Transport Processes I: Heat and Mass Transfer

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Lecture - 30
Unidirectional transport: Balance laws in cylindrical co-ordinates. Heat transfer across the wall of a pipe

We are now in our 30th lecture in our discussion of fundamentals of transport processes, we started off with dimensional analysis, we looked at diffusion and we are now solving problems in transport phenomena. In the previous series of lectures, we had solved problems for transport in one direction; in that particular case, we had used a Cartesian coordinate system in which the surfaces were plane surfaces.

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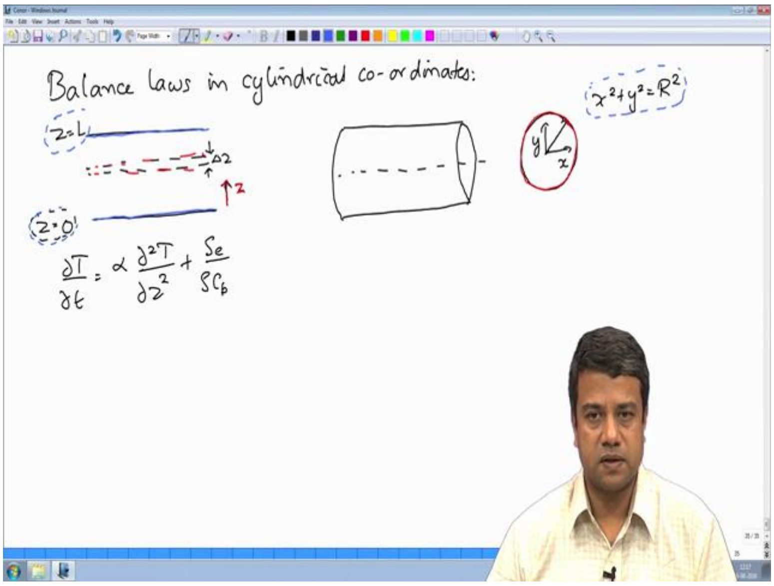

The surfaces in this case where plane surfaces and we had chosen a coordinate that was perpendicular to these surfaces a coordinate that was perpendicular to the surfaces so that these surfaces were surfaces of constant coordinate and then we had chosen a differential volume and written a balanced equation, the balance equation in all cases reduced to the from d T by d t is equal to alpha d square T by d z square plus I am write it below plus any sources.

Now there are applications where these surfaces may not be flat surfaces, case and point is the flow through a pipe or the transport in a cylindrical pipe for example, where the surface of the pipe is a cylindrical, it is a curved surface and if you were to try to you could of course, use a similar balance law in that case as well, but then the boundaries are no longer simple. In the Cartesian coordinate the boundaries were z is equal to 0 and z is equal to l, in this cylindrical coordinate system the boundary is not so simple, so rather than using a Cartesian coordinate system for a curved surface, it is more convenient to use a coordinate system in which the surface is a surface of constant coordinate, to use a coordinate system where the surface is a surface of constant coordinate and write a balanced equation for that surface, so you look at how to do that in this lecture.

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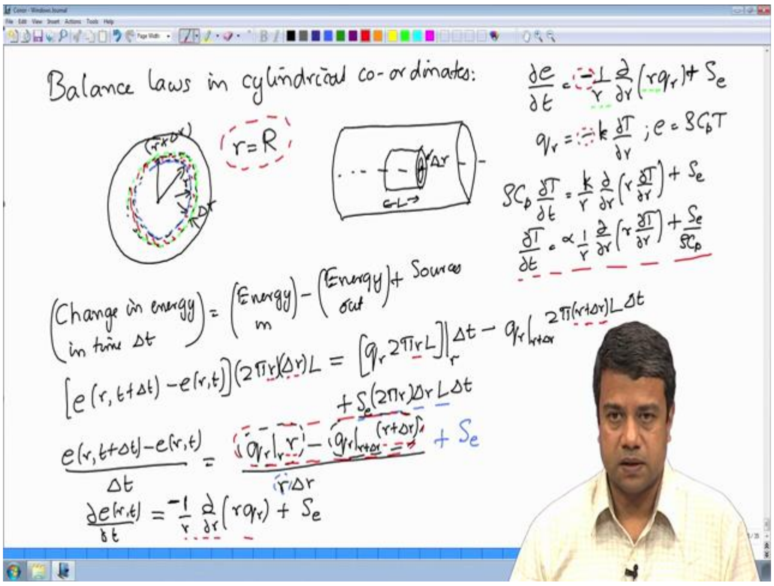

