Transport Process I: Heat and Mass Transfer

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Lecture – 31

Unidirectional transport: Balance laws in cylindrical co-ordinates. Unsteady heat conduction from a cylinder

Welcome to our continuing series of lectures on fundamentals of transport processes where we were going through how one does shell balances to get a differential equation for the concentration temperature of velocity fields, these are typically partial differential equations and then how one goes about solving them.

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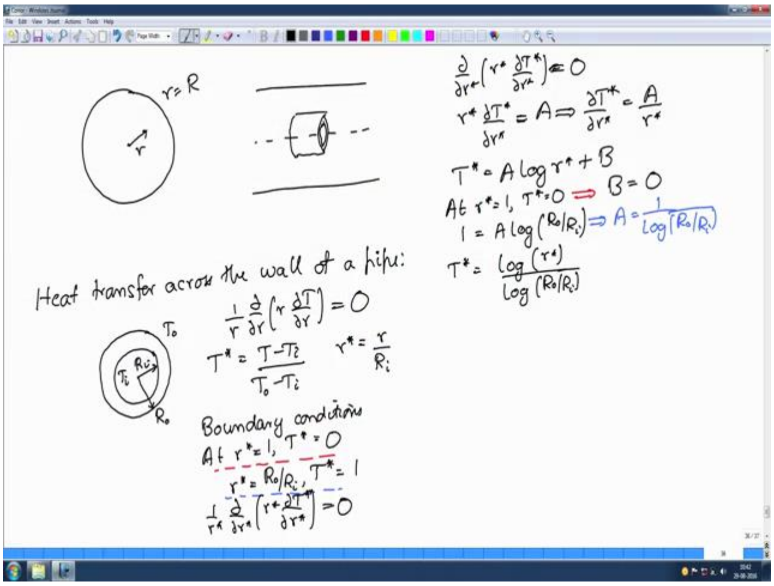

We have seen. So, far 2 different kinds of solutions, one is similarity solutions and the other is separation of variables procedure.

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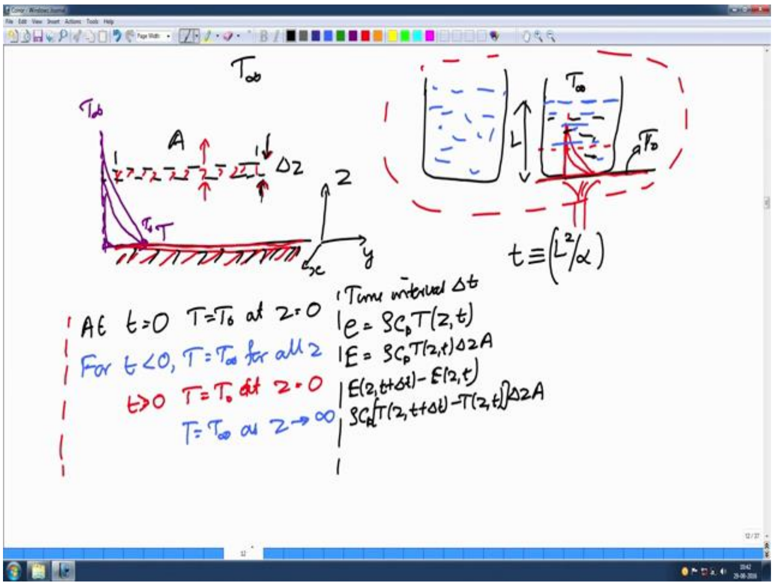

What I had solved for you originally was a Cartesian coordinate system where we just had a flat surface in the x y plane and the z direction was perpendicular to this surface and as you recall we took a small differential volume element with surfaces at locations z and z plus delta z, these surfaces were surfaces perpendicular to the coordinate z and they were surfaces of constant area and therefore, the differential equation that we ended up with was fairly simple and formed.

