Transport Processes I: Heat and Mass Transfer
Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture – 34
Unidirectional transport: Balance laws in cylindrical co-ordinates. Similarity solution for heat conduction from a wire

Welcome to this our continuing series of lectures on Transport Processes, where we were looking at Balance Laws for heat and mass transfer in a cylindrical co-ordinate system.

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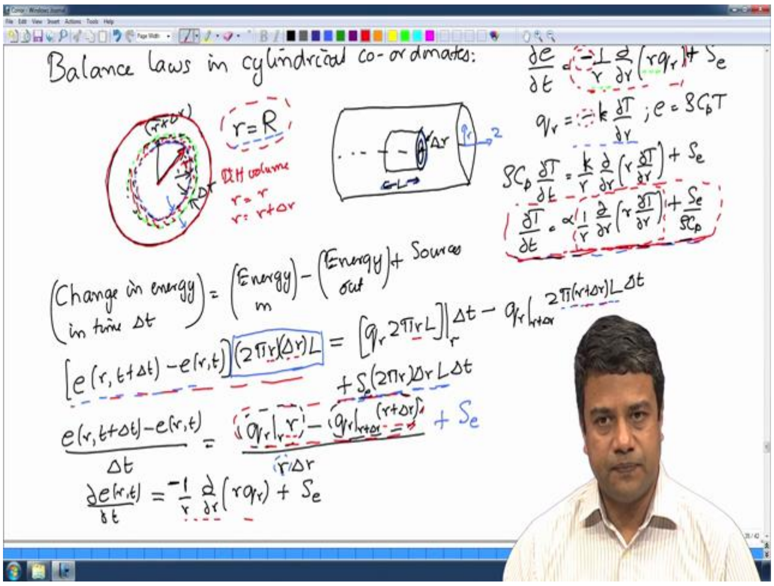

If you recall we had derived the convection of the diffusion equation for a cylindrical co-ordinate system here. We had chosen a cylindrical co-ordinate system because in cylindrical geometries like a pipe flow for example, rather than expressing the surface of the pipe in a Cartesian co-ordinate system which will basically be the equation of a circle. So, if you solve the problem in a Cartesian co-ordinate system one would have to write down the boundary conditions in a co-ordinate system which is not a surface of constant co-ordinate in this particular case it had to choose the co-ordinate system such that x square plus y square is equal to r square; where r is the radius of the pipe.

Rather than do that let us choose the co-ordinate for which the bonding surface is a surface of constant co-ordinate. So, that was the idea. So, in this cylindrical co-ordinate system it is convenient to choose a co-ordinate in which this circular surface of this pipe is as surface of constant co-ordinate. And that quadrant co-ordinate is the distance from the center or from the axis of the pipe. So, we had chosen this radial co-ordinate as the distance from the center of the pipe and we had expressed a balance equation in terms of this radial co-ordinate and this as this one.

So, we have chosen a slightly more complicated co-ordinate system so that we can use the boundary condition as a surface of constant co-ordinate the boundary as a surface of constant co-ordinate. The price to pay is that this differential operator becomes slightly more complicated.

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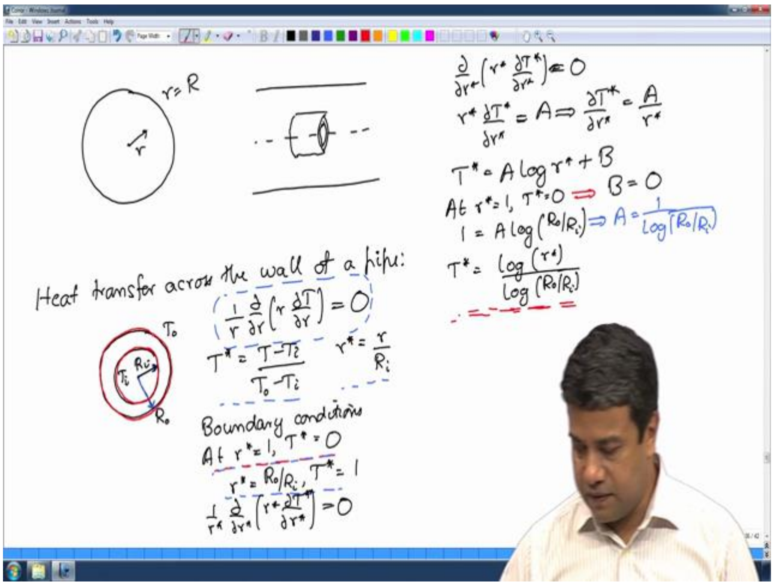

However, we can still do all of the calculations that we had done in a Cartesian co-ordinate system in this co-ordinate system as well. And I had shown you the balance laws and how to solve for a first for a steady problem which is just the temperature across the surface of an annular section of pipe within not to a radii. So, that was the first problem that we had solved.

