Transport Processes I: Heat and Mass Transfer
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Lecture - 29
Unidirectional transport: Separation of variables for transport in a finite domain cont

Welcome to our continuing discussion on transport in 1 dimension, we had gone through the separation of variables procedure for transport in a finite domain in the previous 2 lectures. This is an important procedure and therefore, I will try to summarize this in this lecture before proceeding to the next topic.

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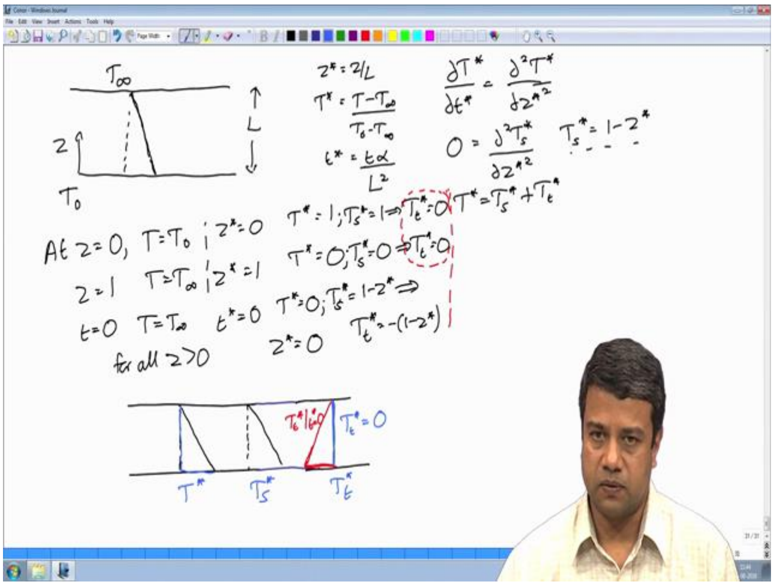

What we had was transport in a finite channel of cross sectional distance L, the top surface was at temperature T infinity the bottom surface at temperature T naught.

Therefore, if I consider this as the z coordinate, the boundary condition was that at T equals T naught and z equals 1 T equals T infinity you defined scale distances that star is equal to z by L T star is equal to T minus T infinity by T naught minus T infinity and in those coordinates z star is equal to 0, T star is equal to 1 h star is equal to 1, T star equal to 0 and at time T is equal to 0, we had T is equal to T infinity for all z greater than 0 because at time T is equal to 0 is when we had switched on the heating at that point T was equal to T infinity throughout the domain only the surface was heated. What that implied was that T star is equal to 0 T star is equal to 0 for h star is equal to 0 where the scaled time was equal to T alpha by L square. So, that was the scale time in the problem.

Now, the equation that I was trying to solve was the convection the diffusion equation which basically reduced to partial T by partial T is equal to partial square T by partial z star square. Now this equation I had expected that in the long time limit you would get a steady temperature profile in the long time limit you would get a steady temperature profile that temperature profile was satisfying the equation because in the long time limit the temperature should no longer depend upon time, it should retain a steady value in the long time domain and that give us a steady temperature is equal to 1 minus z star in this case a linear temperature profile that was the final solution in the case of steady state in the long time limit and then we had expressed the temperature as the sum of the steady part plus the transient part.

And if you look at these boundary conditions here, if you look at these boundary conditions here at z is equal to 0 T star was equal to 1, the steady temperature is also equal to 1 which implied that the transient temperature was equal to 0 and z is equal to 1, T was equal to 0 the steady temperature was here also equal to 0 this implied that the transient temperature is equal to 0 for the transient temperature field and at time T is equal to 0 the total temperature was 0 the steady temperature is equal to 1 minus z star. So, the steady part plus the transient part has to sum to 0 therefore, this implies that the transient temperature is equal to minus of 1 minus z star. So, we have already solved for the steady part the transient part is a part that goes to 0 in the long time limit because the temperature has to tend to the steady part. So, that transient temperature goes to 0 in the long time limit its important to note that it is also 0 on both boundaries, it is also 0 on both boundaries and it is non zero only within at the initial time, it is non zero at the initial time and then it slowly decreases to 0.

