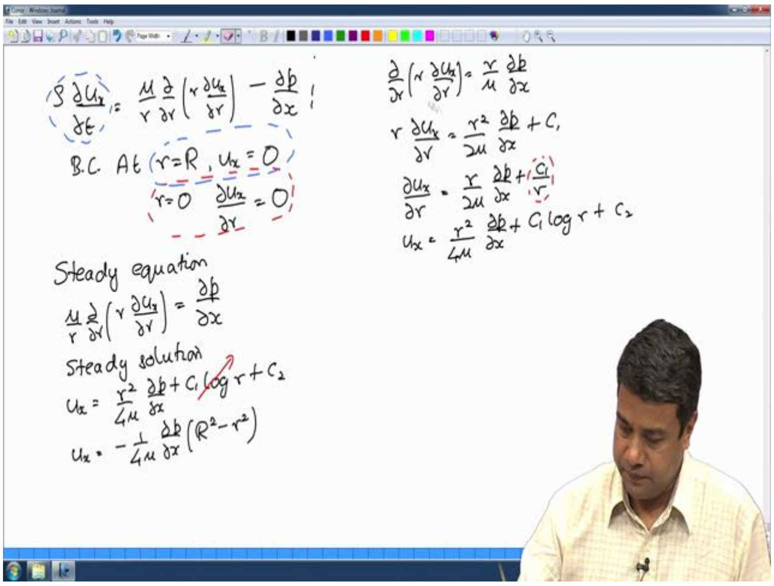

So, let me write that once again for you the total momentum conservation equation for a general unsteady flow has to be of the form mu by r d by dr of r d u x by dr minus dp by dx, so that is the momentum balance equation. First we consider and the boundary conditions at the surface of the pipe r is equal to r u x is equal to 0 and at r is equal to 0;

as I will explained in the previous class we have the symmetry condition partial u x by partial r is equal to 0. So, those are the boundary conditions.

And now we have to solve this; first of course we will take the steady solution. The steady solution the time derivative is 0, so I just get new by r d by dr of r partial u x by partial r is equal to dp by dx. So, that is the equation at steady state, the left side of this equation is equal to 0 because there is no time dependence.

And now you can solve this, first you will get d by dr of r partial u x by partial r is equal to r by mu dp by dx. As I told you the pressure gradient had to be a constant is independent of x direction, the pressure gradient is in the independent of the x direction. The pressure is also independent of the radial quadrant. So, this is once again something that is not easily appreciated; let me just briefly tell you why. The momentum balance that I have written here is for the x momentum, the momentum along the flow. I could write the momentum balance equation perpendicular to the flow in the radial direction. In any direction that is perpendicular to the flow. For any direction perpendicular to the flow the velocity is equal to 0. For any direction that is perpendicular to the flow the velocity is equal to 0, because the velocity is only in the x direction.

So, if I have to write a momentum balance equation for any direction that is perpendicular to the flow, the velocities are all 0 there is equivalence of all of these terms and the shear stresses they are all 0 because the velocity is 0. So, in any direction that is perpendicular to the flow I will be left only with this pressure term. In any direction that is perpendicular to the flow I will get an equation of this form except that all the velocities are 0, because the velocity is only in the x direction. Therefore, I will get only a pressure gradient in that direction is equal to 0.

So, therefore, the pressure has to be a constant in any cross section perpendicular to the flow; that is because there is nothing to balance that pressure gradient, there is no viscous stress or inertial stress to balance that pressure gradient perpendicular to the flow, because the velocity is identically 0 perpendicular to the flow. And for that reason the pressure has to be a constant across the cross section. Therefore, the pressure gradient is also independent of the radial coordinate, therefore I can integrate.

Integrated once and you will get r partial u x by partial r is equal to r square by 2 mu partial p by partial x plus some constant C 1. Simplify partial u x by partial r is equal to r by 2 mu partial p by partial x plus C 1 by r and integrated once more and you will get u x is equal to r square by 4 mu partial p by partial x plus C 1 log r plus C 2. So, for this steady equation the steady solution this u x is equal to r square by 4 mu partial b by partial x plus C 1 log r plus C 2. And the constant C 1 and C 2 have to be determined from the boundary conditions.

We know that at the center d u x by dr is equal to 0 that will be true only if C 1 is equal to 0; at the center d u x by dr is equal to 0 that will be true only if C 1 is equal to 0. Therefore, this logarithmic term has to be equal to 0 because log of r in the limit as r goes to 0 goes to infinity. So therefore, this velocity profile if C 1 is nonzero does not satisfy the condition that the velocity has to be finite at the center. C 2 can be determined from the other boundary condition; it r is equal to capital R the velocity is equal to 0. And the velocity profile that satisfies both of these conditions is u x is equal to minus 1 by 4 mu partial p by partial x into R square minus r square.