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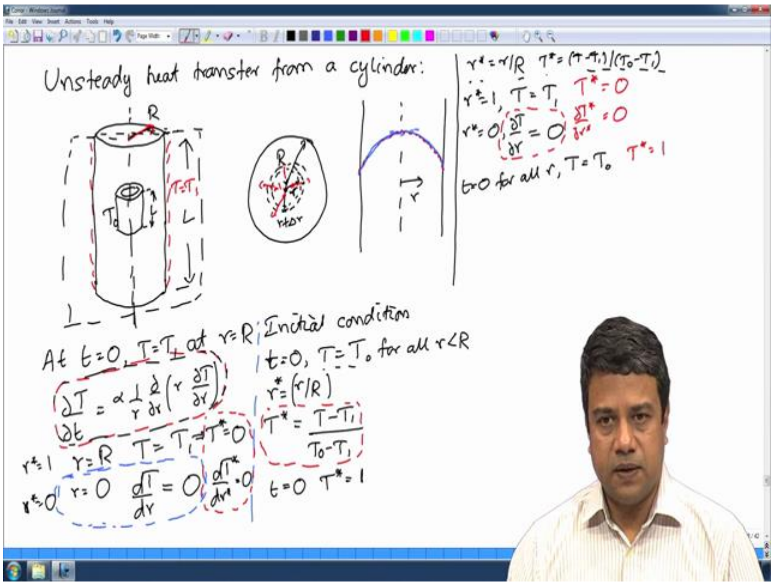

The second problem that we had solved was to use separation of variables for an unsteady problem; where initially the entire cylindrical object was at one particular temperature and at time t is equal to 0 it was brought into contact with the fluid which was at a different temperature. And our task was to find out how this reaches that equilibrium temperature.

And we had posed this problem; in this particular case in the radial co-ordinate there is only one physical boundary at r is equal to capital R which is the radius of this cylindrical object. There was a second boundary which was due to symmetry, which is a natural boundary condition because at the center itself the radius is equal to 0. And at that center we had chosen we had said that since there is no physical boundary there a symmetry condition has to be satisfied so that the temperature derivative from the left and from the right is exactly the same.

In fact, regardless of what angle you approach these center from the temperature field has to be exactly the same, and there should be no discontinuity in the gradient of the temperature at the origin. And that itself gave us a natural boundary condition which was of the form dT by dr is equal to 0 at r is equal to 0. And we had seen how to do the separation of variable solution procedure for this problem.

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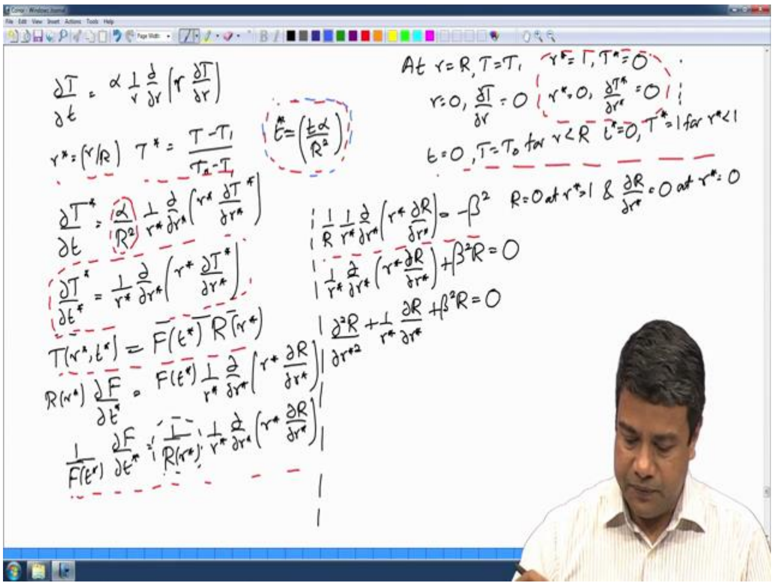

Basically, we scale the radius by the radius of the cylinder the temperature is a scale temperature which is equal to either 0 or 1. It is 0 at the boundaries that T is equal to T 1 and at initial time it is one everywhere within the cylindrical object. And we had got a scaled time as well, and then we had used separation of variables the solutions turned out to be in the form of Bessel functions rather than sine and cosine functions in a Cartesian co-ordinate system.

