Transport Processes I: Heat and Mass Transfer
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Lecture – 36
Unidirectional transport: Effect of pressure in momentum transfer. Flow in a pipe

We continue our past series of lectures now on momentum transfer in one dimension. In the last lecture, I had taken a simple example of the flow down an inclined plane in a Cartesian coordinate system in order to illustrate how the solutions are obtained when you have a body force on the system.

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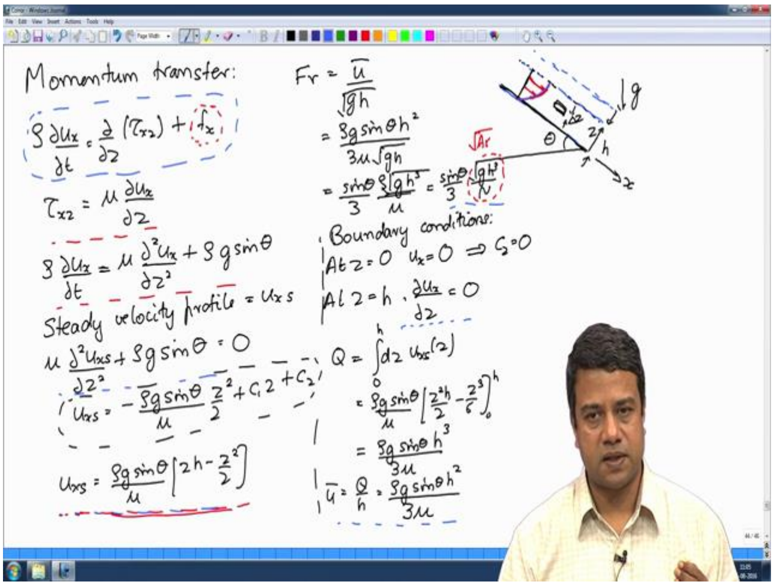

If you recall, we had the flow down an inclined plane which was inclined at some angle theta to the horizontal and because there is a component of the gravitational acceleration along the direction this generates the fluid flow down the plane and what we were trying to do was to find out what is the velocity profile, what is the average velocity and whether we can get an expression for the frond number; the dimensionless number in this case.

The momentum conservation equation was of this form, we had derived it for unidirectional flows earlier, it contains a body force density; a force per unit volume as the inhomogeneous term in this equation and if we use Newton's law of viscosity, we get a second order differential equation and we had solved this; at steady state you get a steady solution where there is no variation in time, recall that when we wrote this equation the implicit assumption was that there is no variation in the x direction, there is no variation in the stream wise direction and therefore, we had written only for a fully developed flow.

If the flow is also steady we solved it in order to get the velocity profile and that velocity profile as you recall was a parabolic flow, the velocity was 0 at the base because it was in contact with the solid surface. At the liquid gas interface, I had made an argument for you previously that the shear stress on the liquid side has to go to 0. Strictly speaking the boundary condition is that the shear stress in the liquid is equal to the shear stress on in the gas and the velocity in the liquid is equal to the velocity in the gas.

However, the viscosity in the gas is so low that for approximately equal strain rates, the stress in the gas is much lower than the stress in the liquid. The only way that the stress balance condition can be satisfied; the viscosity in the gas is much lower than the viscosity in the liquid the only way that the stress balance condition can be satisfied is if the strain rate in the liquid is much smaller than the strain rate in the gas therefore, we can assume a boundary condition which basically states that the strain rate in the liquid is equal to 0 at that top surface; this is always true for liquid gas interfaces.

Subject to those conditions, we got a solution for the velocity profile and then we got an expression for the froude number which is basically based upon the Archimedes number, we got the mean velocity as a function of the acceleration due to gravity, the density, the viscosity and so on, so that was for a steady flow. The velocity profile is parabolic it has 0 slope at the top surface and 0 velocity at the bottom surface and then between it has to satisfy this expression. So, you basically get a quadratic velocity profile when you have a body force; this is in comparison to the linear profile you would have got for example, for the steady temperature field between 2 plates; where there is no source, you get a linear profile.