Is I said my steady temperature profile looks like this and my initial temperature profile looks like this. So, what I have done is effectively to write this as the sum of a steady part some of the steady part plus another part which is the transient part that transient part is basically equal to 0 in the long time limit. So, at if I sum the steady part plus the transient part in the long time limit I will get the total temperature initially also the sum has to be equal to 0 and therefore, the transient part at T is equal to 0 has to look something like this.

Whereas, the transient part in the limit as T goes to infinity has got to be equal to 0. So, that the temperature reduces to the steady part so that is basically the problem that we were trying to solve we already got the steady part we wanted to solve for the transient part.

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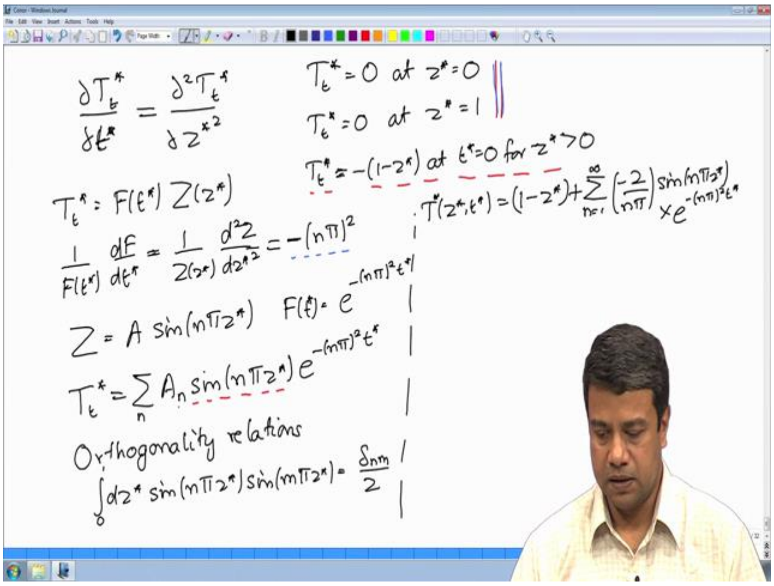

The equation that we had was partial T transient by partial T is equal to partial square T transient by partial z star square that was the equation that we had you have to solve it, subject to the condition that T transient is equal to 0 at z star is equal to 0 at z star is equal to 1 and T transient is equal to minus of 1 minus z star at T star is equal to 0 for z star greater than 0 that was what we are trying to solve.

Note that this direction the z direction is the homogeneous direction the temperature is equal to 0 on both boundaries so that is as far as the transient temperature is concerned, the z direction is the homogenous direction the boundary condition is that the temperature is 0 on both boundaries the time direction is the inhomogeneous direction because I have an initial condition at which T is non 0 and then it decays to 0 as time progresses.

The time dimension is the inhomogeneous dimension, in this case the way we have solved it was to write T, is equal to some function of T some function of z, give it well written it as a function of T and a function of z and using separation of variables we got an equation of the form 1 by f of T d f by d T 1 by z of z d square z by d z square that was the equation that I have got and the both of these sides they have to be equal the left side is only a function of time the right side is only a function of z these will be equal only if both are constants because if both are not simultaneously constants if I keep time a constant and change z then the left side does not change the right side changes and vice versa. So, both of these have to be constants, what constant should they be Ok.

If the constant were positive this function f would be an exponentially increasing function in time gives the constant were positive this function f would be an exponentially increasing function in time that does not satisfy our expectation that in the long time limit T, T star has to go to 0 the transient part has to decrease to 0 that cannot happen if this the temporal variation is an exponentially increasing function.