Let us say that I had a fluid within the cylindrical surface with the origin at the center I could choose a coordinate. So, normally in Cartesian coordinate system we have been choosing the coordinators as x and y; however, I could choose a coordinate system where the surface is a surface of constant coordinate, the surface is at a constant distance from the center if the surface of this cylinder is at a constant distance from the center. So, rather than do it this way I could choose the coordinate as the radius R from the center. So, the boundary of the cylinder is just going to be at R is equal to capital R and then derive conservation equations for this cylindrical boundary.

How do we derive conservation equations? We have to choose a differential volume element. Previously the differential volume element was between z and z plus delta z, it was for a small displacement in the z direction in this case our coordinators r. So, any differential volume that we choose has to be between location r and location r plus delta r of thickness delta r so that should be the volume that we are choosing, in this case. Of course, this has to have some unit length in the direction perpendicular to the surface, if it this should be the volume that we are choosing in this case. Having chosen this volume we have to write a balanced equation in this case if it is for energy, energy transfer, change in energy in time delta T is equal to energy in minus energy out if plus any sources or sink plus any energy sources.

Change in energy in time delta t is going to be equal to the energy density e at r t plus delta t minus e at r t times the volume, if we take it per unit length in the direction perpendicular to the plane I have to multiply by this area between these 2 locations, if I take a particular length perpendicular. So, in this case for example, if I take a shell between r and r plus delta r so that this distance is delta r radius is r and I have some length L in this direction. So, this has to be multiplied by the volume this volume is going to be equal to 2 pi r the circumference into delta r into L. So, there is going to be equal to the change in energy in a time delta t. Energy in, is the energy in at the surface at r energy in with the energy in at the surface at r, this energy is in the r direction is in the radial direction.

It is going to be equal to q r times the surface area surface area is 2 pi r L at the location r that is the energy in. Flux is energy per unit area per unit time. So, I need to multiply this by delta t to get the total energy in energy out is at R plus delta R. So, this out has to be equal to q r at r plus delta r times the surface area which is 2 pi into r plus delta r into l I have take the energy out at the location r plus delta r. So, q r at r plus delta into l into delta t of course, and then I have the sources which are basically of the form s e into volume 2 pi r delta r into L into delta t so that is the balanced equation in the cylindrical coordinate system using the same shell balance. Choose surfaces that are perpendicular the surfaces of constant coordinate surface is perpendicular to the direction of transport these are surfaces of constant radius R.

And then write the rate of change is the change in energy in a time delta t is equal to energy in minus energy out plus any sources or sinks and then I divide throughout by delta t divide throughout by volume. So, if I have to divide by 2 pi r delta, r delta t, if I do that I get the energy density at r t plus delta t minus energy density at r t divided by delta t; 2 pi and L will cancel out. So, I will get q r times r at the location r times r minus q are at the location r plus delta r times r plus delta r that is this term here these 2 terms, these 2 terms are coming in here; 2 pi L will cancel out and I have dividing by delta r times are over here. So, therefore, this whole thing will be divided by r delta r and then even I divide the last term here by the volume n time I will get plus the energy source and then I take the limit as delta t goes to 0 delta r goes to 0, if I take the limit as delta t goes to 0 and delta r goes to 0 the left side will just be equal to partial e of r t by partial t is equal to on the right side I have the difference between in q r times r at the location r and r plus delta r the difference in q r times r.

Therefore, when I take the limit as delta r goes to 0 this factor of r is going to come out I will get one over r times d by d r of r times q r because this is a difference between q r times r at these 2 locations divided by delta r what I have here is the difference between q r times r at these 2 locations divided by delta r.