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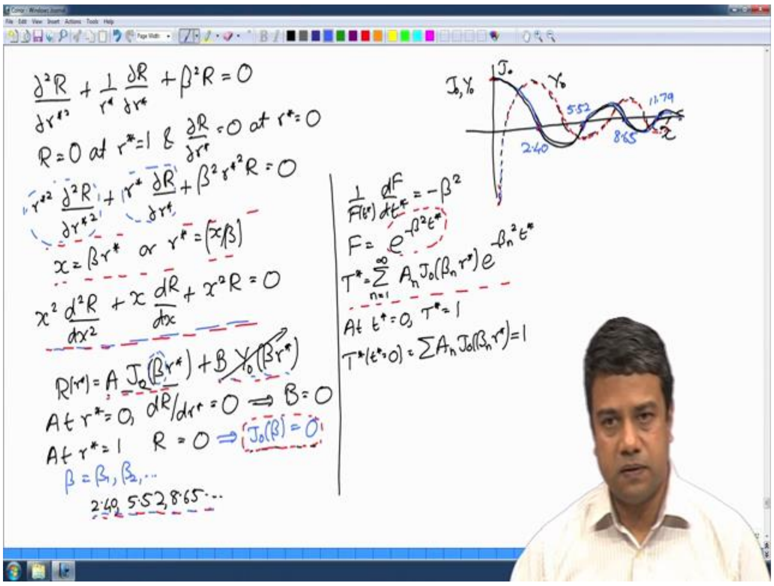

In this case the solutions turned out to be in the form of Bessel functions, the fact that it had to have 0 slope at the center meant that the one of the constants in the expression. You have two Bessel functions: j naught and y naught which was natural solutions for this conservation equation in the radial co-ordinate. The constants multiplying one of those has to be had to be 0 because that one goes to minus infinity as r goes to 0 whereas we require the temperature to be finite or the slope of the temperature to be equal to 0 at the origin.

So, that gave us one constant. The eigenvalues emerged from the requirement that the temperature has to be equal to 0 at r star is equal to 1; homogeneous boundary condition at r star is equal to 1 the temperature has to be 0. And that gave us the value of this eigenvalue beta it had to have discrete values. And this Bessel function solutions, so these were the discrete values that I have written for you.

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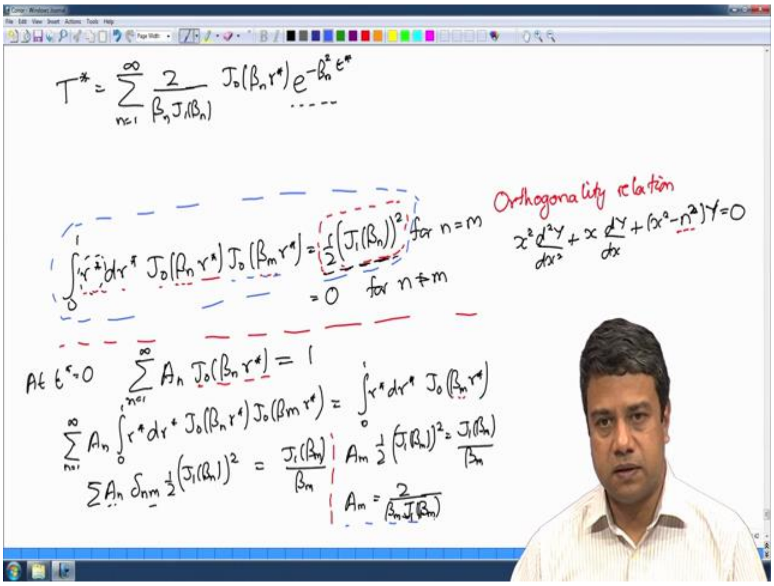

These Bessel function solution has it is own orthergnolity relations, and using that I can calculate all the constants in the solution.

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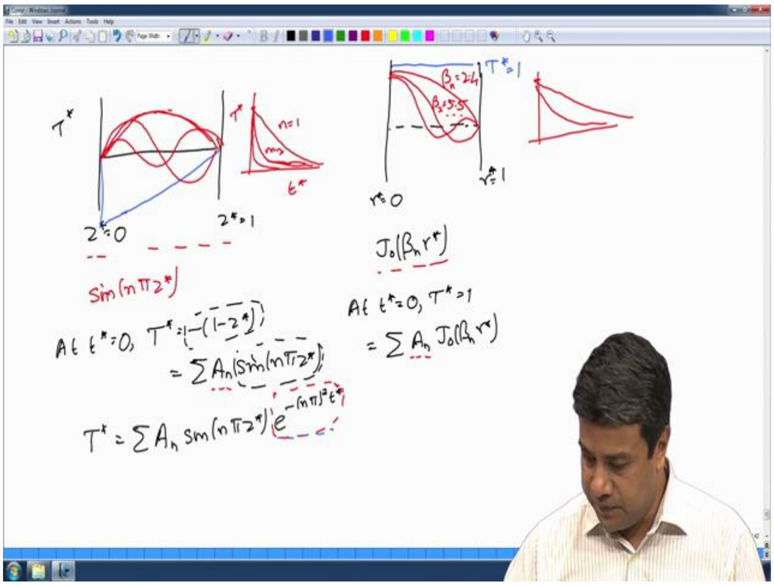

