**Transport Process I: Heat and Mass Transfer**
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**Lecture – 32**
**Unidirectional transport: Balance laws in cylindrical co-ordinates. Unsteady heat conduction from a cylinder continued**

This is a continuing series on lectures on transport in one dimension unidirectional transport where we were looking at transport in a curvilinear coordinate system.

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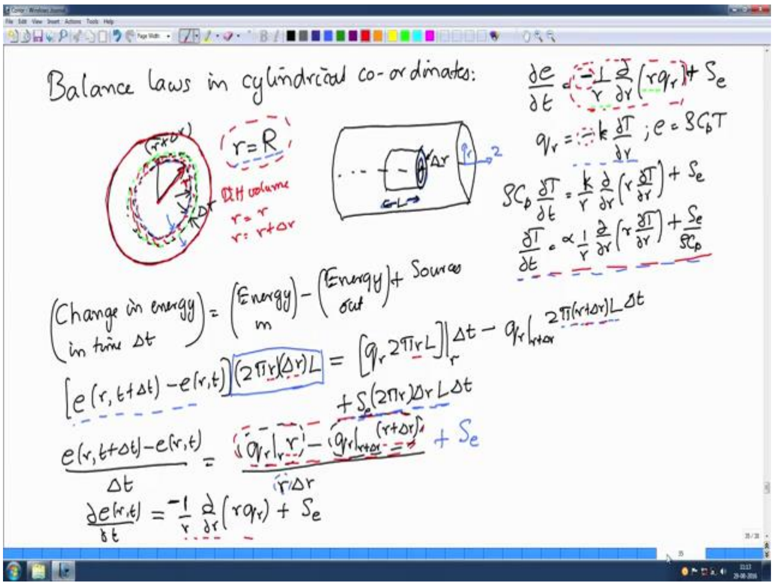


If we recall, we have write down balance laws for a cylindrical coordinate system, this is useful when we are considering transfer in a pipe for example, or an annular region; these are quite often used in practical applications and in these cases rather than defending rather complicated expression for the boundaries in the Cartesian coordinate system. In this particular case, if we take x and y coordinates in the plane; the boundary would be defined by x square plus y square is equal to r square, we could solve the differential equations of course, but then applying boundary conditions on a complicated boundary becomes the problem. It is much preferable to have the boundary as a surface of constant coordinate.

Therefore, we had chosen curvilinear coordinate system; a cylindrical coordinate system where we have two coordinates, one is the radial coordinate which is the distance from the axis of the cylinder and the other is the axial coordinate which is the distance along the axis of the cylinder. In this particular case we have first considered only variations in the radial coordinate and we have derived the balance equations; by doing a shell balance where the rate of change of energy was equal to energy n minus energy out plus any sources. Energies in and out the products of the flux times the surface area, in the Cartesian coordinate system earlier the surface area was a constant, so only the flux was changing. In this cylindrical coordinate system both the flux and the surface area change with position and for that reason the deferential operator which represents the diffusion, in this particular case is straightly more complicated, it is no longer just a second derivative with respect to the special coordinates, it has slightly more complicated structure to it.

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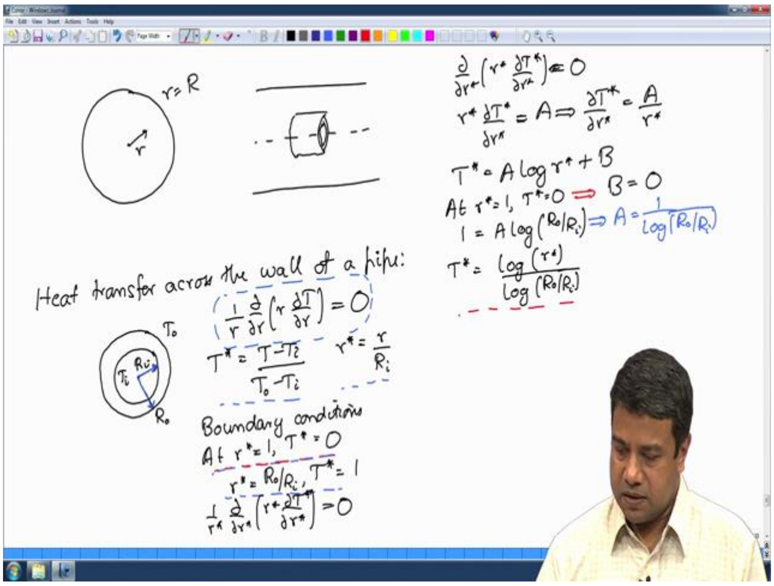


We had solved the heat transfer across the wall of pipe, the study problem and we have got the logarithmic profile for the temperature field which is different from the linear profile you would expect if you had for transport cross flat surfaces.