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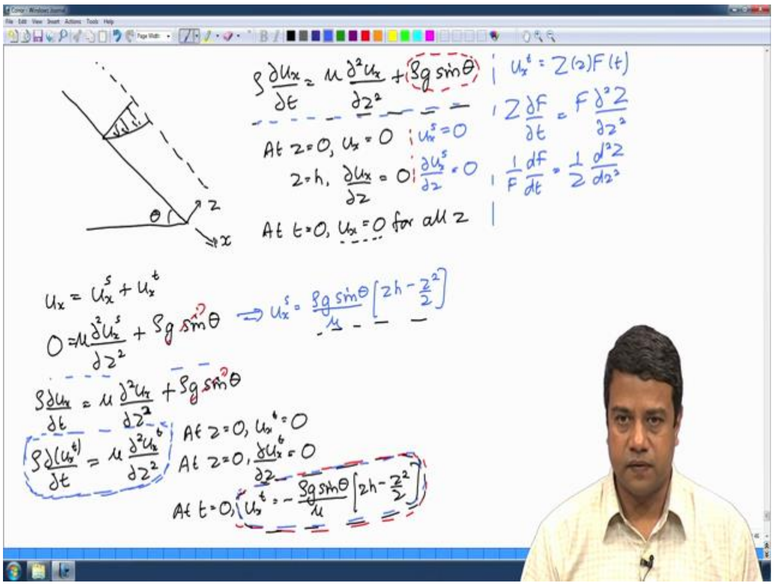

I had briefly given you an indication of how to solve the unsteady problem. In this case as well one has to separate out the velocity into the steady part plus the transient part. The total velocity profile has this body force in it, if I separate this out into the steady and the transient part. If the steady part has the same body force, so when I subtract it out the transient part has is homogenous, there is no body force in the transient part and this is important for us because if you were to use a separation of variables procedure, if I were to write down u x transient is equal to some function of z times some function time and I were to insert it into an equation with an inhomogeneous term, what I will get is that z times partial, I should write this as a F by partial t is equal to F times partial square z by partial z square plus if I have some inhomogeneous term here and if I divide throughout by z times F as I have been doing for separation of variables, what I will get is that 1 by F; d F by d t is equal to 1 by z these squares is set by d z square plus this inhomogeneous term by F times z.

Now you do not have 2 sides of the equation one of which is only a function of time and the other is only a function of space. So, you cannot impose a separation of variables procedure, you can only do it if this inhomogeneous term is equal to 0 only then do you get a solution where one side is only a function of time and the other side is only a function of space and then you can express the solution comes the Eigen functions. So, it is important that in all of these problems, when we go to a separation of variables procedure; we have to subtract out the steady part of the velocity or temperature fields in such a way that the equation itself is homogeneous, it contains no inhomogeneous terms and the boundary conditions are homogeneous except in one direction. In this particular case in the 2 spatial coordinates at z is equal to 0 and z is equal to h, the boundary conditions are homogeneous and the only in homogeneity constant in at initial time, once you have that you can solve it to get the Eigen values and the Eigen functions.

In this case the boundary conditions in the spatial direction are slightly different, it requires a 0; velocity at the base and 0 slope at the top and in the Cartesian coordinate system, the Eigen functions have to be in the form of sin and cos functions only sin functions if the velocity is 0 at z is equal to 0; however, those sin functions have to be such that the velocity is 0 at the base and the slope is 0 at the top. So, I have to have sin functions which have Eigen values that are of this form.

Previously we just had n pi because we had fixed boundary conditions on both sides whereas, in this case we have 0 velocity at the bottom and 0 slope at the top; that means, that the Eigen functions have to be of this form. So, I have to use a solution which contains Eigen functions which are sin functions, which are 0 at the base and have 0 slope at the top; they are maximum at the top and once that is done, the Eigen functions in time is well specified and the coefficients A n over here are determined from this orthogonality relation, which basically says that any 2 Eigen functions if you multiply them integrate from 0 to one that product is 0 unless n is equal to m only if n is equal to m with that product nonzero. So, using these orthogonality relations you can find out what all the constants are.

So, just want to give you a brief summary of how this procedure is done. So, that was in a Cartesian coordinate system, the example that we deal with most often is in a cylindrical coordinate system that is flow in a pipe.