The only possibility in that case is for the temporal variation to be an exponentially decreasing function which means that the constant is negative I had also showed you last time that when the constant is positive the function z admits only trivial solutions if you write it as an exponential function exponentially growing and decaying both of those constants in that equation are identically equal to 0 and you just get a trivial solution capital Z is equal to 0. You get a non trivial solution only if the constant is negative and this is equal to you get a non trivial solution only if this is negative and is equal to minus of n pi the whole square where n is an integer, in that case the non trivial solution for z is a sin of n pi z star non trivial solution for T is equal to e power minus n pi the whole square T star only then do you get a non trivial solution and the special form of this constant if you recall was fixed from the requirement that you should have homogenous boundary conditions.

Only for specific values of this constant only if this constant is an integer times pi thus the solution for the function z satisfy the homogenous boundary condition that it has to be 0 on both walls. So, therefore, the fact that you have a homogeneous boundary condition in one direction fixes the value of this constant and fixes the value of the function these have to be sin functions.

The total equation for the temperature was of the form that was the total equation for the temperature and these coefficients now have to be evaluated from this initial condition these coefficients have to be evaluated from this initial condition and how did we evaluated using what are called orthogonality relations. For these functions which are called Eigen functions if for these functions in the homogeneous direction which are called Eigen functions.

Those orthogonality relations are out the form, integral d z star from 0 to 1 of sin n pi z star sine n pi z star is equal to delta n m by 2 - where delta n m is one if n is equal to m and is equal to 0 if n is not equal to m. So that was the procedure that we had reduced and we had finally, got the solution for the temperature field, the temperature it consists of the steady part 1 minus z plus the transient part the constant was minus 2 by n pi sin n pi z e power I will write that below so that was the final solution that we got there are some important intermediate steps that need to be kept in mind whenever you do a separation of variables solution.

You have to first identify the homogenous and the inhomogeneous direction.

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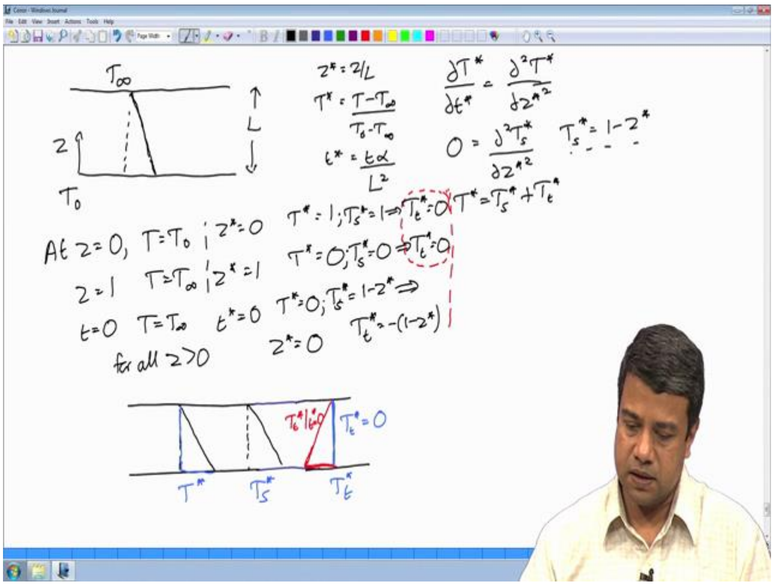

That is the reason that we have separated out the solution into a steady part plus a transient part because the total temperature field and the steady temperature field satisfy the same boundary conditions therefore, the transient temperature field has to be equal to 0 on both boundaries so that is the homogenous direction the in homogeneity is coming in at the initial time.

You could have problems where you have 1 spatial homogeneous direction and 1 spatial inhomogeneous direction, in any case you have to reduce the problem in such a way that the boundary conditions are all homogeneous in all directions except one and that one direction is the forcing direction. For the homogeneous directions the boundary conditions are all 0 either the temperature or the flux whatever it may be those boundary conditions are all 0 in all the homogeneous directions. The fact that you have boundary conditions equal to 0 implies that the solutions have to be of a special form as I showed you in the earlier lecture if I just assume a positive constant for the separation of variables procedure.