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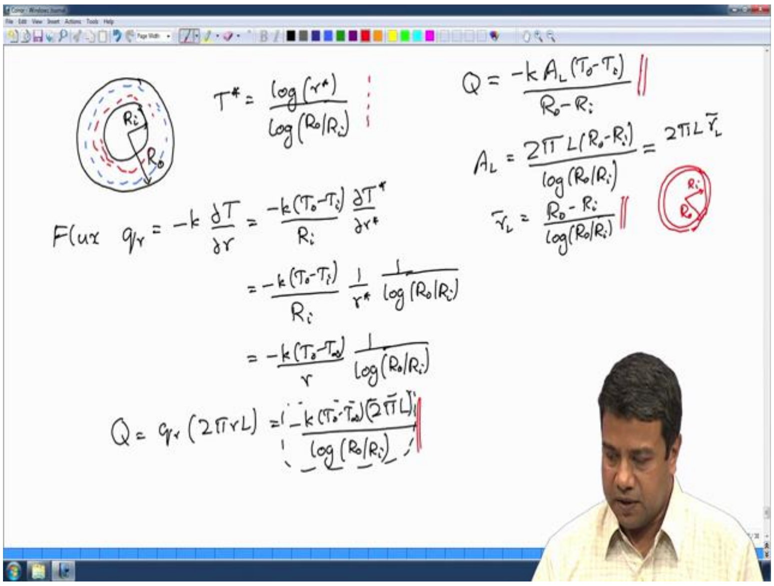

And because of that we had the logarithmic average radius entering into the definition of the area of cross section across which the transport takes place. Strictly speaking the flux should be given just by this, but this coloration it is expressed in terms of the thickness times and average area and that logarithmic average is given by this one.

So, that we are seen in the previous lecture and then we would doing an unsteady problem where initially we had a cylinder of temperature T naught which was immersed in the flow it in temperature T 1 and we wanted to know how the temperature varies with time. As I said this is the problem of particle importance; you very often in industries either you have metal cylinders which we want to cool in water or you have cylinders such as cans of different materials which are immersed water in order to cool them and one would like to know the rate at which the temperature decreases with time or increases to the outside temperature.

So the balance equation if we recall had the same form that I had earlier, the time derivative temperature is equal to thermal diffusivity; times this 1 by r; d by r of r times d by dr, the boundary conditions in this particular case I had only one physical boundary, that physical boundary was at r is equal to capital R; there was only one physical boundary where the temperature was specified; however, the radial coordinate goes from r is equal to 0 to r is equal to capital R.

So, therefore the interval for the radial coordinate r is from 0 to capital R of course, the radial coordinate is the distance. So, it can go any direction we had assumed that there is no variation of the temperature with respect to the angle around the axis, we will come back to cases where we have considered variation, but in this case there is no variation with respect to angle around the axis. So, in this case they interval for r is from 0 to capital R; that means, that there is a boundary at r is equal to 0 imposed by the coordinate system itself because we have taken the origin that r is equal to 0.

So, the interval is from r is equal to 0 to capital R and that r is equal to 0, there is no physical boundary; however, there is symmetric condition as I had explained to you; it is only if the slope of this temperature curve at r is equal to 0 is equal to 0 that the temperature will be continues if you take it from the right side or from the left side; that means, the derivative has to be 0; however, you will have a difference in the derivative from the right to the left side; that means, the derivative cannot be uniquely defined at r is equal to 0. So as long as there is no physical boundary at r is equal to 0, the boundary condition has to come out of symmetry and that symmetric condition as I showed you is equal to that the temperature derivative was equal to 0 at r is equal to 0.

So, we had defined r star is equal to r by R; that was the scaled radius and defined in terms of r star, we find that at r star is equal to 1; t is equal to T 1. The temperature that is imposed on this outer boundary, k t is equal to T 1 and at r star is equal to 0 right dT by dr was is equal to 0. So, that was the temperature I should actually use of partial derivative here this because the temperature or function of both time and position and at T is equal to 0 for all r; the temperature was equal to T naught that is the initial temperature of this cylinder. So, how does the temperature scale I had gone through that in the previous lecture, what is it that you would expect the temperature to be in the limit of long time.