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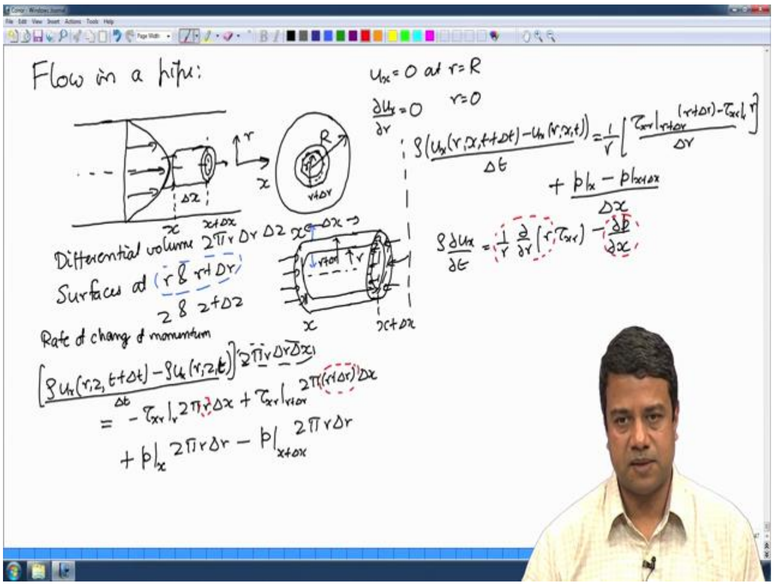

So, we look at the flow in a pipe in some detail. So, I have a pipe the radius of the pipe is R and you have flow in the pipe due to a pressure difference between the inlet and the outlet, so the pressure at different locations in this pipe is not the same. So, let us take a co-ordinate system; I will call the axial coordinate as x and the radial coordinate is r. We will be using a cylindrical coordinate system so that the surface of this pipe and r is equal to R is the surface of constant coordinate, so that I can apply the boundary conditions easily in this case.

So, the velocities u x is equal to 0 at r is equal to R that is the physical boundary; the other condition that I would have is at r is equal to 0, the velocity as I said r is equal to 0 is not a physical boundary, it is a boundary imposed by the coordinate system; there is no physical boundary there. At this location you require that the slope of the velocity be equal to 0, as I told you in the case of the temperature field in a cylindrical object. If the slope were not equal to 0, then you would have different slopes depending upon which direction you come from. If there is no physical boundary there, you cannot have different slopes depending upon the direction in which you approach the boundary.

So, the velocity slope was not 0 if you do something like this that at the center, you come from in this direction the slope is in this direction whereas, we come from the opposite side since there is no variation with the angular coordinate around, the slope has to be something different and this curve cannot have 2 different slopes at a single point because there is no physical boundary there, if there were a physical boundary you might be able to have 2 different slopes as we saw in the case of conduction from a wire, but in this case, there is no physical boundary there therefore, you cannot have two different slopes at that same location. Therefore, the symmetry requires that the boundary condition should be partial u x by partial R, the derivative of u x with respect to r has to be equal to 0.

And that is intuitively obvious we would expect that for a pipe flow, the maximum velocity should be at the center and the maximum velocity is the location where the slope is equal to 0 or the derivative is equal to 0. So, we will write the balance laws now for this pipe flow, for a cylindrical differential volume. As I said in a curvilinear coordinate system; one has to take a differential volume in which the surfaces are surfaces of constant coordinate. So, we take an annular volume in, so if this is the center of the pipe we can take an annular volume.

So, if you look at it here it looks something like this, so this thing in the radial direction; it is bounded by the location r and the location r plus delta r. So, it has a thickness delta r in the radial direction and in the z direction, it will have a thickness delta z and I have to write a momentum balance equation for this. The momentum balance equation will basically tell me that the rate of change of momentum is equal to the sum of forces. Once again I will consider variation only in one direction; I will assume that the flow is fully developed so that there is no variation in the velocity along the x direction.

We will see a little later when how to take into account variations in the x direction. So, for this differential volume, I have to write a balance of momentum. Now the forces, the rate of change of momentum is just equal to the difference in the momentum between time t and time t plus delta t. So, this differential volume it has the total volume in this is going to be equal to this circumference times the thickness 2 pi R delta r times the length delta z. So, that is the volume it is bounded by two surfaces which is surfaces r at r and r plus delta r and at z if I take this as an circuit take this as x. So, I will take this location as x and x plus delta x. So, those are the two surfaces and that will write a balance equation for this. So, the rate of change of momentum, there is the change in momentum at t plus delta t minus times the volume. So, there is the rate of change of momentum times the volume is 2 pi, r delta r delta z is equal to the sum of forces. So, forces are acting at the surfaces r and r plus delta r, if I look at this differential volume if I just cut a section through this, I have one surface at r and one surface at r plus delta r.