And physically I told you that as in both cases whether it is in Cartesian co-ordinate system or cylindrical co-ordinate system like you are expressing the solution as the sum of a series of basis functions each of which is orthogonal. In the theory of linear algebra you are guaranteed that any function in this space can be written as the sum of these basis functions; these natural basis functions for the differential operator that we have. And the coefficients which multiply these basis functions are evaluated from the orthogonality relations.

Once again this is an orthogonal function space, therefore each function is perpendicular to each to every other function with the orthogonality defined either for the Cartesian or the circle co-ordinate system, here different definitions. But with the definition of orthogonality you can find all of the coefficients. So basically, the initial perturbation you are writing as sum of basis functions coefficients you determine from the orthogonality relations. So, once you have excited all of those initial perturbations each of those basis functions each one decays at it is own rate.

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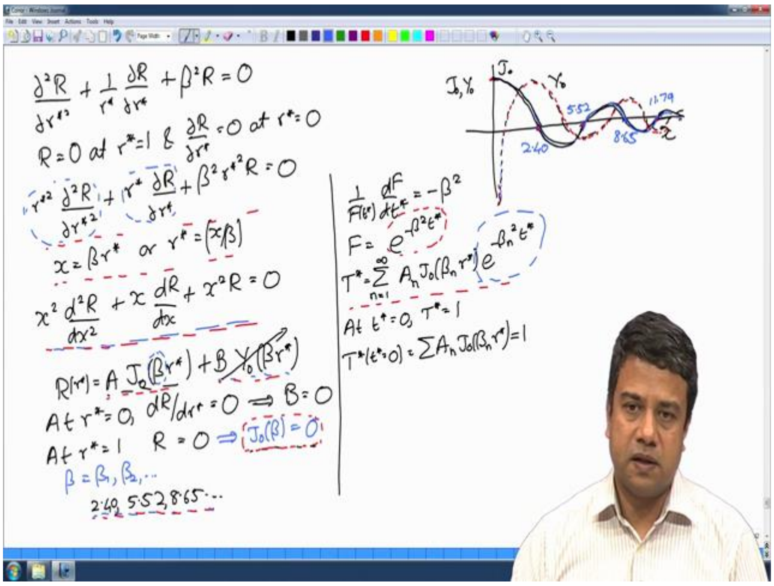

So, the basis functions in the cylindrical co-ordinate system each one will decay at it is own rate. So, you just sum up all of the basis functions times their rate of decay you add it up and you get the temperature at any intermediate point in time.

So, that is the physical understanding of the method of separation of variables. Before we proceed to looking at momentum transfer I will give you one different example of heat transfer in cylindrical co-ordinates which will illustrate some differences from the problem that I have just solved for you.

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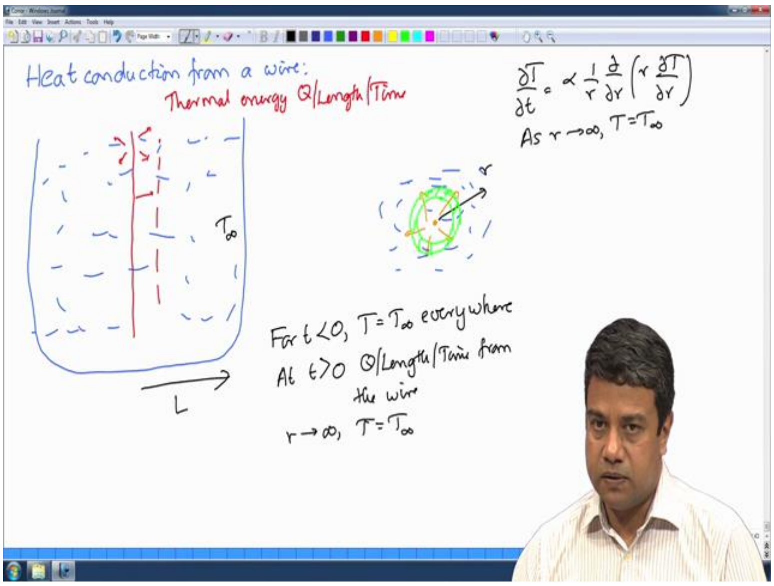

And that is heat conduction from a wire. So, the idea is that you have some fluid in which there is a wire which is heated and it is generating heat in all directions, in this quiescent fluid. For this problem we will consider the fluid to be of infinite dimension. Of course, no real fluid is in finite dimension it is always in some kind of a container.