As T goes infinity, the temperature of the cylinder should be uniform; it should be equal to T 1 because that is the surface temperature. So, the long time limit the temperature entire cylinder should be equal to T 1. So, it is preferable to define the departure from that T 1 as the scale temperature, therefore we have defined T star is equal to t minus T 1 by T naught minus T 1.

So, this is equal to 0 in the long time limit at all r because the temperature everywhere would have equilibrated to T 1. So; that means, that r star is equal to 1, the scale temperature T star is equal to 0 and the r star is equal to 0; partial T star by partial r star is equal to 0. At T is equal to 0, for all r; the temperature is equal to T naught which means that T star is equal to 1. If I have to define my scale temperature as T star is equal to T minus T 1 by T naught minus T 1, so those are the equations and the boundary conditions.

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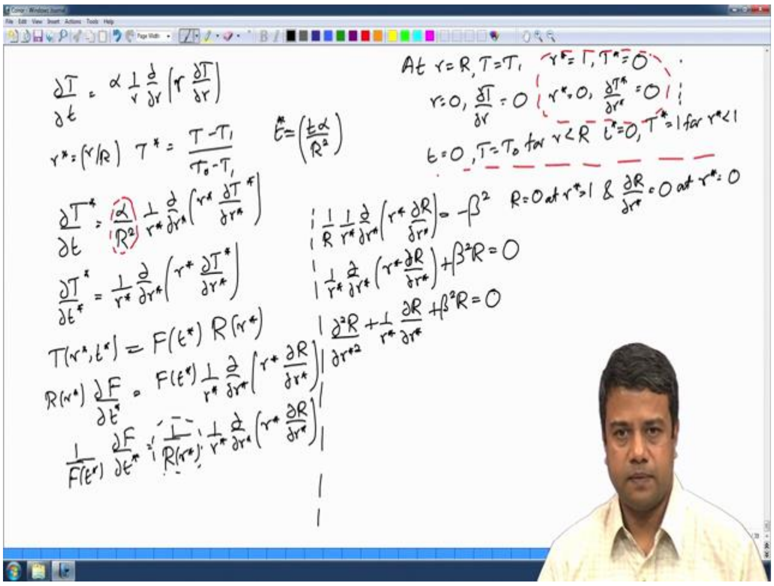

So, my differential equation partial T by partial t is equal to alpha 1 by r. At r is equal to R; t is equal to T 1, r is equal to 0; partial t by partial r is equal to 0 and at t is equal to 0 t is equal to T naught for r less than R; it defined the scaled variables, r star is equal to r by R and T star is equal to T minus T 1 by. So, this became at r star is equal to 1, T star is equal to 0; r star is equal to 0, partial t by partial r is equal to 0. Once I scale it this way, you can easily verified the equation becomes of the form partial T by partial t is equal to alpha by R square, the equation is linear in T and therefore, it is equidimensional therefore, I can just straight away convert t to T star and I have two derivative susceptive r and they if I get 1 over r square, we have not yet scaled with time and this equation makes it clear what at the scale should be because if I scaled by this factor then the equation becomes dimensionless.

So, therefore, I can define at time scale as t is equal to T star is equal to t times alpha by R square. Once again R square by alpha is the diffusion time, the time takes energy to defuse over the distance capital R. So, that is the natural time scale by wise to scale this problem, if I use that naturals times scale; I will get, written in this way this is equation is independent of the temperatures, the thermal defeasibility and so on.

So, it becomes universal equation any problem that I had can be transformed into this just by scaling if called as what the material defeasibility and so on are and the scale this way the initial condition becomes that at T star is equal to 0; T star is equal to 1 for r star less than 1. So, therefore, I have the forcing in the initial conditions and the two boundary conditions set I have here for this problem are both homogeneous; this temperature is equal to 0 or the temperature derivative is equal to 0. So, therefore, this becomes an Eigen value problem which can be solved using separation of variables.

So, back to the separation of variables procedure T of r t can be written as the product; some function of t and some function of r; insert that into the balance equation, what you get is that R of r star times partial F by partial t is equal to F of t star times 1 by r . So, separately write this function as a function of r time function of t insert that include balance equation and then divide throughout times by r times; F divide both side of the equations by r times F. So, you get 1 by F; divide through by R times F. Now in this equation, the left side is only function of time, the right side only function of position.

So; that means, of both of these the left and right sides individually have to be equal to constants if that were not so then I could change the location while keeping the time a constant and this equality will no longer be valid. So, only way that this equality will be valid is if both of this the left and the right are both equal to constant. Should the constant be positive or negative, we have already looked at that when we with heat transfer in finite domain. If the constant is positive, the left side if you solve; it gives you a function that is exponentially increasing in time. So, it never satisfies the condition that as t goes infinity the T star should be equal to 0.