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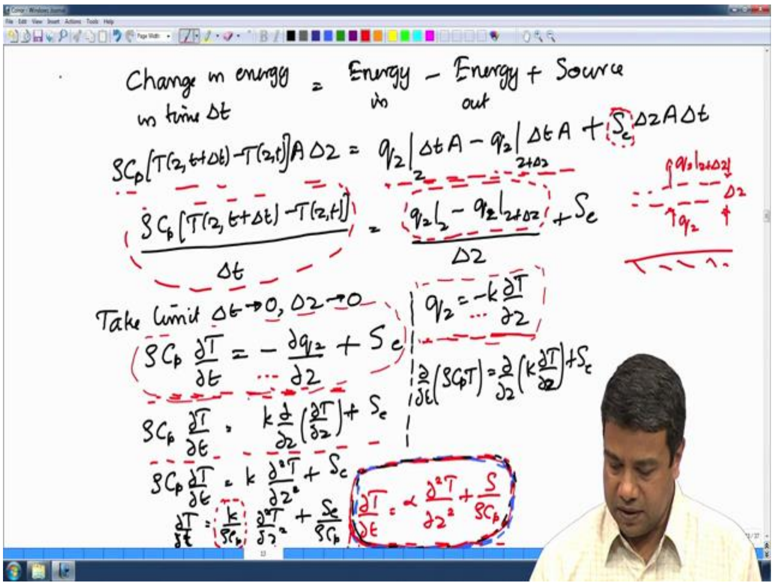

If you recall here, we just had one time derivative and the second order derivative in the spatial coordinates. This is in a Cartesian coordinate system where the coordinates themselves are straight lines and the planes, the surfaces perpendicular to the coordinates are surfaces of constant area.

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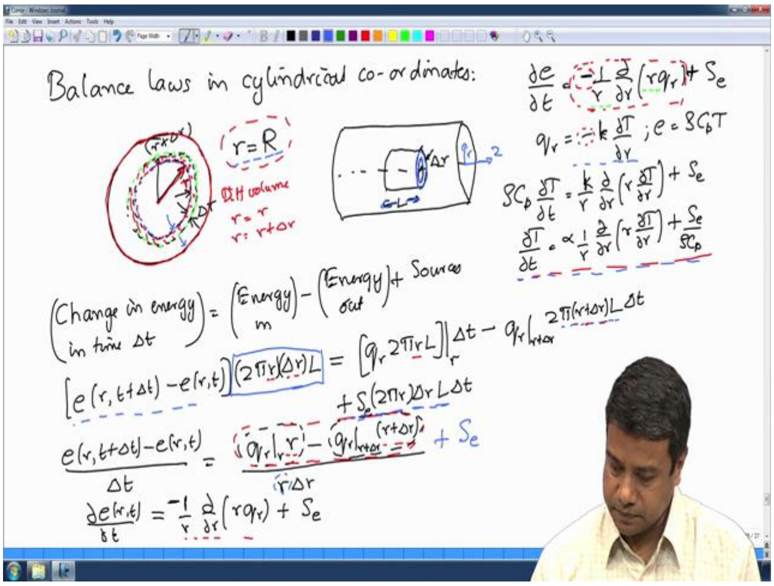

In the last lecture, we had looked at how to do this for a coordinate system where the surfaces are no longer flat, one would think that even if the surfaces were curved one could still use exactly the same approach, use a Cartesian coordinate system and in that coordinate system the surfaces will be defined by some equation and then solved in that equation in that coordinate system, turns out that that is more complicated it is easier to consider surfaces such that the boundaries are surfaces of constant coordinate, there the boundaries just becomes surfaces at which the coordinate is equal to a constant value.

However the differential equations turn out to be more complicated and we had looked at one such a cylindrical coordinate system in the previous lecture. Cylindrical coordinate systems are widely used in all pipe flows the coordinate system is a cylindrical one. So, anytime that you have transport of mass movement or energy in a pipe and annulus and other such configurations, the natural coordinate system to use is the cylindrical coordinate system.

In this cylindrical coordinate system rather than having the x and y coordinates, one has instead the radial coordinate which is basically the distance from the origin, the distance from the origin is a constant at this boundary surface. So, at this surface of this pipe for example, in the cylindrical coordinate system, the distance from the origin is a constant, therefore, it is convenient to use this distance from the origin as a coordinate and once you have chosen the distance from the origin or the radial distance as the coordinate, the differential volume that has to be chosen is a volume having constant coordinate, a volume which is perpendicular to the direction of variation of the coordinate. In this particular case the distance from the origin is the coordinate r and therefore, the coordinate system that I will choose is a system I am sorry the differential volume for carrying out the balance is a volume between 2 surfaces of constant r.

The boundary is at the location r is equal to capital R and the differential volume will be between surfaces of constant r at the location r is equal to some r and r is equal to r plus delta r. So, therefore, in this cylindrical coordinate system the surfaces that I have chosen are surfaces at r and r plus delta r.