What is the net force at the surface at r, this is going to be is equal to the shear stress times the surface area at the location r. If you recall the shear stress tau x r was defined as positive if it acts at the surface whose outward unit normal is in the r direction. If you recall we defined tau x r is equal to force per area in x direction at surface with outward normal in r direction, if you look at the surface at r the outward perpendicular is in the minus r direction. Therefore, the force is equal to the stress times the area and this will be equal to minus tau x r at r times the surface area; 2 pi R delta x, the surface area of this is equal to the circumference 2 pi r times the width which is delta x; that is the force at the inner surface. At the outer surface, the outward unit normal is in the plus r direction, therefore, the force exerted is equal to plus tau x r at R plus delta R the 2 pi into R plus delta R into delta x.

So, those are the forces at the 2 surfaces r and r plus delta r; however, there are also forces exerted on the surfaces at z and z plus delta z that is the pressure force; pressure always acts inward along the inward unit normal because pressure is always compressor. So, if I take for example, the surface at x plus delta x here at this surface the pressure is acting inward the pressure is compressor. So, at the surface at x plus delta x, the pressure is acting in the minus x direction. So, at this surface the pressure is acting in the plus x direction and because there is a pressure difference across the pipe these two pressures are not the same.

Therefore, this pressure difference will enter into the momentum balance equation. So, at the location x the pressure is acting in the plus x direction. So, the total force is plus the pressure at the location x times the surface area; surface area is this curved surface area which is equal to the parameter times thickness, maybe the surface area is a curved surface area there is a parameter times the thickness. So, is basically equal to 2 pi r times delta r, so there is a force exerted at the surface at the location x, there is a force that is exerted at this location x plus delta x; this location is x and this location is x plus delta x on the right side. The force on the right side acts in the minus x direction, the pressure is compressive so it is acting inwards. So, is equal to minus p at x plus delta x; 2 pi R plus tau.

There is also momentum coming in and going out, in this particular differential volume because of the mean velocity there is momentum coming in at the location x and momentum going out at the location x plus delta x. The momentum coming in at the location x is equal to the momentum density times the flux. So, at the location x there is momentum that is entering because the fluid is entering give you the differential volume the momentum entering at the location x is going to be equal to the momentum density times the velocity is the momentum flux; momentum transported per unit area per unit time, times the surface area, 2 pi R delta r; there is a momentum entering at the location x because the fluid is entering. This momentum leaving at the location x plus delta x because the fluid is leaving, so this is going to be at the location x and this momentum leaving and that is equal to rho u x; the momentum density times the normal velocity; velocity perpendicular to the surface it is also u x times 2 pi R at x plus delta x.

So, that is the total momentum balance equation; in this particular case since the flow is fully developed, we have assumed that there is no variation of velocity in the x direction and therefore, these two are exactly the same. If there is no variation in the velocity in the x direction then the momentum in and the momentum out are exactly the same and for this particular case we will have no contribution to the momentum entering and leaving, but in general that contribution should be there even though there is fluid flow in addition to the forces exerted there is also a convective transport of momentum and that has to be incorporated in general.

The pressure; however, is not equal at the locations x and x plus delta x because of the pressure difference between the ends of the pipe and now we divide throughout by the volume, if we divide toward the volume I made a mistake here; I should write down the rate of change of momentum is equal to the sum of the applied forces. The rate of change of momentum is the difference in momentum density divided by delta t, so kindly correct that.

So, this is the balance equation I have to divide throughout by the volume; volume in this case is 2 pi R, delta r times delta z. If I do that, I will get rho times u x at r x; t plus delta t minus u x at r x t by delta t is equal to; now we have to be careful here because once again you can see that the surface area is changing as the radial coordinate changes, the surface area at r and r plus delta r are not the same. So, there is a radial dependence of the surface area and that has to come in here. So, I will get 1 over r into tau x z times r at

r. So, whenever it divided by 2 pi R delta r delta z, the delta z terms will cancel out, now we will divide by 2 pi r delta r times delta x, the delta x terms will cancel out on both sides; I would have a 1 over R 2 pi once again will cancel out on both sides. So, I will have 1 over R and 1 over delta r; times tau x r at r plus delta r minus tau x r times r at r.

