Transport Processes I: Heat and Mass Transfer
Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture - 35
Unidirectional transport: Effect of body force in momentum transfer. Falling film

So this is a continuing series of lectures on fundamentals of transport processes. In the last few lectures, we had solved some problems on heat and mass transfer the framework for both of these is the same. We had seen heat transfer in an infinite fluid in a Cartesian coordinate system, where we used the similarity solution in order to obtain the temperature field both for a flat surface at fixed temperature as well as for a pulse input. We had solved in finite domains using separation of variables procedure and I had shown you how the same procedure can also be adopted to a cylindrical coordinate system. The differential operator in a cylindrical coordinate system is slightly more complicated, but the solution procedure is exactly the same.

Before we go on to deriving conservation equations in 3 dimensions, I would like to go through some examples of momentum transfer because in the case of momentum transfer, there are 2 things that are different; one is that one could have body forces such as gravitational force, centrifugal force and so on and one could also have pressure differences which could exert a force on a volume element of flow.

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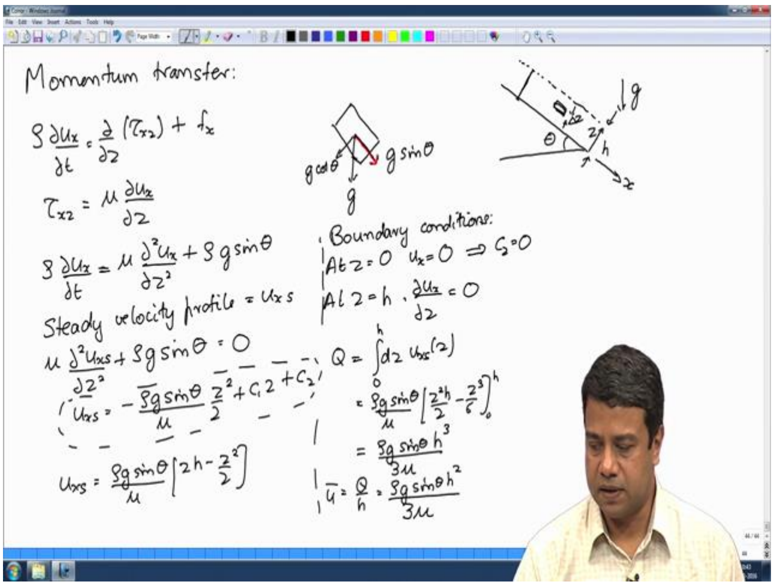

So, in this lecture we will start momentum transfer and I will consider first the simplest case where you have an inclined plane, inclined at an angle theta to the horizontal and the gravitational force is downwards and then you have a film of fluid that is flowing down this incline plane with let us say some height h and one would like to know what is going to be the flow rate of a film, flowing down the incline plane with a thickness h.

It is flowing down because in each volume element, there is a gravitational force. So, if I look at that volume element of fluid, it is convenient for the present purposes to consider the x coordinate along the inclined plane; the direction of flow and the y coordinate perpendicular to the inclined plane; the x coordinate along the direction of flow and the y coordinate perpendicular to the direction of flow.

So, for this volume element I have a gravitational force acting vertically downwards; that means, that I have a force density, the acceleration along the plane is going to be equal to g sin theta and perpendicular is going to be equal to g cos theta. So this acceleration is driving the flow, there is no velocity perpendicular to the plane of course, we will consider this to be of a fully developed flow; that means, that there is no variation along the stream wise x direction. So for a fully developed flow, it may not be steady; steady means there is no variation in time fully developed means there is no variation in the stream wise direction, the only variation in the velocity is in the cross stream direction; in the y direction.

So, the momentum balance equation that you get for this kind of a coordinate system for the momentum in the x direction, after all we are interested in the velocity in the x direction. The momentum balance equation we have already solved, this is a Cartesian coordinate system, so if I take a differential volume of height. So, in order to be in conformity with what I had done earlier, I will call this as z instead of y. So, I take a small differential height delta z and with this differential thickness, if I do a balance then I will get the familiar equation rho partial u x by partial t is equal to partial by partial z of tau x z plus the force in the x direction, so this is going to be my balance equation. I had derived from this for you earlier by doing a shell balance, in case you would like to refresh it; you are welcome to go back and look at that derivation which I had used.