It is the value at r minus the value at r plus delta r, what I have written here is the value at r plus delta r minus the value at r therefore, I should have a negative sign here plus the source. So, therefore, this has a slightly different form from what we had earlier the form of the energy balance equation that I get in this case this is of the form partial e by partial t is equal to minus 1 by r, d by dr of r q r plus s e why did this additional term proportional to r come out it came out basically because if you look at these 2 surfaces, if you look at this inner surface here and the outer surface here that is r changes the surface area is changing the surface area is changing proportional to r because the perimeter is proportional to r. So, the surface area is changing proportional to r. The total transporters the flux times the surface area, since the flux is a function of r surface area is also a function of r therefore, you get this additional term in the balanced equation just likely different form of the balanced equation.

If you recall in our earlier Cartesian coordinate system, the surface area was not changing as the z coordinate changed the surface area was not changing and therefore, you got a simple form, in this case the surface area is changing and therefore, the form is slightly more complicated and then if you use the relation q r is equal to minus k d T by d r the conduction equation and e is equal to rho C p times T the energy density then you get the equation of the form rho C p partial T by partial t is equal to minus when I put this in I will get a positive sign because I have there is one negative sign in there is one negative sign in the balanced equation itself and there is one negative sign the constitutive relation the reason is because heat is transferred from regions of high temperature to low temperature therefore, my balance law will be of the form plus S e with thermal conductivity this of course, once again assumes that the thermal conductivity is independent of the spatial position this once again assumes to the thermal conductivity is independent of the spatial position.

Alternatively I can write this as divide throughout by rho C p is equal to alpha 1 by R plus the energy source by rho C p. So, this is the conservation equation in a cylindrical coordinate system, this is a conservation equation in a cylindrical coordinate system and for that reason the cylindrical coordinate system the solutions will also be slightly different.

So would have basically done here is that we said that since the surface is a curved surface if I use a Cartesian coordinate system as the reference coordinate system the equation for the surface becomes complicated the equation for the location of the boundary conditions becomes complicated therefore, I would prefer to choose a surface a coordinate system in a surface which itself is curved and in that coordinate system the fundamental coordinate is the distance from the center or the radius when I do that the equation for the surface becomes quite simple, but for a curved surface the surface area changes as the coordinate changes. So, even though the flux may not change is the surface area change there is going to be a change in the total amount transferred because of that change in surface area that comes into the balanced equation and therefore, becomes slightly more complicated. This is all once again in one dimensional transform in a cylindrical coordinate system.

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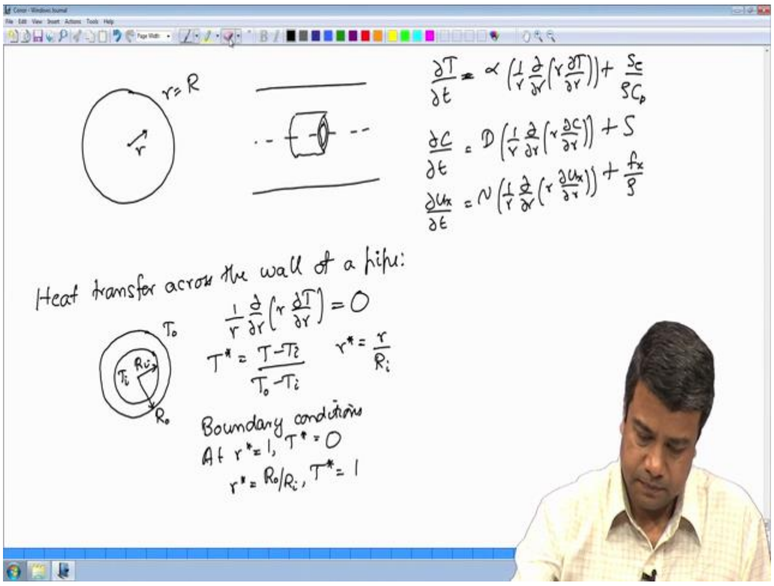

The coordinate is the distance from the surface R the boundary condition is applied at constant coordinate, the balance equations are of the form d T by d t is equal to alpha into 1 by r d by d r of r d T by d r plus s e by rho C p if you did the same thing for mass balance you would get an equation of the form d c by d T is equal to D into 1 by r plus any mass source per unit volume per unit time and similarly for momentum the kinematic viscosity times plus. So, these are the equations that you would get if it did for mass and heat transfer and these are all done for cylindrical volumes.