And the second term I will get plus p at x minus p at x plus delta x divided by delta x and if I take the limit of delta r and delta x going to 0. The resultant equation becomes rho times partial u x by partial t is equal to minus delta p by delta x. So, that is the equation additional term here the pressure gradient which is not there in heat and mass transfer, this is another mode of momentum transfer which makes momentum transfer different. Note that this has the exact same form as a body force; the pressure gradient is equivalent to a body force so long as the density is a constant.

So, this additional pressure gradient term and now once again since we are doing cylindrical coordinates, this operator is slightly different. If you recall when we solved the same problem in a Cartesian coordinate system, what we had got was d square tau x z by d x square. In this case in a cylindrical coordinate system this operator is slightly different, the reason for that is that there in a cylindrical coordinate system because you have curved surfaces, the surface area changes as the radius changes and the total force is equal to product of the stress times the surface area and the surface area is changing and the stress is also possibly changing and because of that; you have this additional differential operator here. In the case of a Cartesian coordinate system, it was just one derivative on tau x (Refer Time: 29:16).

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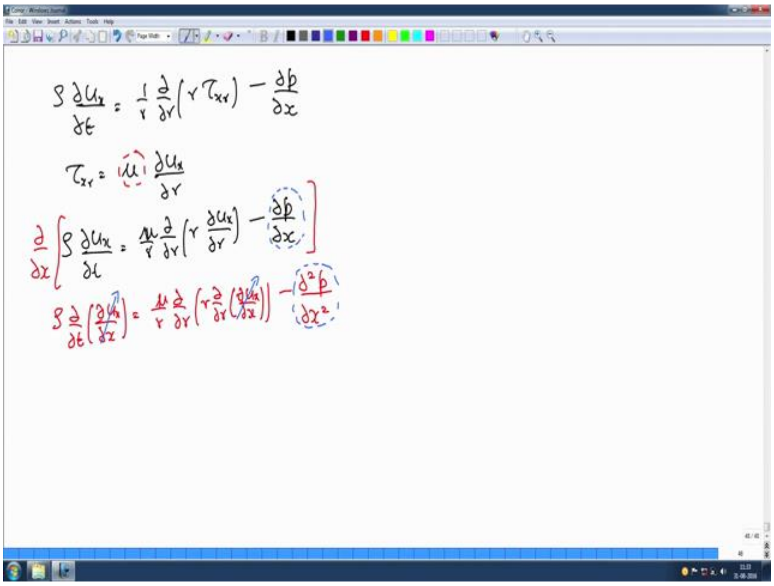

And then if we use Newton's law for viscosity, the differential equation I had was rho times partial u x by partial t is equal to minus partial p by partial x and tau x r is equal to mu partial u x by partial r, the differential equation becomes rho d u x; by d t is equal to minus d p by d x. So, this is the final equation for momentum transfer in a pipe; the velocity profile in a pipe. This pressure gradient, it turns out it as to be a constant, if the flow is fully developed; we are often taught that it is equal to a constant, but we have never told the reason why it is a constant.

If the flow is fully developed, I can take the derivative of this equation with respect to the x coordinate and since it is a partial derivative you can interchange the order of differentiation. So, you will get rho d by d t of partial u x by partial x is equal to 1 by r; I should have a viscosity here I apologize, I shall viscosity here the viscosity comes out of the viscosity and this equation for the stress.

Just taking the derivative and then interchanging the order of differentiation, this is what I get. Since the flow is fully developed, the variation of velocity with respect to x 0 on both sides since the flow is fully developed d u x by d x is equal to 0 on both sides and what that implies is that the second derivative is equal to 0; d square p by d x square is 0; that means, that d p by d x is equal to a constant. So for a pipe flow; if it is fully developed, d p by d x has to be equal to a constant or the pressure has to be a linear function of position, if the flow is fully developed.

So in the next lecture we have obtained the equation here, now we have to solve for the velocity profile and then see if we can get results for the friction factor as a function of Reynolds number and so on. So that part, the velocity profile and the friction factor that we will continue in the next lecture using this particular equation for the momentum balance, so we will continue with pipe flow in the next lecture. First we will derive the equation for a steady flow and then we will look at some examples of unsteady flows, so we will continue that in the next lecture, we will see you then.