So, long as the penetration depth of this heat within the fluid is small compared to the total width of the container, I can assume that the heat conduction is in to an infinite fluid. And this wire is generating thermal energy of Q per unit length per unit time. So, given a total amount of thermal energy that is generated per unit length per unit time of this wire.

This wire is considered of infinitesimal thickness. If I look from above what I will see is a wire of infinitesimal thickness which is generating heat in all directions within the fluid. So, the problem is that initially for t less than 0 the temperature is equal to T infinity everywhere in the fluid. And exactly at t is equal to 0 right the total heat generated is Q per unit length per unit time from the wire. And if you go a long distance away; if r is a radial co-ordinate or let me say that this is generating a constant amount of heat so for all t greater than 0 it is generating this heat. So, I switched on the wire at time t is equal to 0; however, for all time in the limit as r goes to infinity T is just equal to the ambient temperature. But the ambient temperature in this case is T infinity. So, that is the problem that we have to solve to find out what is the temperature profile in this case.

So, I said for t greater than 0 the amount of heat generated is Q per unit length per unit time from the wire. So, this is an idealization of for example, an immersion heater. So, if you have wire in a fluid and it generates heat, if the thickness of the wire is much smaller than the characteristic dimension of the container you can consider the wire to be of infinitesimal thickness; and that is generating some amount of heat per unit length per unit time.

So, the mass of the energy conservation equation in this case I can do the same balance for any differential volume around this wire. This is spherical; I am sorry cylindrically symmetric configuration it does not depend upon the location it only depends upon the distance from the center, it does not depend upon the angular location. So, the energy balance equation that I will get will be the same; partial T by partial t is equal to alpha 1 by r t by dr of r dT by dr. So, that is going to be the energy balance equation the boundary conditions as r goes to infinity right the temperature is equal to T infinity. Now what is the boundary condition at r is equal to 0. It turns out that it was not as trivial in this case, so let me take a minute to explain that.

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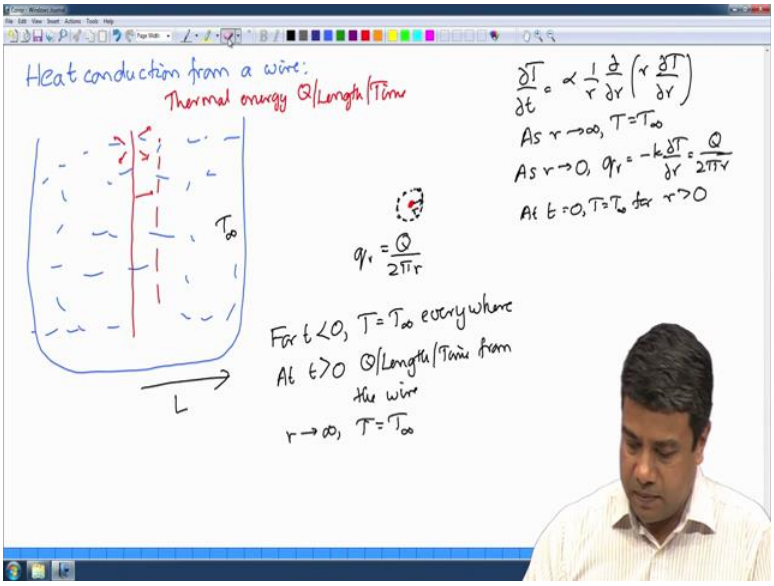

So if I have a wire here, it is generating a certain amount of energy per unit length perpendicular to the plane per unit time. So, what is going to be the heat flux at any distance r? At any distance r where r is small what is going to be the heat flux. The heat flux is obviously q r has got to be equal to the energy generated per unit length per unit time divided by the circumference; here the circumference of this circle tells me the energy generated per unit length per unit time divided by the circumference of the circle.

So, q r has to go as q by 2 pi r in this case in the limit as r goes to 0. So, as you get closer and closer the amount of heat generated per unit length per unit time is a constant, but the circumference the perimeter of the circle reduces down to 0. So, the heat flux actually goes to infinity in this case. It might seem strange to have a heat flux that is tending to infinity, but as I will show you a little later this is quite a natural boundary condition for this particular problem.

So, the other boundary condition is that as r goes to 0 q r which is equal to minus k dT by dr; that has to be equal to q by 2 pi r. Note that we cannot apply it exactly at r is equal to 0, because at r is equal to 0 this thing is infinite; but as r becomes smaller and smaller we require that the heat flux should go as q by 2 pi r. And then you also have the boundary condition that at t is equal to 0 t is equal to T infinity for r r greater than 0. So, those are the equations and the boundary conditions for this problem.