So, that is the velocity profile which satisfies both the boundary conditions and satisfies the momentum balance condition.

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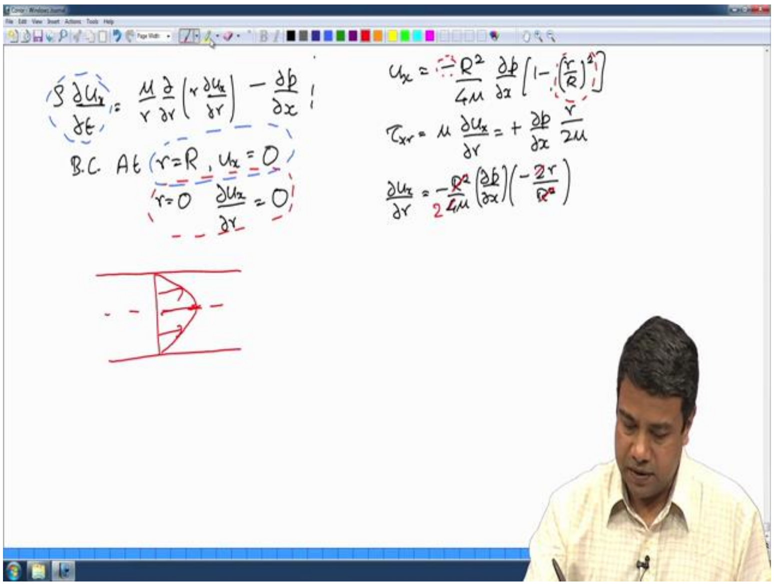

Therefore, my velocity u x is equal to I can rewrite it slightly as r square by 4 mu partial p by partial x into 1 minus r by r the whole square; so that is the velocity profile. Note that you x is positive is dp by dx is negative. Basically, if the pressure at the inlet is greater than the pressure of the outlet the pressure gradient is negative; in case the pressure the inlet is greater than the pressure at the outlet the pressure gradient is negative, therefore u x is positive. So, the flow is from region of high pressure to a region of low pressure and that is the reason for this negative sin here.

The velocity profile is parabolic in the radius. This is the reason that the velocity profile in the flow through a pipe is a parabolic velocity profile at steady state you get a parabolic velocity profile with the maximum at the center. This is for a laminar flow, it is a maximum at the center the velocity profile is a parabolic velocity profile.

Now, you can calculate the shear stress, tau x r is equal to mu times partial u x by partial r; should be equal to minus partial p by partial x into r by 2 mu if I take the derivative, so they should have a positive sin here. So, let is just work it out a little bit. So, partial u x by partial r is equal to minus R square by 4 mu dp by dx into minus 2 r by R square, I take the derivative of this term here I will get minus 2 r by R square. So, there is the shear stress and these two will cancel out to give me plus dp by dx and this 4 and 2 will cancel out to give me a to hear. So, that is the expression for the shear stress. Shear stress is 0 at the center, because the slope of the velocity is equal to 0. However, it is nonzero at the wall.

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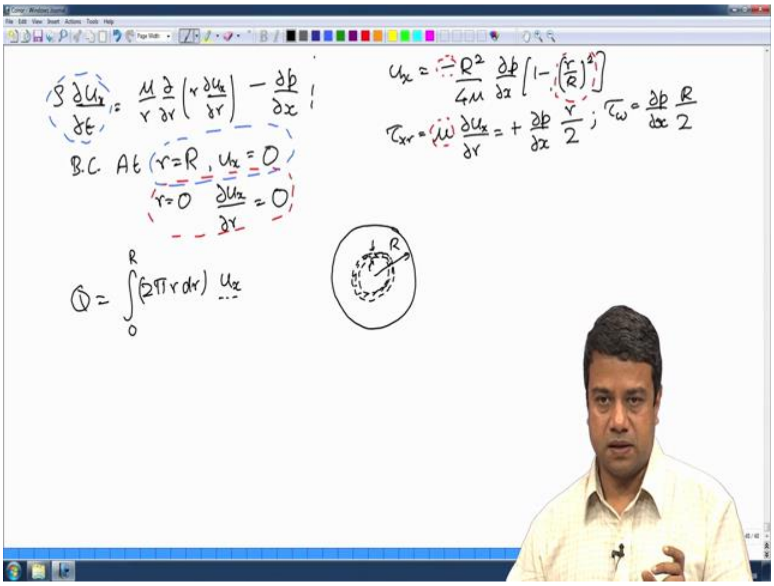

At the wall of the pipe the shear stress tau w is equal to dp by dx into r by 2 mu; I am sorry there should be no viscosity there because I multiplied by this viscosity here. When

I multiplied by this viscosity I do not get a factor of the viscosity there. That is the wall shear stress, I exerted on the wall of the pipe.