Along the axis of course, the coordinate that you will choose is what is called the z coordinate along the length direction for the present we will assume that there is no variation in that direction. So, we have basically a differential volume bounded by 2 cylindrical surfaces, one at r and the other at r plus delta r and the earth and bounded by 2 flat surfaces in the direction along the axis, this is very often called the z direction along the axis and r is the distance from the center in the plane of the cylinder.

Therefore, we will to choose 2 coordinates at location r and r plus delta r in the z direction between 0 and l there is no variation in the z direction therefore, that volume will just be equal to r.

We went ahead and did the energy balance for this curvy linear coordinate system the change in energy in a time delta t is the change in energy density times the volume which is basically 2 pi r delta r times l, 2 pi r is the circumferential length the length of the perimeter times the thickness delta r is the area times the length along the axis l is the total volume so that change in energy in time delta t is equal to q r what comes in at the surface what comes in at the surface r at r minus what goes out at the surface at r plus delta r.

What comes in is the flux times the surface area of this curved surface, the surface area of the curved surface is 2 pi r times l at the location r, at the location r plus delta r it is equal to 2 pi into r plus delta r times l, since we have choose chosen curved surfaces for carrying out our differential volume the surface area on these 2 curved surfaces is no longer equal and that has to be incorporated in the balance.

Plus of course, any source for sink of energy within this differential volume and now when we divide throughout by time and by volume, when we divide throughout by time and by volume we get a slightly more complicated differential operator here, acting on the flux it is not just a sec first derivative as we had earlier, when we did it in Cartesian coordinates rather it is a slightly more complicated operator and then if we use the constitutive relation for the heat flux, we get this as the conduction equation in a cylindrical coordinate system.

Rather than the second derivative of the temperature on the right side multiplying the thermal diffusivity, you have a slightly more complicated operator at the radial coordinate and that complication arises because surface area is changing with the radial coordinate in this curvy linear coordinate system.

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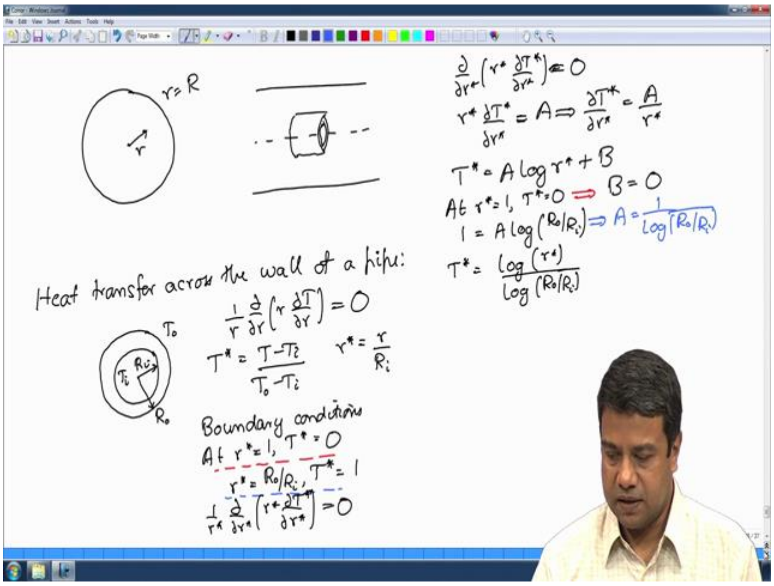

We had seen how to solve this for a relatively simple problem, heat transfer across the wall of a pipe the pipe had 2 radii, one was the inner radius r I and the other was the outer radius r 0. The temperatures were fixed and one had to find out what is the heat flux, this is a typical conduction problem across the wall of a pipe because very usually in the case of heat exchanges and so on, there is transport across the wall of this pipe the pipe is solid. So, the transfer is due to heat conduction alone and this transfer takes place across curved surfaces usually and it is necessary to accurately estimate what is the flux due to a temperature difference or alternatively what temperature difference is required to generate a desired heat flux across the surface to achieve the necessary transport of heat, this is at steady state there are no variations in time and we will assume that there are no variations in the axial direction either along the axis of the pipe.

