Transport Processes I: Heat and Mass Transfer
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Lecture – 33
Unidirectional transport: Balance laws in cylindrical co-ordinates. Unsteady heat conduction from a cylinder cont

Welcome to this continuing series of lectures on Fundamentals of Transport Processes, where we were looking at heat conduction from a cylinder. Previously we had considered heat conduction from between two flat surfaces and we had seen how to use similarity solutions as well as separation of variable solutions. And then we had progressed to a cylindrical geometry, where it is more convenient to have the surfaces if you solving a problem in a pipe for example; have the surface as a surface of constant co-ordinate.

Therefore, in this case we had defined one of the co-ordinates as the distance from the axis of the cylinder.

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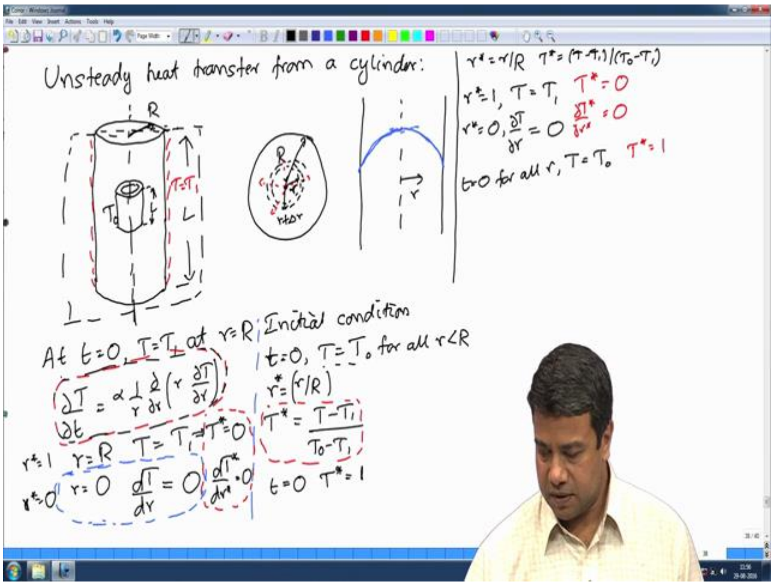

And we had derived differential equations for that; the distance from the axis of the cylinder was the co-ordinate, the second was the distance along the axis that we had assumed that there is no variation in that second direction and we had got an equation for the temperature field.

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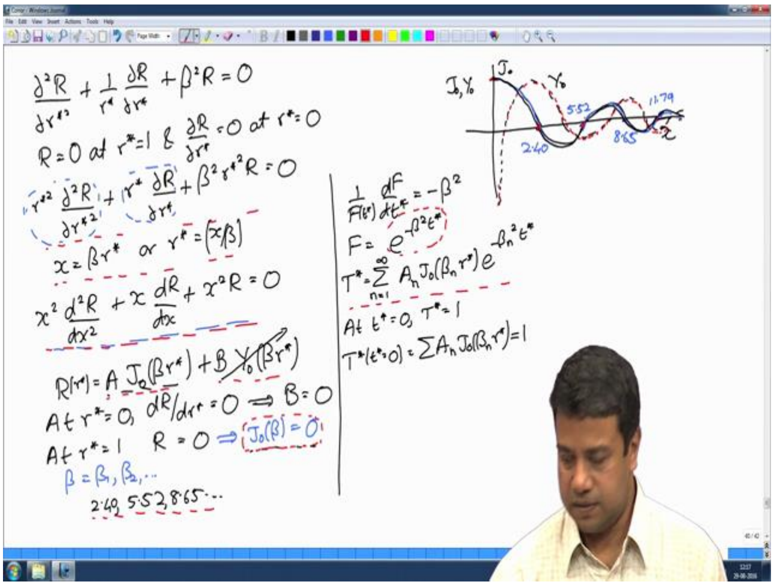

And we were solving it for the case where we have.