One can now calculate the average velocity. The total flow rate Q is equal to integral 2 pi r dr from 0 to r times u x. So, how do I calculate the flow rate? So, if I have a pipe of radius r, I take a differential element between r and r plus delta r, the velocity u x is only a function of the radius. Therefore, I have to take the velocity and multiplied by the surface area to get the flow rate, the amount flowing per unit time. If I take the velocity and multiplied by the area it is a cross sectional area here. To find out the total amount that is crossing one particular cross section per unit time you can see that the velocity times the area has dimensions of volume per unit time. So, there is the flow rate.

So, I take the velocity at one particular radius multiplied by the area which is 2 pi r times delta r the area is equal to the thickness of this slice times the parameter, so as 2 pi r times delta r and have to integrate it from 0 to r; so that gives me the total flow rate.

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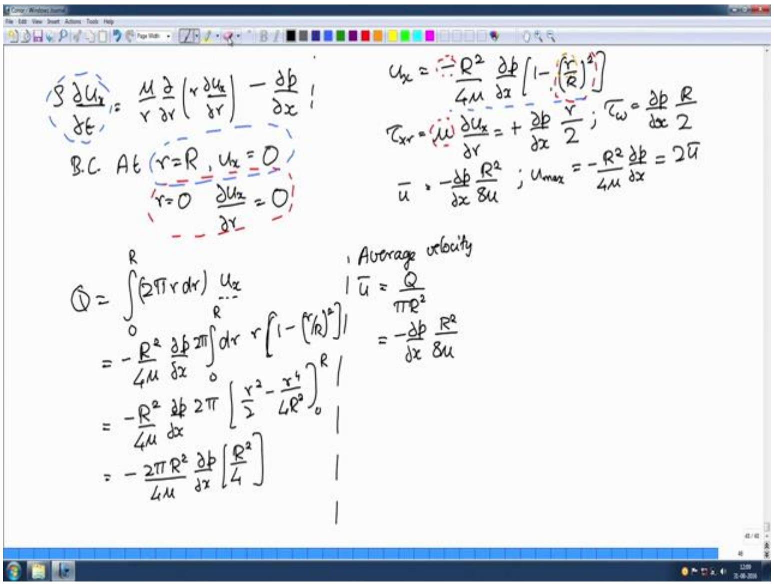

So, this will be equal to minus r square by 4 mu delta p by delta x integral 0 to r of dr into r into 1 minus r by r whole square, and I have an additional factor of 2 pi here 2 pi is just a constant. So, I have integral of r square minus r; I am sorry integral of r minus r to the 4. So, this will be equal to minus r square by 4 mu dp by dx times 2 pi into r square by 2 minus r power 4 by 4 r square between 0 and r. So, that is what I get by doing the integral. So, this gives me minus 2 pi r square by 4 mu into dp by dx, at 0 both of these terms are 0, at capital R I just have to substitute capital R here.

So, I will basically get r square by 2 by 4; I am sorry 1 by 2 minus 1 by 4 just gives me 1 by 4. So, that is the flow rate in this spike volume transferred per unit area per unit time. And the average velocity u bar the average velocity is equal to the flow rate divided by there a cross sectional area is equal to Q by pi r square. The average velocity is just the ratio of the flow rate and the cross sectional area. So, if I do that the pi is will cancel out and I will get minus partial p by partial x into to r square by 8 mu because one, one factors are square cancels out to the denominator I will just get minus partial p by partial x into r square by e t. So, that is the mean velocity. Therefore, the mean velocity u x bar is equal to minus partial p by partial x into r square by 8 mu.

What is the maximum velocity? The maximum velocity is the velocity when r is equal to 0 because the velocity has a maximum at the center of the pipe. So, the maximum velocity is the velocity at r is equal to 0; I just use u bar for the mean velocity. The maximum velocity is just the velocity if I substitute r is equal to 0 in this expression, the substitute r is equal to 0 here I will get the maximum velocity. So, this maximum velocity is equal to minus R square by 4 mu partial p by partial x which is just equal to 2 times the mean velocity. So, the maximum velocity at the center of the channel is just equal to two times the mean velocity in this particular side flow.

So, now that we have the flow rate, the mean velocity as well as the shear stress; we can now calculate what is the friction factor for a pipe flow.