In that case the temperature equation reduces to a rather simple form, 1 by r d by d r of r times d T by d r is equal to 0 we had scaled the equation so that the temperature is 0 at the inner surface and is equal to 1 at the outer surface so that we get this boundary condition we had scaled the radial coordinate by the inner radius. So, we get a at r star equal to 1, T star is equal to 0 and when the radius is the ratio of the radii of the outer and inner cylinder T star is equal to 1.

The solution that we get for the temperature field is a logarithmic solution the solution that we get for the temperature field in this coordinate system is a logarithmic solution it is not the linear variation as we would have got in a Cartesian coordinate system and this difference is because for a curved surface the surface area is changing with the radial coordinate.

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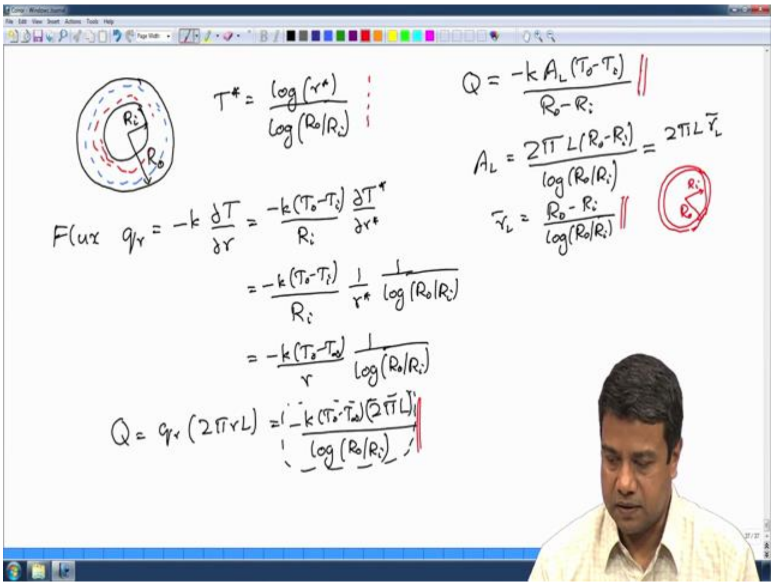

And finally, from this we had calculated the radial flux; this flux itself is varying with radius, but the total heat that is being transported across turns out to be independent of radius, the total heat that is being transported across turns out to be independent of radius.

That is expected because in our balance equation there was only the heat conduction term there was no sources or sinks of heat and consequently the total heat that is transported has to be independent of the location at which we look regardless of whether you look at one particular surface which is closer to the inner surface and another one that is closer to the outer surface the flux will change the surface area will change in such a way that the whole total heat transported is the same.

Now, in correlations the total heat is usually expressed as the thermal conductivity times an area times T naught minus T i divided by the difference in radius so that is usually written in this fashion. In order to bring it approximately of the same form as you would have for flat surfaces and you can see by comparing this equation and this one if you just compare with the 2 this effectively defines the area for us A L has to be equal to 2 pi L into R naught minus R i divided by log of R naught by R i or I can also write this as 2 pi L into R L, the average logarithmic radius log average radius where R L is equal to.

This is the logarithmic average of the radius between the inner and the outer surfaces and this is what should be used for calculating the heat transfer from across this pipe the area is modified the area is no longer either the inner area or the outer area rather it is this, this fact which should be used for the area of the pipe. So, this is the logarithmic log for the heat transfer across the wall of the pipe. As the difference becomes much smaller than either the inner or the outer radius that is if I have a pipe which is very thin such that R naught minus R i is much smaller than either R naught or R i.

This expression will reduce to what you had for a flat surface so that the thickness becomes much smaller than the radius you will get back the expression that you had for a flat surface because log or R naught by R i that times you can take the limit as this goes to r naught goes to R i and you will get back the expression that you had for the heat transfer or for the flat surface.

This was a simple 1 dimensional problem heat conduction.