Tau exact is the shear stress, the force in the x direction acting at a surface whose unit normal is in the z direction, so there is a shear stress. In this particular case, the shear stress is just given by Newton's law of viscosity. So, therefore, my momentum balance equation now becomes rho partial u x by partial t is equal to once again if we assume that the viscosity is not a function of the z coordinate, if you assume that the viscosity is independent of distance then I will get mu into d square u x by d z square plus f x the force is equal to f x is a force density in the x direction, the acceleration in the x direction is given by g sin theta therefore, the force density is equal to the mass density times the acceleration in the x direction. So, force density is equal to the mass density times the acceleration in the x direction. So, that is the conservation equation in this case for this flow down an inclined plane.

And now we can ask what is the velocity profile; first what is the steady velocity profile. So, effectively we consider fully developed flow that is we assume that this plane of the infinite extent so that there is no variation in the velocity at different downstream distances and then if it is at steady state. So, that there is no variation in time as well, the conservation equation will become mu times d square u x by d z square plus rho g sin theta is equal to 0. So, that is the solution for the steady velocity profile; I will call that as u x steady, the steady part of the velocity profile; the steady part of the velocity in the x direction. This equation can be solved quite easily, you will get that u x steady is equal to minus rho g sin theta by mu into at integrated 2 times with respect to z. So, I will get into z square by 2 plus c 1 z plus c 2 and these constants after integrating it 2 times, these constants have to be determined from the boundary conditions.

So, what are the boundary conditions for this fluid element; at the base of course, you have a solid surface therefore, the velocity has to be equal to 0 at the base. So, the velocity has to be equal to 0 at the base at z is equal to 0 u x steady is equal to 0.

Now, what happens at the top surface; the top surface the liquid is in contact with air or the gas. At that surface you have the continuity of velocity and the continuity of stress, the stress on the liquid side has to be equal to the stress on the gas side, the velocity on the liquid side has to be equal to the velocity on the gas side. For a liquid gas interface one can make a simplification; the stress boundary condition will be of the form mu liquid times d u x in the liquid side by d z is equal to mu gas times d u x gas by d z and therefore, if I write the velocity gradient on the liquid side, I will get that d u x liquid by d z is equal to mu gas by mu liquid; d u x gas by d z.

Now, if the velocities on both sides have to be approximately equal, you require that as well the velocity condition is that u x steady in the liquid side has to be equal to u x steady on the gas side. Both of these are at z is equal to h; at the top surface of the film we are looking at the boundary condition for the gas and the liquid interface. If the velocities have to be approximately equal, we know that the viscosity of the gas is much smaller than the viscosity of the liquid; liquid has a viscosity of about 10 power minus 3 Pascal seconds, water is 1 into 10 power minus 3 Pascal seconds. Gas is typically, have viscosities of 10 power minus 5, so this ratio is about 1 in 100.

Then to a very good approximation, you can assume that this velocity gradient on the liquid side has to be equal to 0 at a liquid gas interface. There is the 0 shear rate condition at a liquid gas interface; you can apply the 0 shear rate condition that is that the velocity gradient is equal to 0 at a free surface because the gas is not able to exert any shear stress on the liquid because its viscosity is so low. Therefore, if you want to have stress continuity, the shear stress in the liquid as you approach the surface has to go to 0 and; that means, of the strain rate in the liquid as you approach the surface has to go to 0 that is called the 0 shear stress condition.

Rather than having a fixed velocity the requirement here is the velocity gradient has to be equal to 0 at that surface. So, with that the second boundary condition becomes the first boundary condition was that at z is equal to 0 u x was equal to 0 and the other condition is that at z is equal to h partial of partial u x by partial z is equal to 0. So, those are the two boundary conditions which we have to solve in order to get the steady velocity profile. These conditions apply both for the steady as well as the total velocity profile, so these conditions are common; they apply for the steady velocity profile and the apply for the total velocity profile.

So, the first condition at z is equal to 0 u x is equal to 0 implies that this constant c 2 has to be equal to 0 because at u x is equal to 0, this expression states that the velocity is just equal to c 2. So, at z is equal to 0 if you x is equal to 0 this c 2 is equal to 0 and that z is equal to h, if d ux by d z has to be equal to 0, d u z by d x is equal to c 1 minus rho g sin theta h by mu; that is a derivative at d u x by d z.