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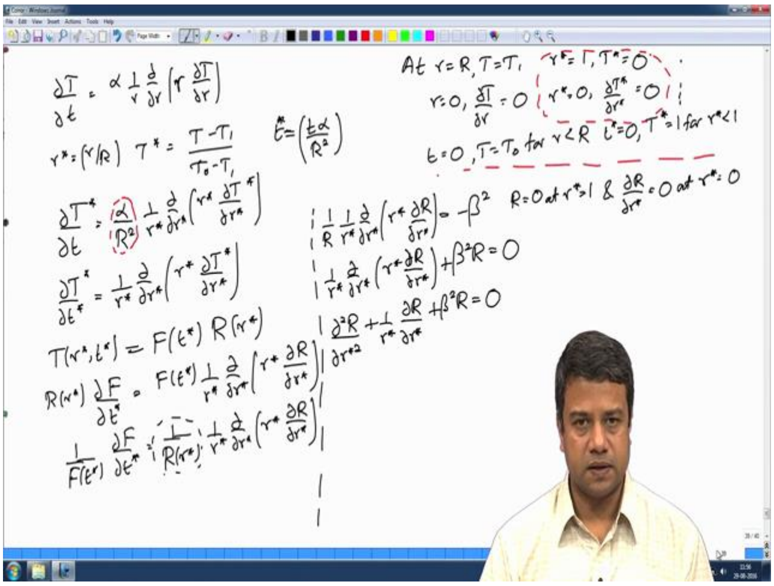

A cylinder in which the temperature at.

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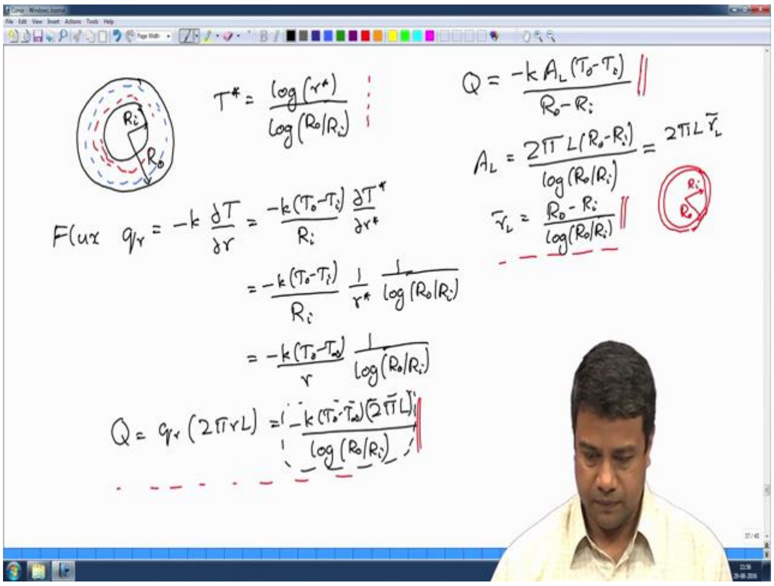

In which the temperature of the cylinder is T 0 initially and it is brought into contact with a fluid which is at temperature T 1. So, that at time T is equal to 0 temperature is T 1 on the surface of the cylinder. And on that basis we are trying to find out how the temperature equilibrates to 0 to T 1, because in the long time limit you would expect that the temperature of the cylinder is equal to the temperature of the surroundings; if T 0 is greater than T 1 heat will be conducted out until at very long times the temperature is the same between the inside and the outside.

So, how does this attain that steady temperature and that is what we are trying to solve. So, we had defined scaled co-ordinates T star is equal to T minus T 1 by T 0 minus T 1 we had defined it in such a way that T star is equal to 0 in the long time limit as T going to infinity T star is equal to 0, so that we only find out what is the correction to that due to the initial forcing. We scaled the radial co-ordinate by R. And on that basis we had derived an equation for the temperature field. The time in the scale was scaled by the diffusion time, the time it takes for the thermal energy to diffuse over a distance comparable to R. So, that was the thermal diffusion time, the time scale was scaled by the thermal diffusion time.

Once that happened we got an equation which did not depend on either capital R or on the thermal diffusion coefficient, but it was a universal equation, because we scaled R by capital R. And the time by the thermal diffusion time over a distance r. We adduced separation of variables where we had expressed the temperature field as a function of time times a function of r, and we had got an equation in which the left side depends only upon T, the right side depends only upon r and therefore both of these have to be equal to constants. In this case just by physical addition we can say that this constant has to be a negative constant, because only if it is a negative constant will the function F of t decrease to 0 in the long time limit exponentially. If it were positive the function F of t would increase exponentially. And it would not satisfy the condition that the temperature has to go to 0 in the long time limit.