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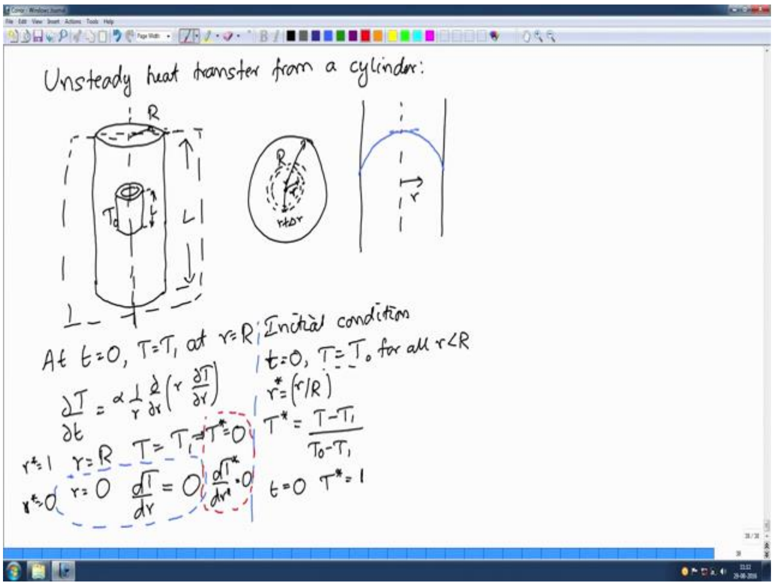

Next I will look at the heat transfer from a cylinder to the unsteady heat transfer from a cylinder the unsteady heat transfer from a cylinder. So, I have some cylinder of whose radius is R and length is L, at a particular initial temperature T naught, at a particular initial temperature T naught. So, the cylinder is entirely at a constant temperature and at time t is equal to 0, I impose the boundary condition that the temperature is equal to T l at r is equal to R. So, the entire curved surface of the cylinder is the temperature is changed to a different temperature T 1.

This is once again, a practical problem very often you have situations where for example, in the food product industry then you have cylinders cans if you will which are at some particular temperature and you want to cool that to a lower temperature and. So, what you would do is to immerse it in a fluid having a lower temperature and one would like to know how long it takes for the entire can to cool so that is a problem of practical importance how long does it take for the entire can to reduce it is temperature from the initial temperature that it was at to a lower temperature. So, this is an unsteady heat conduction problem for the present we will assume that the entire heat conduction is happening within the cylinder the surface is always at the temperature T 1 so that the entire heat conduction is happening at the within the cylinder.

There is no thermal resistance outside. So, we will assume that the temperatures are constant at T 1; the other thing that we will assume is that there is no variation in the axial direction as I said you will have a coordinate system if I look from that the top. you have a radial coordinate system the cylinder surfaces at r is equal to capital R and you also have an axial coordinate system an axial coordinate along the axis of the cylinder and for the present we will assume that variations are entirely into radial direction there is no variation in the axial direction.

How do we solve this problem? We take differential volume of course; with surfaces perpendicular to the direction of the coordinates surfaces of constant coordinate, you will have surfaces at r and surfaces at r plus delta r. So, if I look at it here and I have one surface at r and another surface at r plus delta r the inner surfaces at r the outer surfaces at r plus delta r and it has a length L and as I said if you do the balance you will get back the same balance equation that had got back earlier partial T by partial t is equal to alpha 1 over r d by d r of r plus if there are any sources of sinks of energy in this particular case we consider that there are no surface of or sinks of energy and therefore, this thing is equal to 0.

Now, what is the boundary condition on this? At r is equal to R, we know that T is equal to T 1. Now note that r is equal to 0 is just a point r is equal to 0 is just a point within this is the location along the axis. So, it is just a point in the radial plane at this you require that the temperature has to be finite alternatively the derivative of the temperature has to be equal to 0.

In this particular case, we have only 1 boundary and that boundary is at r is equal to capital R, we have a second order differential equation and strictly speaking we need 2 boundary conditions the other boundary in this case is at the origin itself at r is equal to 0 the other boundary is at the origin itself at r is equal to 0 what should be the boundary condition there. So, this is a symmetry condition which there is no physical boundary at the location r is equal to 0 because it is just a homogeneous solid. If I had something like a wire or something at that location which was heating up the solid then there would be a real boundary, there would be a physical boundary in this case it is just 1 homogeneous material everywhere. So, there is no physical boundary.

However there is a boundary condition that is required by symmetry at this location. So, if were to take a cut across the cylinder, if I were to take a planar cut across this cylinder, if I were to take a planar cut across the cylinder let us say I take it along a plane along a plane I cut the cylinder along a plane then I will get temperature profiles as a function of radius r and the right side is the distance from the axis in a in right word on the left side is the same distance after all r was defined as the distance from the axis. So, it is the same distance from the axis.