Assume a positive constant over here I do not get a non trivial solution the solution just tells me that the entire function has to be 0 everywhere even if I assume a negative constant if I did not assume a specific form of this constant as an integer the solution would tell me that the solution has to be 0 everywhere its only for specific values of this constant that the solution is non zero, those are a finite set I mean. So, they have to be a discrete set of values.

They cannot be just a continuous variation in values it has to be only for a discrete set of values in this particular case, it is an infinite set n can go from one to infinity, but it is still a discrete set and can have only integer values these are the Eigen values and the functions the functions here are what are called the Eigen functions.

For these Eigen functions with homogenous boundary conditions you can show that there do exist orthogonality relations that is the subject of applied mathematics which I will not be able to get in to in this course, but you can actually show that these orthogonality relations applied. So, therefore, even though I had an infinite set of constants I can evaluate each one of them because each constant is orthogonal to every other constant I am sorry, each function is orthogonal to every other function. So, it is like if I had a vector field.

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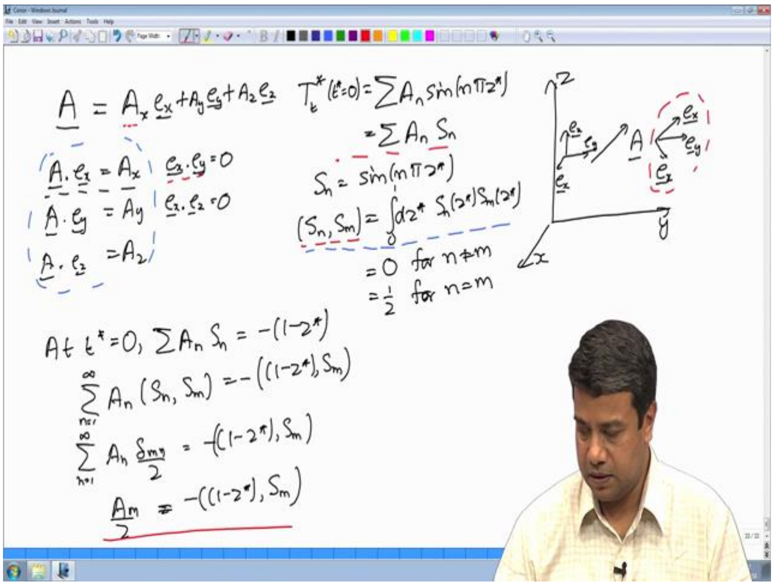

Let us say that the vector A I will put it with an under bar I can write it in terms of A x e x plus A y e y plus A z e z in some 3 dimensional coordinate system a is a vector in a 3 dimensional coordinate system.

Now, I can get each of these constants A x, A y and A z by dotting A with e x if you take the dot product A and d x I will get the component A x if I take the dot product of A dot e y I will get the component A y similarly if I take the dot product of A dot e z I will get the component A z. The reason I get this is because these vectors are orthogonal I know that e x dot e y is equal to 0 e x dot e z is equal to 0 and e y dot e z is equal to 0. So, they do form an orthogonal coordinate system they are perpendicular to each other e x, e y, e, z there are all perpendicular to each other. So, if I dot any 2 of them I will get 0.

I could not have derived this I could not have derived this, if I had used a coordinate system in which the vectors were not orthogonal to each other, if I used a coordinate system in which the vectors were not orthogonal to each other in which they were not perpendicular to each other I could not have got this. So, that is the importance of orthogonality relations in vectors.

Similar a place in this case except that it is an infinite series of functions I had z of z star, but other the transient temperature. So, let me see this was just a transient temperature at T equals 0 is equal to this function I can write this as the summation of A n times S n where A n is equal to a function sin n pi z.