So, therefore, the only way that you will get a solution that is decreasing to 0 as times goes to infinity, is if the left side or the right side are both the equal to negative constants. So, as got to be equal to negative constant; I will call that as beta square where beta is any number, so this has to be equal to a negative constant. With boundary conditions R is equal to 0 at r star is equal to 1 and partial R by partial r star equal to 0 at r star is equal to 0. Those just come out of this boundary condition; these are two homogeneous boundary conditions. So therefore for the radial variation; I have solve this equation, I can expand this out what I will get is d square R by dr star square plus I am sorry I made mistake here kindly note I should have a 1 over R that 1 over R comes with this term here I have 1 over r there and therefore, if I multiplied both the left and right sides by R; what I will get is; so, that is the equation that you will get, expand it out the derivative you will get d square R by d r star square plus 1 over r star, so that is the differential equation that has to be solved.

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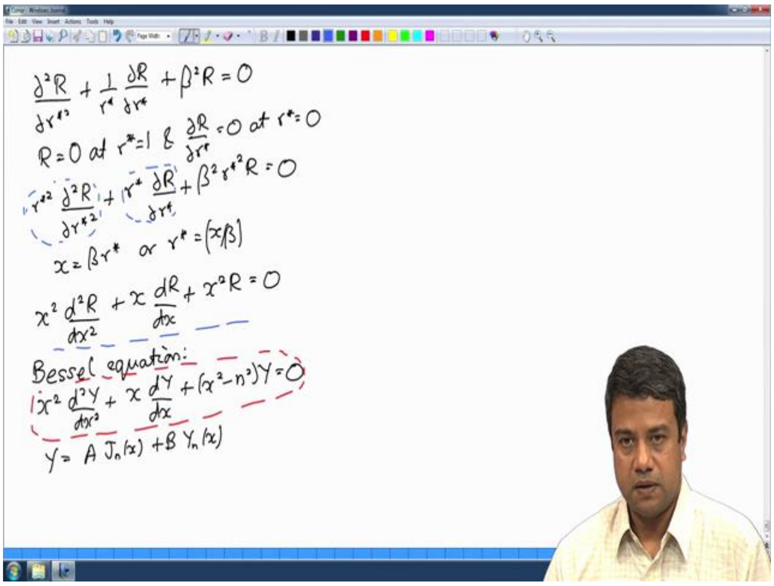

So, let me write it again the equation becomes of the form; d square R by d r star square plus 1 over r star subject to the boundary conditions, R is equal to 0 at r star is equal to 1 and I can rewrite this equation by multiplying throughout by r square and I will get r square; partial square R. Now if I change variables into beta or r star equal to x by beta and the change of variables r star is equal to x by beta or beta is equal to beta times r star is equal to x, but I change the variable you will see that these two terms have 0 net dimension r; r square times the second derivative, r times the first derivative.

So, the net dimension in r of the first two terms is equal to 0 which means that if I scale r by any coordinate these two terms do not actually change and I get equation form x square d square R by dx square plus x square r is equal to 0. Now this equation has to be solved in order to find out what is this function R, it turns out the there is no analytical solution for this equation. The solution for this equation is what is called a Bessel function, the Bessel equation is of the form x square, d square y by dx square plus x plus x square minus n square; y is equal to 0 and for that the solution is the form y is equal to A; Jn of x plus B Yn of x. This j and y are what are called the Bessel functions and this equation is called the Bessel equation and J and y are called the Bessel functions. These equation are solved, they are tabulated you find them standard tables these are similar to in the case of finite channel, we got sin in cos functions.

Sin and cos function of course there also special function; we can plot them we know what the values are the different values of x, the sin and cos functions. Similarly in this cases as well this solution can be plotted and we know what a values of different value of x and y and using these we can construct solution, so instead of sin and cos functions in the case of in finite channel, these Bessel function from the bases set for this infinite channel. So, you can see that this equation correspond to the Bessel equation with n is equal to 0; that means, that solutions for the Bessel equation with n is equal to 0.