It shows surfaces of constant coordinate, it differential volume between surfaces of constant coordinate and then do the balance and now that we have these balance equations, we can apply it to any geometry in cylindrical coordinates. So, let us just consider the simplest case in of heat transfer, this is heat transfer across the wall of the pipe heat transferred across the wall of a pipe let us say that you have a pipe of inner diameter R i and outer diameter R o. At the inner surface you maintain a temperature T i and the outer surface is maintained a temperature T o.

And one would like to find out what is the heat flux in this pipe, what is the heat flux within this pipe, we will assume for the moment that this is a steady solution so that there is no variation in time, you will see next what happens when there is a variation time, but for the present context will assume that there is no variation in time. Therefore, the temperature equation that has to be satisfied is 1 over r d by d r of r d T by d r that has to be equal to 0.

We can do scaling here the simplest thing you can scale the distance with either the inner or the outer radius you will get equivalent results in both cases similarly the temperature can be scaled either with the inner or the outlet temperature. We will for the moment we will scale it such that T star is equal to T minus T i right by T o minus T i and r star is equal to R by T inner radius. So, therefore, the boundary conditions are at r star is equal to one T star is equal to 0 and at r star is equal to the ratio R 0 by R I, T star is equal to 1 is scalene in this way this is what it reduces to.

Now, what is the scaled equation? The original equation that I had was just that 1 by r d by d r of r d T by d r is equal to 0. So, therefore, the scaled equation your just scaling T your scaling r. So, in terms of the scaled variables the equation will still be the same.

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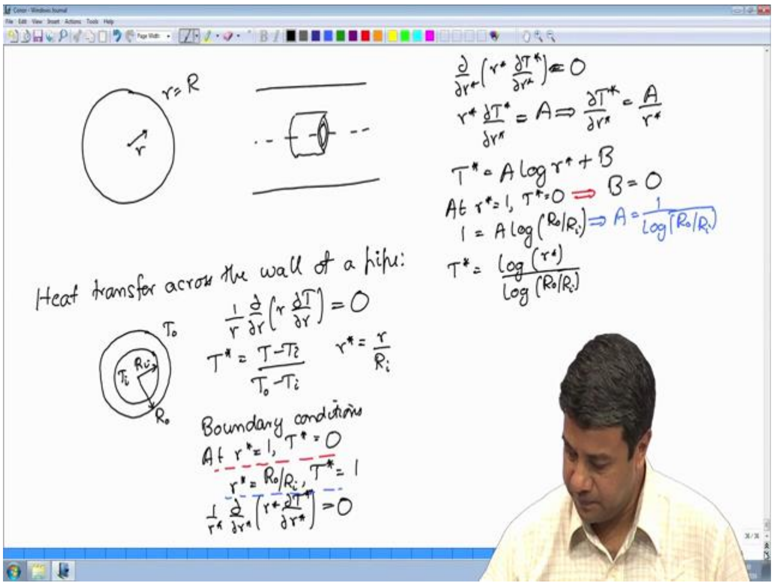

The equation that I get will be of the form 1 by r star d by d r star of r star d T by d r that has to be equal to 0 because I just multiplying r by a constant T by a constant and when I do that on the right side, there is 0 anyways so I will let us get this solution. This we can solve, you will get d by d r star of r star d T is equal to 0, r star d T by d r is equal to some constant at which implies that d T by d r is equal to a by R and this gives me T star is equal to A log r star plus B this is the natural logarithm A log to be e of r star times B plus B.

Now in order to get the boundary conditions in order to get the constants I need to use the boundary conditions at R star is equal to 1 T star is equal to 0 the boundary condition at the inner surface the boundary conditions the inner surface at r star is equal to 1, T star is equal to 0 what that implies is that B equal to 0 because an r star is equal to 1 the logarithm is 0 and T star is equal to 0 therefore, B equals 0.

And the other boundary condition is that at r star is equal to R 0 by R i, T star is equal to 1. So, I get the second boundary condition is that 1 is equal to A log R outer by R inner, 1 is equal to A log R outer by R inner and that would imply that A is equal to 1 log R outer by R inner therefore, this gives us the final solution. T star is equal to log of r star by log of R o by R i so that is the final solution for the temperature field - note that is the logarithmic solution.