So, this gives us an expression for c 1 and we can use that in order to get the expression for the steady velocity profile into z h minus z square by 2. So, this is the expression for the velocity profile; from this we can find out what is the flow rate and what is the average velocity. So, the flow rate q; if this is of course, per unit length perpendicular to the plane of the surface, the flow rate Q is equal to integral 0 to h; d z times u x steady of z and you can do this integral easily, in this case you will get rho g sin theta by mu into z square h by 2 minus z cube by 6; between the limit 0 to h is doing the integral. So, there is a flow rate per unit length perpendicular to the plane and this just becomes equal to rho g sin theta; h cube by 3 mu and the average velocity is just the ratio of Q by h is equal to rho g sin theta h square.

So, that is the average velocity and from those you can get dimensionless numbers.

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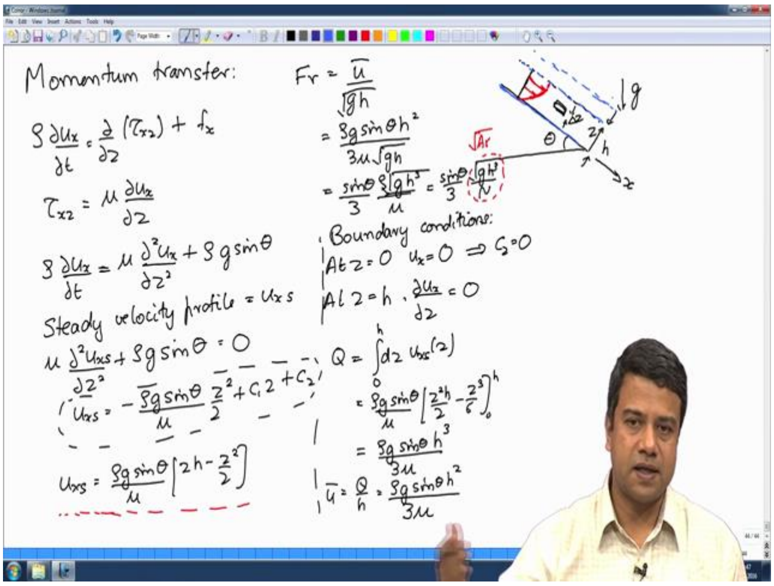

So, for example, in this particular case, the appropriate dimensionless number is what is called the froude number u bar by g h I am sorry square root of g h. So, this is equal to rho g sin theta h square by 3 mu route of g h and you can simplify this; basically you will get sin theta by 3 into square root of g h cubed divided by the kinematic viscosity and this ratio, so this ratio is basically is equal to the density by the viscosity. So, I can write this as sin theta by 3 root of g h by kinematic viscosity. You can easily verify that this parameter; dimensionless the numerator has dimensions of length square per time and the denominator has dimensions of length squares per time.

This is actually the square root of a dimensionless number that is called the Archimedes number; square root of Archimedes number. So, that is the correlation for the froude number for a steady flow down an inclined plane and this is the velocity profile for the steady flow. So, this is the velocity profile for the steady flow down an inclined plane, if I plot it, it has 0 velocity at the base and it has 0 slope at the two. So, it is basically a parabolic velocity profile, it has 0 gradient at the top and it has 0 velocity at the base. So, this is the standard velocity profile that you will get whenever there is an body force acting, you will get a parabolic profile and that is because the solution of this equation turns out to be quadratic in z because the second derivative balances the body force therefore, the solution turns out to be quadratic in z.

You could think of more complicated problems for example, you could think of a film that contains two fluid layers for example, in this case you will have 0 velocity gradient at the top surface if you had 2 films; you have 0 velocity gradient at the top surface. At this interface between the 2 liquids, you have to have equality of velocity and equality of stress; both have to be equal at the liquid interface, there has been equality of the velocity and the equality of stress; 0 velocity boundary conditions at the bottom and 0 stress boundary conditions on the second layer on top and you have to solve all of those simultaneously in order to get the velocity profiles.

So, that was for a steady flow; one could also think of the developing flow, so in that case what you would think of is a problem of the following type.