How do we solve it? As in the case of conduction from a surface that we had seen in this case as well if the fluid is considered to be of infinite extent; that is if you neglect the presence of this boundary. This as I said is valid when the extent of heat conduction is much smaller than the radius of the container in that case there are no length scales in the problem there are no time scales in the problem. So, I can non dimensionalize length and time in only one way and that is to write r star is equal to r by root alpha t.

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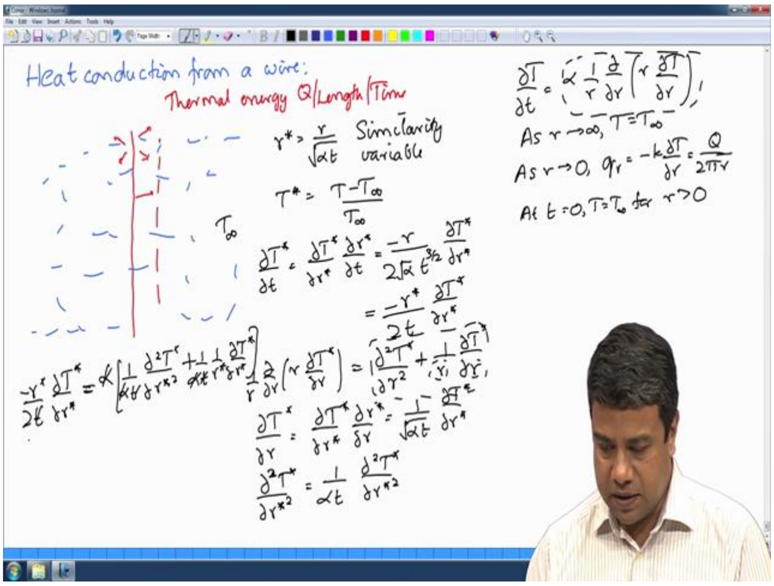

So, that is my equation for the similarity variable. You recall that we had used that in the case of conduction from a surface; in this case we are using it for conduction from a wire in a cylindrical co-ordinate system.

How do we scale the temperature? We would like the temperature to be expressed the difference between the temperature and the temperature far from the surface. So, I would write T star is equal to T minus T infinity by T infinity that would be the natural way to scale the temperature. In this case note that we have not specified a temperature on the wire, but we have specified instead is the flux as you approach the wire. So, expressed in this form the equation I have to once again do it by differentiating using chain rule. So, partial T star by partial t is equal to p divided similarity variable now is r star. So, this becomes equal to minus r by 2 root alpha t power 3 by 2; so let me write that a little bigger.

And this once again can be expressed in terms of r star, so this is just minus r star by 2 t. And then I have the term on the right; I have 1 over r, I am sorry I should express it in terms of r. You can see that dT by dr is equal to 1 by root of alpha t; and similarly the second derivative. So now, if I put these two together what I get is that in this equation in the left side I have minus r by 2 t partial T by partial r is equal to alpha into, I have two terms one is d square t by dr square and the other is 1 over r dT by dr. So, the first term will basically give me 1 over alpha t d square t by dr star square.

The second term is 1 over r dT by dr that will once again give me a factor of 1 over alpha t, because this r has a factor of root alpha t and the derivative as well. I get 1 over alpha T 1 over r star dT by dr. So, that is the equation that I get. And once again these alphas will cancel out in both places; the 1 over t cancels out. And I get an equation in terms of r star alone. And let me write that once again over here.

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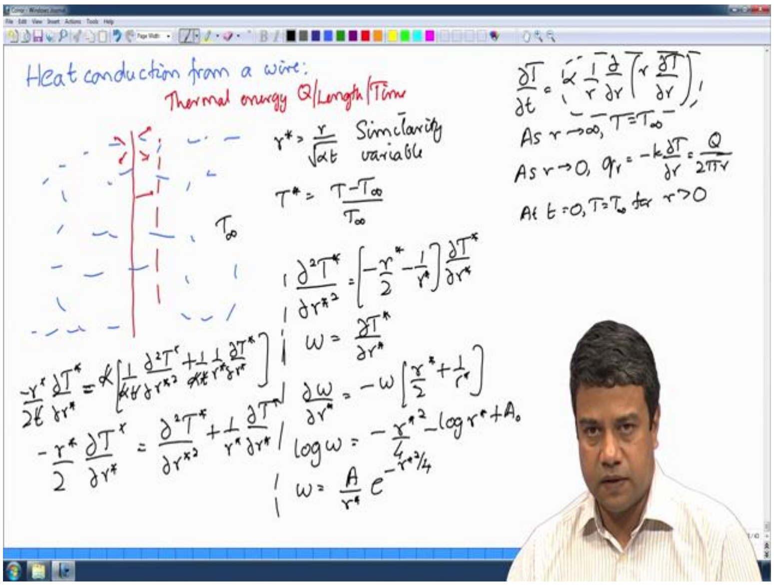

So, after doing the chain rule differentiation the final equation that I get is of the form minus r by 2 partial T by partial r is equal to d square T by dr square plus 1 over r dT by dr. So, that is what I get in a cylindrical co-ordinate system or I can write d square T by dr square is equal to; so that is the conduction equation.