So therefore, we had set this constant as equal to a negative number and we had got a differential equation which we had solved. So, this was the form of the differential equation and I had done a change in variables I have written x is equal to beta times r star. Since the first two equations in this, are not have net 0 dimension in r they do not change even if you scale r and we got an equation of this kind. This as I told you is a Bessel equation; a Bessel equation of 0th order. And the solutions are Bessel functions; Bessel functions of 0th order. These are not analytically solvable; however there are numerical solutions that exist. And these have been well studied their properties are all known and you can find tables which tell you what the values are at specific values of x.

Out of these two functions I have plotted those for you here J naught when something like this that is schematic and y naught on the other hand went something like this; if J naught was equal to 1 at r is equal to 0, whereas y naught goes to minus infinity; we require that the temperature has 0 slope at r is equal to 0. As I told you because there is no physical boundary there it is only a symmetry condition. And therefore, y naught does not satisfy this boundary and therefore the constant b has to be equal to 0. That is one boundary condition at r is equal to 0.

The second boundary condition is that at r is equal to 1 capital R is equal to 0; that means, that J naught of beta has to be equal to 0. And what that implies is that beta has to be one of these locations where this curve passes through 0, has to be one of these locations where this curve passes through 0. And there is a discrete set of values for that. Therefore, the general solution in time of course the solution is in the form e power minus beta square t because it is an exponential in time, therefore the general solution has this form some n is equal to 1 to infinity of An some constant times J naught of beta and r times e power minus beta n square t.

At time t is equal to 0 we have the initial condition that t star is equal to 1, because the temperature was t naught throughout the domain. Therefore, the scaled temperature t minus T 1 by t naught minus T 1 was equal to 1 at the initial condition. So, this implies that t star is equal to at t is equal to 0 which is equal to sigma An J naught of beta n r this has to be equal to 1.

So, that is the initial condition. Now how do we solve this problem for the initial condition? In the case of Bessel functions also we have orthogonality relations.

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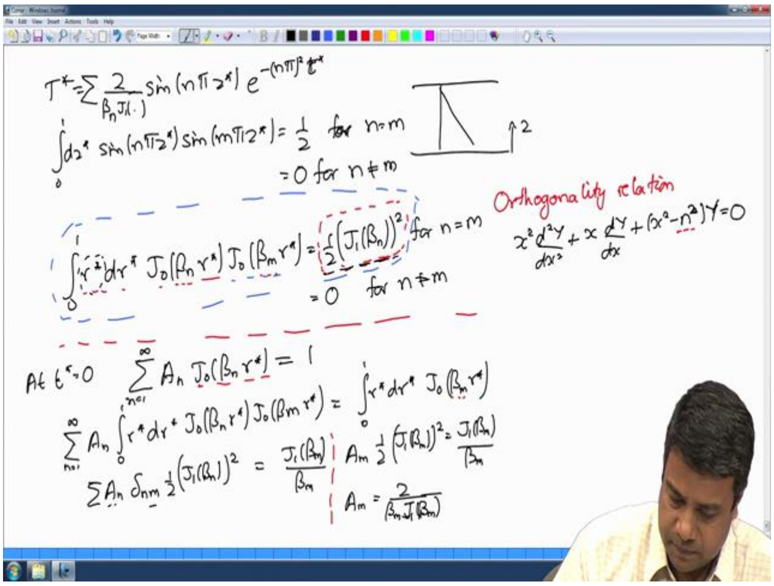

If you recall when we did sin and cosine functions or function the temperature field in that case was equal to summation of An sin n pi z star e power minus n pi the whole square t star. So, that was the scale temperature field for the heat transfer across a finite domain; if you recall. And there we had the orthogonality relation 0 to 1 d z star is equal to half for n is equal to m is equal to 0 for n is not equal to m. So, that was the orthogonal relation which enabled us to solve this equation.