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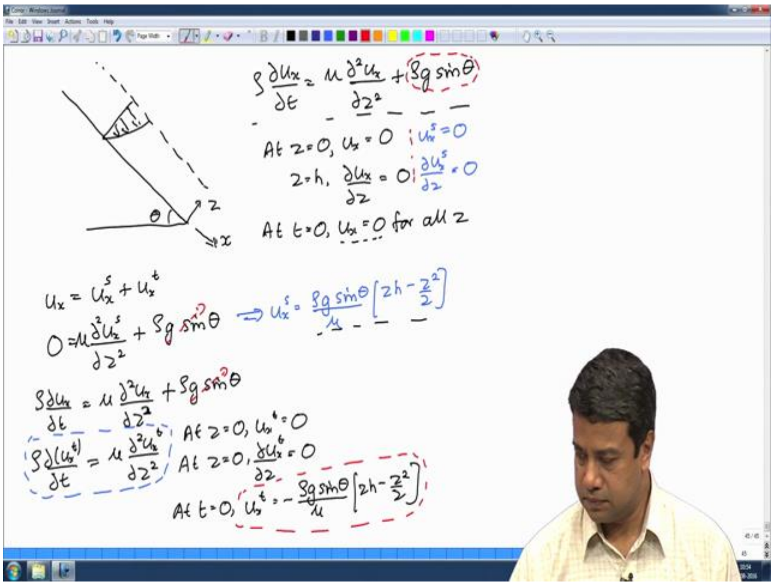

I have an inclined plane inclined at some angle theta and I have a film that is flowing down, this incline plane this is the x direction and that is the z direction; I have a film that is flowing down this incline plane it is fully developed, there is no variation in the x direction; however, it is not steady anymore, I have a flat surface and I tilt it at time t is equal to 0, so that at time t is equal to 0, there is no the velocity is equal to 0.

So, in that case my conservation equation becomes partial u x by partial t times the density is equal to plus rho g sin theta; that is my differential equation. The boundary conditions at z is equal to 0, u x is equal to 0, z is equal to h; u x is that the derivative is 0 and at t is equal to 0; u x is equal to 0 for all z. So at t is equal to 0, the gravitational acceleration is just switched on and therefore, the film starts to flow.

So how do we solve this equation, once again we have to write the velocity u x is equal to a steady part plus transient part, where the steady part satisfies the steady equation; the steady equation is that 0 is equal to partial u x steady by partial z square plus rho g sin theta. We have already got the steady solution; u x steady is equal to rho g sin theta by mu; z times h minus z square, we have already got the steady solution.

The total equation is of the form rho times partial u x by partial t is equal to I should have viscosity here; that was the total equation, if I subtract these 2; I will get an equation for the transient part. In the equation for the transient part, we know that partial of u x transient by partial t because I subtract out these 2, the steady part is independent of time. So, I will just get rho times partial u x transient by partial t is equal to mu times partial square u x transient by partial z square and you can see that these body forces actually cancel out. The original equation that I had was an inhomogeneous equation, the original equation that I had was an inhomogeneous equation where there was a forcing a body force within the fluid itself.

The boundary conditions were both homogeneous; the boundary conditions that z is equal to 0 and at z is equal to h were both homogeneous. When I express that in terms of a steady in the transient part and I subtract out the transient equation from the total equation, the resulting equation that I have for the transient part if I subtract the steady equation from the total equation is now homogeneous, it does not have a forcing in space or in time. So, therefore, since it is homogeneous I can use separation of variables procedure.

Boundary conditions at z is equal to 0; u x is 0 and u x steady is also equal to 0. If the steady parts satisfies the same equation as the total velocity profile, so therefore, for the transient part, the boundary condition will be there at z is equal to 0 give x transient is equal to 0. At the top surface at z is equal to h d u x by d z is equal to 0, the steady part also satisfies the same equation. Therefore, the boundary condition for the transient part is also that at z is equal to 0 partial u x transient by partial z is equal to 0 and finally, at t is equal to 0, u x is equal to 0 for all z; however, the steady part is nonzero. So, therefore, the boundary condition for the transient part should be at t is equal to 0, u x transient is equal to minus; that should be the boundary the initial condition for the transient part. So, as expected in a separation of variables procedure, the fourth thing is at initial time u x transient has to go to 0 as t goes to infinity, it is nonzero at the initial time.