And once again you can solve it by writing the variable w is equal to partial T by partial r which means that t w by dr is equal to minus w into r by 2 plus 1 over r. Integrate once I will get log of w is equal to minus r square by 4 minus log r plus some constant k naught; which means that w is equal to A by r e power minus r square by 4. So, this w has a functional dependence which goes as 1 over r.

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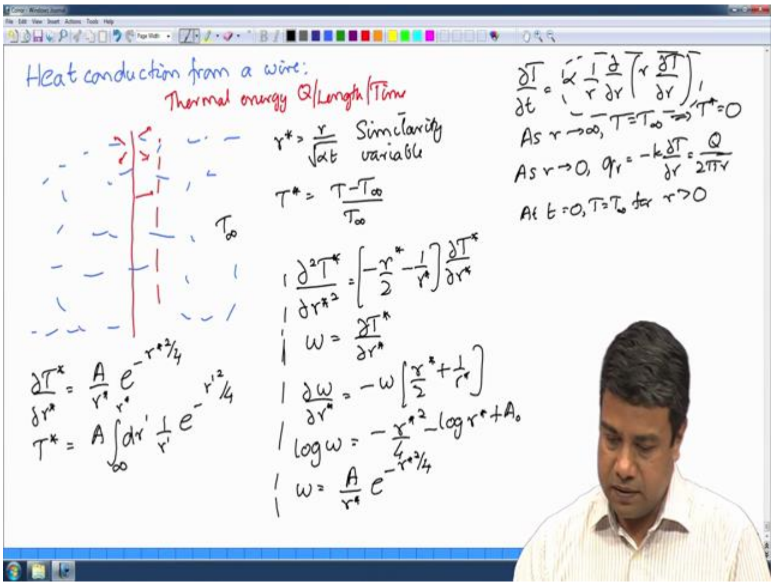

Now this w is dT by dr, which means that dT by dr is equal to a by r e power minus r square by 4. And T star will be equal to integral of A times t r star 1 by r star e power minus r square by 4. I should actually use a dummy variable of integration here, so I will just use r prime instead of r star. Therefore, of course in order to integrate a function the upper limit of integration has to be the value of r itself, the upper limit has to be r star what should be the lower limit. And that is something that we have to be careful about, because I have 1 over r prime and as r prime goes to 0 this integrand goes to infinity.

In this case we know that t star has to be equal to 0, as r goes to infinity t star was defined as T minus T infinity by T infinity which implies that t star has to be equal to 0 as r goes to infinity which means I could as well use the lower limit of integration as infinity. So, you know that if the lower and upper limits of integration are the same then the integral is 0. If I choose the lower limit of integration as infinity then this integral becomes 0 in the limit as r star goes to infinity. And therefore I recovered this boundary condition that t star is equal to 0 as r star goes to infinity.

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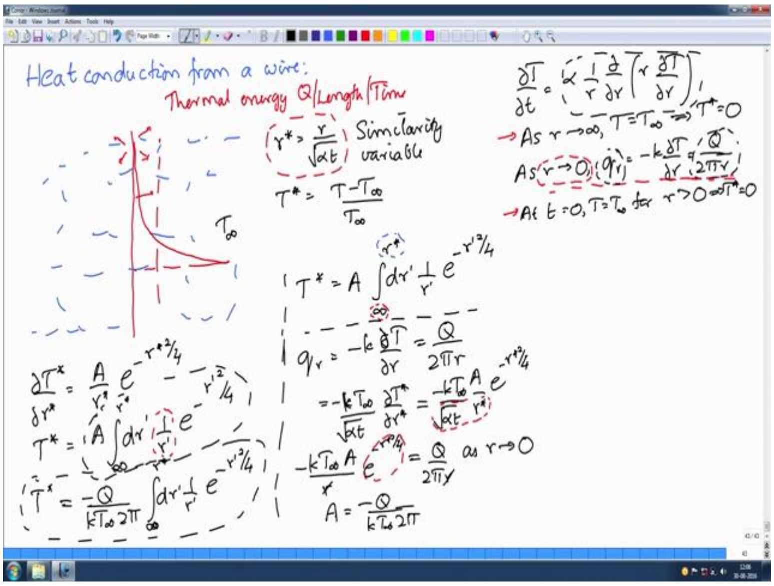

How do I find the constant A? We know that T star is equal to A integral infinity to r star dr prime 1 over r prime e power minus r prime square by 4. Obviously, the value of A has to be determined from this heat flux condition because that is the only condition that we are not used so far. We know that when r star goes to infinity at either r going to infinity or at t going to 0 and in both cases as r goes to infinity and t goes to 0 t is equal to T infinity which means that t star equals 0. So, that boundary condition has been applied by choosing this lower limit of integration.