Similar orthogonality relations also apply for Bessel functions. In fact, for any orthogonal eigenfunctions that you get from a separation of variables procedure there is an orthogonality relation. If these functions are orthogonal you can show that these functions are orthogonal to each other. In this particular case for Bessel function the orthogonality relation is of the form integral 0 to 1 r star dr star of is equal to 0 for n not equal to m. So, this integral the integrand here is r dr because as I told you surface area cylindrical co-ordinate system goes as r dr. Present Cartesian co-ordinate is just equal to delta z.

Since surface area goes as r dr it turns out that the integral has to be done over r times dr times J naught of beta m r and J naught of beta m r this is nonzero only when n is equal to m. And when n is equal to m it has a particular value at a particular constant value, when n is not equal to m it is identically equal to 0. So, this is the orthogonality relation for Bessel functions. Now, the initial condition that we had to solve at t star is equal to 0 summation An j naught of beta n r star; this has to be equal to 1. How do I determine these constants, by using the orthogonality relation which is this relation here?

So, what I do is I multiplied both sides by J naught of beta m times r and integrate. If I multiplied both sides as by J naught of beta n times r and then I will multiply it by 1 factor of r as well. So, I will get summation is equal to integral this is from 0 to 1; multiply both sides by J naught of beta m r times r dr and integrate from 0 to 1. We know what the value on the left side will be, the value on the left side comes out of this orthogonality relation. So, I will get summation of An write, this is nonzero only when n is equal to m so I can multiply that by this delta n m this delta n m is equal to 0 for n not equal to m and is equal to 1 and n is equal to m; times this constant here this constant on the right side half J 1 of beta n the whole square.

On the right side once again one can do the integral, and if you do this integral I would not go through the details, but these are available in standard tables of Bessel functions. The integral of the Bessel function from 0 to 1; the result you will get is J 1 of beta n by beta. Where I should note that this J 1 is a Bessel function of first order, the solutions that I had these J naughts for Bessel functions of 0th order. So, to recall the equation for the Bessel function is of the form x square d square y by d x square plus x d y by d x plus x square minus n square y is equal to 0 that was the original equation for the Bessel function. When n is equal to 0 the solution for that equation is this Bessel function J 0. The J 1 that I have here is the solution with n is equal to 1; that is another solution for the Bessel function with the index n equal to 1. So, the integrals of J 0 can be written in terms of the solutions for J 1 that is the idea.

So, this is what I get from the orthogonality relation and this effectively delta m n I should note on the right side it is all m's; so on the right side I have beta m here because I multiplied it by beta m on the side. So, I have beta m on the right side. So, this An times delta m n is nonzero only when n is equal to m. So therefore, the left side I get only one term that is Am times half beta m square. So, this effectively gives me Am times 1 by 2 J 1 of beta m the whole square is equal to J 1 of beta m by, and this gives me a value for m is equal to 2 by. So, this is how the orthogonality relation is used in order to get the solution for the equation.

Therefore, in a original equation if I substitute the I finally get the solution for T 2 by it times this I am sorry should put m there. So, that is my final solution for the temperature field; this is n I am sorry. So, this is my final solution for the temperature field.

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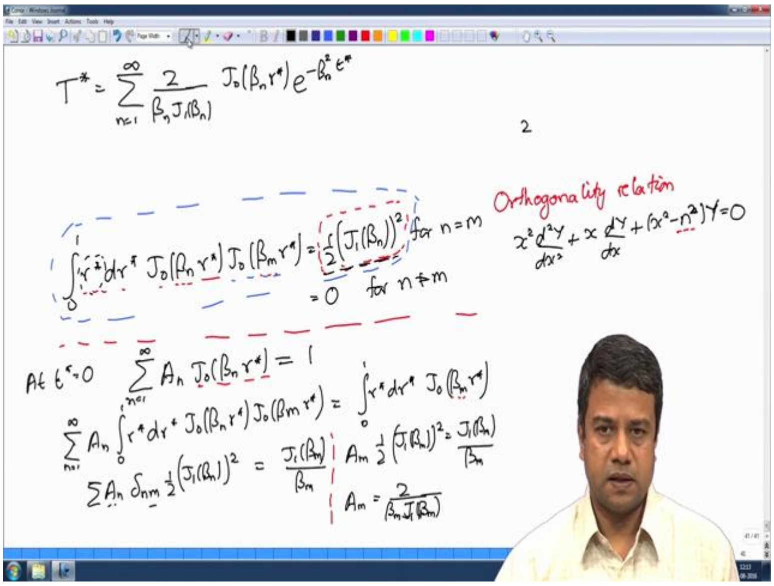