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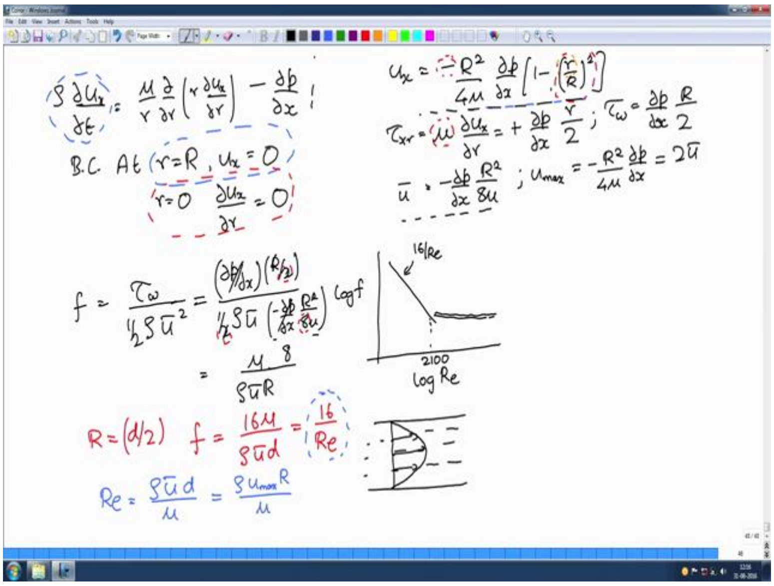

The friction factor is equal to tau w by half rho u bar square that is how it is defined; the friction factor is the wall shear stress divided by half rho u bar square. Now we know that tau w is equal to partial p by partial x into r by 2 divided by half rho, rather than writing you bar square in terms of this factors here you will see that it will be convenient for us to write this as u bar times minus partial p by partial x r square by 8 mu. So, I am just writing this in this form just for convenience. One new bar I have expanded out and the other u bar I have just retaining as a constant.

So here, this pressure gradient cancels out, and one factor of r will cancel out; so I will get mu by rho u bar r times a factor. This 1 over 2 will cancel out and I have 1 over 8, so I will just have a factor of 8 here. If I simplify this expression, you can see this factor of 8 is there the factor of two cancels out in the numerator and the denominator. So, 8 mu by rho u bar r. If I write the radius is equal to one half the diameter of the pipe then in terms of this the friction factor is equal to 16 mu by rho u bar d this is equal to 16 by the Reynolds number.

So, this is the friction factor versus Reynolds number correlation for the flow in a pipe. The friction factor was defined as the wall shear stress scaled by the inertial scale where I told you in the very first lecture then whenever we define friction factor it is scaled by the inertial skills. Therefore, we had defined the friction factor as the wall shear stress divided by the inertial scales. We are calculated explicitly, what is the velocity profile everywhere in the flow; it calculated what is the wall shear stress. And from that we had written down the friction factor as the ratio of the wall shear stress divided by half rho u square. And once I did that I got the friction factor entirely in terms of the Reynolds number for a laminar flow. And that expression is the expression the standard expression for friction factor in the pipe friction factor is equal to 16 by Reynolds number, where the Reynolds number was defined as rho u bar d by mu.

It could be defined in terms of the mean velocity. And the diamond another way to define it is in terms of the maximum velocity and the radius. For a pipe flow these two definitions are the same, because the diameter is two times the radius, the mean velocity is one half of the maximum velocity. So, for the particular case of a pipe flow these two definitions of the Reynolds number coincide, it does not happen for other geometry. In the case of the flow in a channel for example, these two definitions will not in general coincide. For the particular case of a pipe flow these two definitions coincide.

And therefore, for a pipe flow we have got the velocity profile the friction factors and if I plot log of friction factor versus log of Reynolds number it has a minus 1 slope in a log log graph and this one is 16 by Re for the pipe flow. So, this is all for what is called a laminar flow, where the velocity profile in the pipe consists of nice smooth streamlined. The velocity profile is parabolic and the fluid motion is a long straight streamlines and momentum diffusion across the stream lines can take place only due to; I am sorry, momentum transport across the stream lines can take place only due to momentum diffusion. So, this is the special case of a pipe flow for a laminar flow. And we know that a laminar flow the Reynolds number up to about 2100 the pipes the flow in a pipe is laminar. At some part there is a transition to a turbulent flow in that point you get different result. For the friction factor is function of Reynolds number a result that good in general depend upon the roughness of the wall of the pipe.

So, I have got you the relation only for a laminar flow from momentum balances. For a pressure gradient the parabolic velocity profile across the pipe. The details of the velocity profile as a function of the pressure gradient. The shear stress, the wall shear stress, the average flow rate, the mean velocity and from the shear stress in the mean velocity expression we can get an expression for the friction factor; all of this is for laminar flow.

I will continue a little bit on pipe flow looking a little deeper at what happens after the transition to turbulence. There I will not be able to do an exact calculation for you, but I will try to give you some idea of how a turbulent flow is different from a laminar flow and qualitative picture. And after that we will look at unsteady flows. So, we will continue this discussion of pipe flows in the next lecture. I will see you then.