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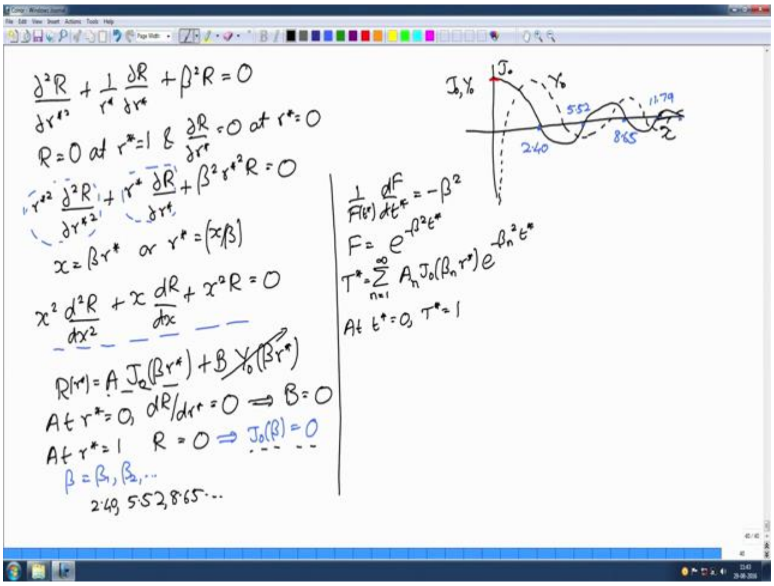

So, the solution form of Bessel function r is equal to A times J 0 of x; x is beta time r star plus B times; y 0 of x where x is equal to beta times r star.

So, these are the Bessel function solution for the radial coordinate; for the function r which is the function of r star and of course, these solution can be plotted for example, as a function of x if I plot J 0 and y 0 J 0 actually has a 0 slope and then goes through 0, it oscillates the amplitude of the oscillation decreases and the distance between 0 also decreases as x increases. This is the equivalent of the sin solution that we had; the sin solution as well or oscillated, but it is cos in amplitude in the cylindrical coordinate system; this is the equivalent that sin solution.

The other one goes to minus infinity and once again it oscillates; this is y 0 and this is J 0. Boundary condition we required the slope of the temperature profile has to be 0 at the horizon; J 0 does satisfy that condition y 0 does not. So, therefore, this constant B as to be set equal to 0 because that solution does not satisfy the boundary condition, that at r star is equal to 0; d r by dr star is equal to 0. This would imply that B is equal to 0; you have one other condition at r star is equal to 1, R has to be equal to 0; does that condition require that A should be 0 because if A is equal to 0; you will get back a trivial solution, turns out it does not because this Bessel function goes through 0 at multiple locations, just like the sin function goes through 0 at multiple locations; the Bessel function also goes through 0 at multiple locations.

Therefore the requirement is that J 0 of beta as to be equal to 0; therefore, beta as to have these discrete values and these value once again a tabulated, you can get values from standard tables, you can get what are the locations at which this goes through 0, this is about 2.40; the next one is about 5.52; the next one is at about 8.65, the next one is at about 11.79 and so on. So, there is discrete set of betas for which the Bessel function is equal to 0 at R is equal to at R star is equal to 1.

Therefore, the valid solutions are only those values of beta, so therefore, beta is discrete; beta can be equal to beta 1, beta 2 etcetera that is infinite set of such solution. So, these are the Eigen functions and beta as the Eigen values, the discrete Eigen values. I told you can get homogeneous boundary conditions; you get discrete set of values and discrete set of Eigen function. In the previous case it was the sin functions and Eigen values for n pi, in this case the Eigen values are not equally spaced, but you still have discrete set of values. So, that is solution for R; what is the solution in time, the other equation is equal to minus beta square; that would implied that F is equal to e power minus beta square t star; once again exponential.

Therefore, the solution for the temperature becomes A J naught of beta n r star; e power minus beta n square T star. A combination of the solution for the radial coordinate with beta having discrete values so that boundary condition that r is equal to 1 is satisfied and the exponential in time of course, any function with one of this discrete beta n that is 2.45, 5.52, 8.65 etcetera; any one of these satisfies both the conditions referred to the valid solution, any of this discrete beta n, beta n the location at which this Bessel function goes through 0 satisfy the equation that was most general solution is one where n is equal to 1 to infinite. So, that is the most general solution for the temperature filed as satisfied the boundary condition in the special direction, the two homogeneous boundary conditions at r is equal to 0 and r is equal to 1; how do I get these constants.

These constants A n have to be obtained from the initial condition because at T star is equal to 0, you know that T star is equal to 1 that is the initial condition. So, from the initial condition; we will satisfy, we will obtain the constants in this expression; using of orthogonal to relation, similar to what we have done in the previous lecture, I will continue this and show you how the orthogonality relation are obtained in the next lecture; I see you then kindly revise what is done so far, both for the flux geometry; we have sin functions as well as for this particular case and I will continue this in the next lecture we see you then.