You can see that as n increases as n increases; maybe write that solution once again for you; so that is the final solution for the temperature filed for the cylindrical. As I told you this beta n’s they increase as n increases. We saw that in the previous case the locations where these beta n’s go to 0; 2.4, 5.5, 8.6, 11.7 and so on. The spacing increases, and therefore beta n square increases the square of that; that means, that these higher order terms in this expansion they decrease much more rapidly with time; they decrease much more rapidly with time. And therefore, if you are interested in the solution only to certain accuracy you can neglect the terms that increase beyond that exponential term. So, this shares common features with the solutions that we had for a Cartesian co-ordinate system.

The procedure is the same, we are expressing this solution for these partial differential equations as the product of two solutions: one of which is only a function of time, the others only a function of space. In this case the function of space in the case of a Cartesian co-ordinate system is a sin function; that is a natural basis in that case, cylindrical co-ordinate system their Bessel functions. These are the natural co-ordinates in this case.

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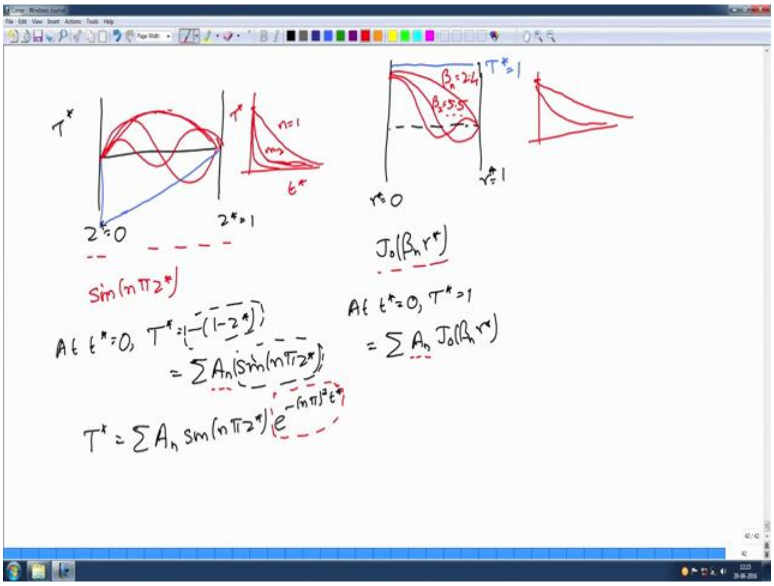

So, if I want to compare the solutions that I had in the Cartesian co-ordinate system between; there I had z stars equal to 0 z star is equal to 1 and I had the temperature field t star. And I had an initial temperature field which is basically equal to minus of 1 minus z star if you recall, so it looks something like this. And I was writing that as the sum of a series of functions, though series of functions was sin of n pi z. So, for m is equal to 0 it looks something like that for n is equal to 1 it looks something, sorry n is equal to 1 n is equal 2 3 and so on I was writing this as a series of functions. And I knew that at time t is equal to 0 t star was equal to minus of 1 minus z, and this was equal to summation of An sin n pi z; if you recall.

Basically, this function here I was expanding in terms of these basis functions and finding out what these constants were. And the way I found out those constants was using the orthogonality relations. Similarly, the procedure is similar in the radial co-ordinate system. In this case, so this is r is equal to 1 r star is equal to 0. In this case my functions are J naught of beta and r star. If you recall the form of the Bessel function was of this form, if the Bessel function looks something like this. And these were the locations where it was passing through 0. So, each of these functions beta is equal to 2 is 1 basis function 5.52 is one basis function and so on higher and higher basis functions.

Basically, my basis functions will look something like this. So, beta is equal to 2 it looks something like this, the next one will look something like this, the next one will look something like this. They are basically scaled functions; scaling of these functions if I put r star is equal to 1 here I will get the first one; I will put r star 1 there I will get the next one and so on.