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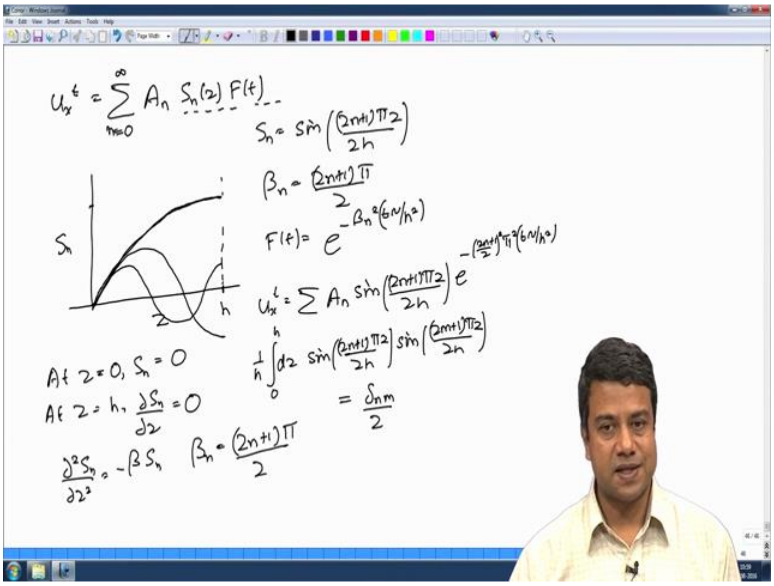

So therefore this u x transient the solution for that is going to be of the same form that we had earlier; u x transient will be equal to the summation of a set of Eigen values A n times the set of coefficients n is equal to 1 to infinity times coefficients A n times some function in space and some function in time, well how should those functions look like, as a function of z, each of these functions let us call it as S n of z times; sum function F of t, each of these functions should be equal to 0 at z is equal to 0 S n is equal to 0 and at z is equal to h S n has to be equal to satisfy the derivative condition.

So, I will there being 0 on both surfaces there to satisfy the condition partial S n by partial z is equal to 0 and they have to satisfy an Eigen function problem of the form d square S n by d z square is equal to minus some coefficient beta times S n; the solutions are; obviously, sin and cosine functions and cosine functions the coefficient has to be 0 because they are not 0 at z is equal to 0, what about sin functions; you have to choose sin functions in such a way that sin function has 0 value at z is equal to 0 and it has 0 slope at z is equal to h, it has have 0 slope at z is equal to h. The sin functions that satisfy this are of the form S n is equal to sin of 2 n plus 1 by z by h.

Previously we had sin of n pi z star; in this case it is of the form sign of 2 n plus 1 pi z by h. So, the functions beta n is equal to 2 n plus 1 into pi I am sorry by 2 h by 2. So, n is equal to 0 beta I am sorry n is equal to 0 beta is equal to pi by 2 n is equal to 1 beta is equal to 3 pi by 2 and so on. So, I should strictly have n going from 0 to infinity in this case you can write it in this way. So, the first 1 will be a sin function that goes from 0 to pi by 2 slope is 0 at pi by 2. The next one will have something that looks like this the third 1 will be looks like that and so on.

So, these are the basis functions that I will be using and the Eigen values beta n is equal to 2 n plus 1 pi by 2 and therefore, the function F of t will be equal to e power minus beta n square times t scaled by the viscous time scale. The viscous time scale in this case is going to be equal to the kinematic viscosity divided by h square there is going to be the viscous time scale in this case.

So, therefore, I can express this in this form, so the basic function that I will get is equal to sigma A n sin 2 n plus 1. So, I z by 2 h times e power minus plus 1 by 2 square pi square; t mu by h square and these coefficients I will determine from the orthogonality relations; the orthogonality relations for this is also the same 0 to h d z, I should take 1 over h sin this is non 0 only if n is equal to m. This is non zero, only if n is equal to m is equal to delta n m by 2 if it is half if n is equal to m and it is 0 if n is not equal to m.

So, expressing it in the terms of this basis function that satisfies the 2 boundary conditions, it has to be 0 at z is equal to 0 just as 0 slope at the top because of that we get slightly different functions compared to the previous case where we had 0 boundary velocity temperature boundary conditions at both surfaces. So, these are the modified functions in this case and using the initial condition here and the orthogonality relation, all of the coefficients in this expansion can be evaluated.

So, I just wanted to give you an outline of how this is done in this case. It is very similar to the previous case except that both the equation and the boundary conditions in the spatial coordinates have to be homogeneous when you do the separation of variables procedure. So, I will just briefly go over this once again in the next class to emphasize what has been done and then we look at flow in a pipe. So, we will continue this in the next lecture, I will see you then.