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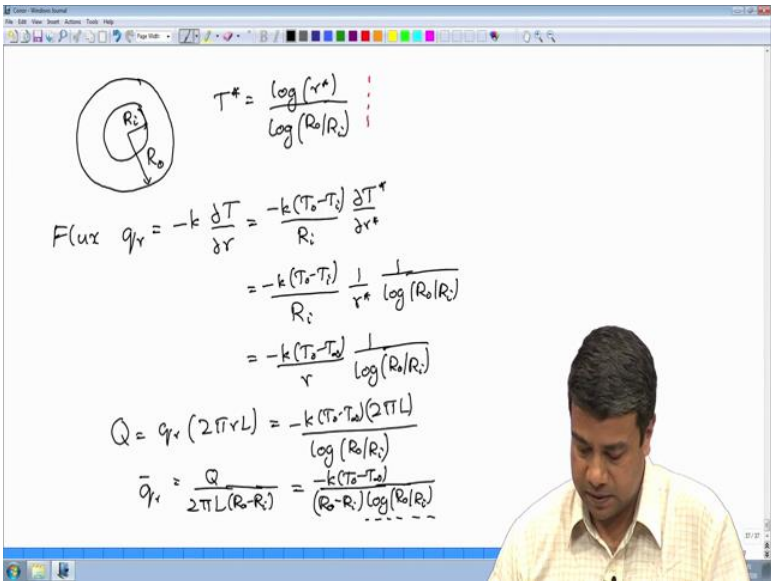

My solution for T star is equal to, now what is the flux? The flux in the radial direction q r right at any location is equal to minus k partial T by partial r.

Now, if you recall, I had said that T star is equal to T minus T i divided by T o minus T i. So, therefore, the derivative of the temperature I can write it as minus k into T o minus T i the difference in temperature across the 2 surfaces and the radio side scales by the inner radius times partial T star by partial r star. Now this partial T star by partial r star I can get quite easily from here this partial T star by partial r star, it is just equal to 1 over r star. So, this is equal to minus k into T o minus T i by R i into 1 over R star into 1 over log of R o by R i and since R star is equal to the ratio. Since R star is equal to the ratio, this is becomes equal to minus k T naught minus T infinity divided by r, 1 by log of this is the flux, the total heat transported across any surface the total heat transport across any surface is the flux times the surface area therefore, the total heat transporters is equal to q r times 2 pi r L. When I multiply this I will get 2 pi L into; I will get minus k into T naught minus T infinity into 2 pi L.

Note that the total heat transfer to is now independent of the radius that is what you would expect the balance law that we had written down basically made a balance between the total heat at the location R in the load total heated location R plus delta R. At steady state these 2 have to be equal therefore, the total heat transported has to be the same regardless of position.

Therefore, if I can write this I can write an average heat flux, an average heat flux if I write it as q r average is equal to q by 2 pi L into the thickness across which the heat is transported, this I write it as an average heat flux is equal to Q by 2 pi L into the distance across which the heat is transported then I will get this is equal to minus k into T naught minus T infinity divided by minus R i into log of R naught by R i. This is the log mean of the average heat flux across a distance delta t. So, this is the flux in a pipe when there is change in radius the flux is not just equal to the difference in temperature divided by the thickness types of conductivity.

That is because the surface area is changing as the radius changes in this pipe therefore, you get a different expression that expression contains this in it as well so that has to be incorporated in the expression for heat transfer in a pipe. So, this is a simple steady state solution came 1 dimension, what about unsteady solutions? That we will continue in the next lecture, I will show you how to do on steady solutions using separation of variables in cylindrical coordinates, the principle is exactly the same, the details differ the conceptual methodology is exactly the same. So, kindly try to revise the separation of variables at done for Cartesian coordinate system then I will go through how it is done for a cylindrical quality system.

This will continue in the next lecture - cylindrical coordinates, we look at spherical coordinates and then we will talk about how to derive balance equations for a general coordinate system so that is going to be the plan for the next few lectures, I will continue this in the next lecture, I will see you then.