How about the other condition? The condition is that q r which is equal to minus k partial T by partial r. This has to be equal to q by 2 pi r. This was in the limit as r goes to 0. So, this q r can be written as minus k times T infinity by root of alpha t partial T star by partial r star; just using the scaled variables. And this partial T star by partial r star is just the derivative of this function which is the value of the integrand at the upper limit. The derivative of an integral is just the value of the integrand at the limit of integration. So, it is just going to be the value of the integrand at the upper limit. So, this is equal to minus k T infinity by root of alpha t into 1 over r star; this is function this factor if the constant is there A by r star right times e power minus r star square by 4.

Now, r star times root alpha t is equal to just r, so this product because r star is equal to r by root alpha t this product is just equal to r. And so this becomes equal to minus k T infinity A by r e power minus r star square by 4. This has to be equal to q by 2 pi r as r goes to infinity. So, this now gives you the value of k because as r goes to infinity this thing is just equal to 1; I am sorry at please correct it I made a mistake, this condition has to be applied as r goes to 0 here. So, as r goes to 0; in the limit as r goes to 0 this thing e power minus r square by 4 is just 1. And r cancels out on both sides, and I get a finite value for the constant A it was equal to minus q by k T infinity times 2 pi. So, this gives me the solution for the temperature field is equal to minus q by k T infinity 2 pi integral from infinity to r star the dr prime 1 over r prime e power minus r prime square by 4.

So, this is the similarity solution in this particular case, it might seem a little unusual for you. In the previous case we had said that the temperature derivative has got to be equal to 0 at r is equal to 0 because there is no physical boundary there, so leave symmetry boundary. In this particular case there is a wire at the center which is generating a certain amount of heat. So, there is a physical boundary there and for that reason the flux actually goes to infinity as r goes to 0; for the similarity solution. We have to satisfy a condition that the flux is proportional to 1 over r as r goes to 0.

However, when we solve the equation we did get a solution in which the flux did go to infinity as r it goes to 0; however, the solution in this case is well defined. At r is equal to 0 itself the temperature has diverges logarithmically; however the temperatures were well defined for all r greater than 0. Of course, an infinitesimal wire is an idealization all, wires have a finite thickness and therefore this divergence in the temperature as you reach the origin if you plot the temperature as the function of distance it starts from 0 and will start to diverge, that divergence is logarithmic because you can see that I have an integral of 1 over r that goes as log r, and therefore it increases logarithmically as you approach the center. That logarithmic increase will get cut off at some finite thickness corresponding to the thickness of the wire.

Despite having infinity in a boundary condition we were able to successfully solve this equation. So, when there is a physical boundary the temperature does not have to the slope does not have to be 0, in this particular case the temperature was tending to infinity as r goes to 0, the flux was also tending to infinity; however the total heat generated was finite in this case. And therefore, we were able to get a similarity solution where the temperature diverges, the flux diverges, the heat generated is finite and we did get a solution for the temperature field in this case.

So, you could have situations where the temperature is diverging the equation predicts that, but still the boundary conditions are well posed and we were able to solve those to get all of the solutions. Such things happened in cylindrical co-ordinates because the surface area goes to 0 as the radius goes to 0. However, the amount of heat generated is finite and therefore the flux has to go to infinity in this case. So, this is a similarity solution in cylindrical co-ordinates even though the temperature is diverging we still managed to get a solution. Similar things will happen if you had for example a wire which was reacting, so that you had a finite mass flux at that surface or if you had a wire which was being pulled at the center of a tube; in those cases you will get solutions that look like this.

So, I have shown you both separation of variables and similarity solutions in cylindrical co-ordinate system. Next class we will go on to momentum transfer. In momentum transfer one of the things that we do not have in mass and heat transfer is the transport of momentum due to pressure gradients, due to the hydrostatic pressure. That we will see in the next lecture, we will solve for the flow in a pipe where we include the pressure difference along with the convective and the diffusive transport of momentum and obtain balance equations for momentum transfer. So, that we will continue the pipe flow in the next lecture, I will see you then.