So, these are now the basis functions. At t star is equal to 0 I had t star is equal to 1 and I was expressing this as a summation of An times J naught of beta n r star. So, this t star is equal to 1, this initial condition that t star is equal to 1 a constant. I was expressing it as a sum of these functions. And these constants were then determined from the orthogonality relations. The orthogonality relations for Bessel functions have a different form than the orthogonality relations for Cartesian co-ordinate systems, but the principle is the same; in both cases you get solutions of these kinds.

So, I hope I have given you some insight into how the separation of variables procedure is done. In this particular case we had two co-ordinates: one time and one space. In the spatial direction we reduce the problem to a homogenous problem; it had either boundary condition for the temperature equals 0 or the derivative of the temperature equals 0. In the first case where we did this problem we are actually solved it, we had actually reduced it to a transient problem where the boundary conditions on both boundaries are homogeneous and the forcing is at initial time. In the homogeneous boundary condition direction you will get discrete eigenvalues and a set of orthogonal eigenfunctions.

So, that the initial condition can be expressed as a sum of these orthogonal eigenfunctions in that summation the constants are determined from the separation of variables procedure. That procedure is the same, whether it is in a Cartesian co-ordinate system or a cylindrical co-ordinate system. The operators change, the differential operators change in the case of a Cartesian co-ordinate system we just had a second derivative in the z direction. Cylindrical co-ordinate system little more complicated. The operators change the eigenfunctions and eigenvalues change. Cartesian co-ordinate system it was just n pi, whereas in this case it is equal to beta n locations where the Bessel function goes through 0.

So, the eigenfunctions change, but in both cases you do have orthogonal eigenfunctions and those orthogonal eigenfunctions can be used to solve for the constants in the equation. These constants decay at different rates. As I said that the first one, the rate of decay so that the temporal part of this; if you look at the temporal part of the decay for n is equal to 1 it decays as minus pi square t square, n is equal to 2 it decays as minus 4 pi square t square. So, for n is equal to 2 it decays much faster than n is equal to 1. So, if I plot the decay rates of each of these in time. If I plot the decay rates of the maximum we see each of these in time for n is equal to 1 I will get a relatively slow decay, n is equal to 2 it will decay much faster, n is equal to 3 will decay faster still.

Therefore, the slowest decaying one is the one for is the transient will be there for the longest time, and that is this basis function. N is equal to T 2 decays much faster n is equal to 3 decay cases e power minus 9 pi square, so that decays faster still. And, so if you wait for a long time the dominant perturbation to the temperature dominant transient contribution to the temperature is going to be equal due to n is equal to 1. So, depending upon the accuracy with which I want to calculate the temperature I can neglect these higher functions because they decay much faster, unless we are at times that are very close to t equals 0.

Similar thing holds in this case as well, if you should recall for n is equal to 1 beta n is equal to 2.4. So, this will decay slowly n is equal to 2 beta 2 is equal to 5.5, so this peak is actually much faster. It decays as e power minus beta square t, so it is actually this will decay is roughly e power minus 6 t, this one will decay as e power minus 25 t because it is e power minus beta square. So therefore, the fundamental mode is the one that will decay the slowest and so if I am interested in the long time limit I can retain the fundamental model.

So, this completes our discussion of the separation of variables procedure. I tried to give you a physical insight into how this is actually carried out. This is an important procedure because we will see it again and again as we go through the course; the fact that you can separate out a solution into a set of basis functions then get the constants out. And if you want a numerical solution this series decays exponentially fast as the number as the number of modes increases. Therefore, we can read it truncate the solution at the finite value of eigenfunctions and get a good approximation of the solution.

So, this completes our discussion of separation of variables in two different co-ordinate systems. We will see the separation of variables procedure in a cylindrical I am sorry, in a spherical co-ordinate system a little later. But before we progress our next topic is to look at fluid flows. Everything that I have done here I could as well substitute temperature with concentration and thermal diffusion with mass diffusion and everything remains the same.

In the case of fluid flows there is one other complication that can arise and that is the generation of fluid flow due to pressure gradients. You look at that the flow in a pipe next, but before that I would like to solve one problem in cylindrical co-ordinates using the similarity solution. The heat transfer from an infinitesimal wire, I will do that in the next lecture and then we will progress to looking at flow in a pipe. So, that is the plan for the next few lectures, I will continue this in the next class. I will see you then.