If I plot the temperature profile on the right side for example, I could get different sorts of temperature profiles I could imagine a temperature profile in which the temperature becomes flat as I approach the axis I could. In fact, imagine a temperature profile where the temperature becomes flat as I approach the axis. I could imagine another temperature profile where the temperature does not become flat as I approach the axis these are possibilities on the right side alone. Since there is no variation around the axis in this case I have considered the system to be symmetric in that direction so that there is no variation around the axis.

Therefore, for this blue temperature profile the temperature on the other side will look something like this it will be symmetric about this point. Similarly for the red profile if I were to rotate this temperature profile by 180 degrees on the other side I would get something that looks like this, which of these is physical? Turns out only the blue temperature profile is physical because it is continuous at that point r is equal to 0.

This function is continuous because whether you go from the right side or the left side, the derivative of the function is the same. So, this is a continuous function with a continuous derivative at the axis, the other one that I had imagined is actually not continuous, the slope from one side has 1 value, the slope from the other side has another value at the same location, a function cannot have 2 different slopes, 2 different derivatives at the same location therefore, this red profile is not physical in particular, if you had a discontinuity in a slope; that means, you will have a discontinuity in the temperature field at that point and the flux would effectively be infinite because the flux is the derivative of the temperature which you cannot define, the derivative of the temperature you cannot define the flux at that point.

In general if you have a homogeneous material, the requirement is that the temperature the slope of the temperature or the derivative of the temperature has to be equal to 0. That is the boundary condition that emerges from the symmetry requirement of the coordinate system itself even though we do not have a physical boundary at this location.

Therefore, this red temperature field is not physical, the only physical temperature is the blue one at the axis at r is equal to 0 is therefore, you require this from the coordinate system itself.

Once again this is because there is no physical boundary at this location, if I did have a physical boundaries such as a thin wire or something like that you could have a non zero slope because there will be a heat flux from that we will come and see that a little later. In this case because it is all 1 homogeneous material, the boundary is only due to the coordinate system there is no physical boundary this becomes the boundary condition at the origin. Alternatively another possible boundary condition is just to say that the temperature itself has to be finite at this location.

And then I have an initial condition at t is equal to 0; T is equal to T 0 for all r less than R. The initial time the temperature is uniformly t 0 everywhere only a times t is equal to 0 the temperature has been increased to t1 at the boundary r is to r. So, those are the initial and the boundary conditions.

Now, we can scale the equations, we can scale the equations, the natural scaling for the radial coordinate r star will be equal to r by R because then this boundary condition becomes r star is equal to 1 and r star equals 0, those are the boundary conditions the locations of the boundaries when I scale r by capital R.

How should I define a scaled temperature, how should I define a scaled temperature, what would you expect in the limit as times goes to infinity you have the cylinder whose surface temperature is T 1. As you wait for a very long time you would expect the temperature to all become equal to this surface temperature therefore, the temperature in the long time limit will be equal to the surface temperature T 1.

You would expect the scaled temperature field to go to 0 in the long time limit if you recall when he had done the heat conduction in a finite slab, we had said that in the limit of t going to infinity the transient part of the temperature should go to 0 the transient part is the total temperature minus the steady temperature. In this particular case the steady temperature is just equal to T 1 it is a constant everywhere.

So, therefore, it is best to define T star is equal to T minus T 1 by T naught minus T 1, so that when T is equal to T naught so that at this at r is equal to R, this would imply that t star is equal to 0 at r is equal to capital R, this will imply that t star is equal to 0 because t is equal to T 1 at the surface and at r is equal to 0 you can scale it quite easily and you get d T star by d r star is equal to 0 and if I defined it that way that would mean that at time t is equal to 0, T star is equal to 1, at t is equal to 0, T star is equal to 1 because T is equal to T naught at t is equal to 0 therefore, T star is equal to 1.

Define this way the forcing is at initial time and then the temperature field evolves in time. The boundary conditions in the spatial coordinates are both homogeneous the boundary conditions in the spatial coordinates are both homogeneous either temperature is 0 or it is derivate is 0 what is forcing this temperature field is the initial difference between the temperature in the cylinder and that temperature of the surrounding fluid.

So, this problem; this is basically the transient part of the temperature, we can solve using a separation of variables procedure almost identical to the separation of variables procedure that we had used for the heat transfer in a finite channel. I will continue in the next lecture how this separation of variables procedure is implemented. So, we will continue this in the next class.