Now, these functions satisfy orthogonality relations in the sense that the product of S n and A m if I define this product of A n and A n as integral 0 to 1 d z star S n of z star S m of z star I define this as the product of these 2 this is equal to 0 for n is not equal to m and is equal to 1, in this case it is 1 by 2.

This thing, in the case of these functions is the equivalent of the dot product in the case of vectors. So, since the product of 2 functions is equal to 0 if n is not equal to m right I can quite easily invert this equation here I know that that T star is equal to 0, I know that summation A n S n is equal to minus of 1 minus z star. If I take this inner product on both sides with respect to m where the inner product is defined inner product is defined in this manner then I will get sigma A n S n comma S m equal to minus the inner product of one minus z star comma S n.

Similar to what I am doing over here, I take the inner product of this with some unit vector similarly this with some function. This inner product only gives me A m. So, this gives me basically summation of A n times delta m n by 2 n is equal to 1 to infinity is equal to minus of one minus z star comma S m and since delta n m is nonzero only when n is equal to m this reduces quite easily this just gives me a m by 2 is equal to; ok.

That is the contextual underpinning, but behind writing these in terms of these basis functions the Eigen functions of the basic functions since we know that they are orthogonal we can determine each coefficient. On the other hand if we had taken some other function which was not orthogonal similar to some other set of vectors which was not orthogonal we could not have got each one of these individually.

In the case of vectors, there are only 3 relations you will get therefore, you can invert it to get 3 constants A x, A y and A z, in this case that an infinite number n. So, therefore, you cannot invert it quite simply essentially what we are doing is.

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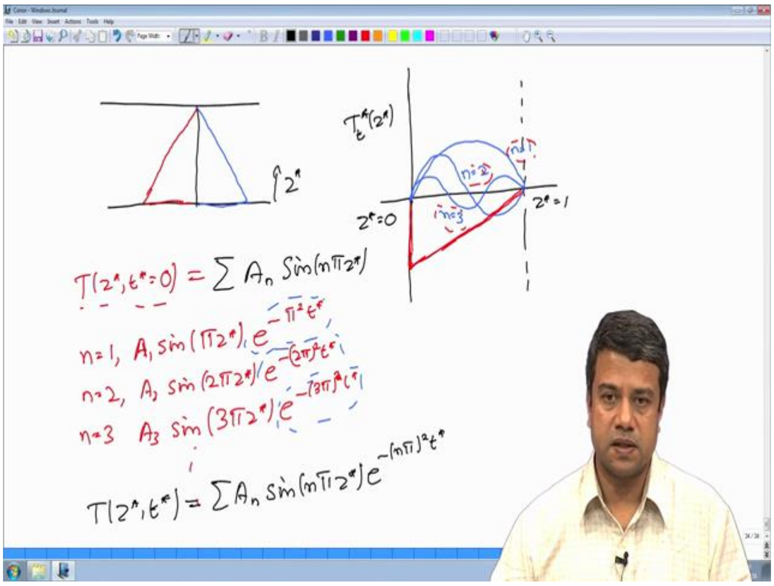

We know that at time T is equal to 0 at time T is equal to 0 the total temperature is 0 the steady temperature is given by this function the steady temperature plus the transient temperature has to sum to 0 therefore, the transient temperature looks something like this the sum of these 2 will be equal to 0.

If I plot this transient temperature, if I take this z star coordinate as a power of z star, I plot the transient temperature; z star goes from 0 to 1 so plot T T star of z star, I get a function that is, it should actually be negative in this case so, the way I get a function that looks something like this, (Refer Time: 26:11) transient part of the temperature get a function that looks something like this.

Note that this is 0 over here as well the reason is because you just increased the temperature. So, both the steady and the total temperature field are equal to 1 at this point. So, the transient part is actually equal to 0 at z star is equal to 0. So, it is a sharply varying function that looks something like this, it is equal to 0, z star is equal to 0 where it has as a linear function for any z star not equal to 0.

What you are doing is to express this function as the sum of sin functions, this is 1 function for n is equal to 1, this is a function for n is equal to 2. So, for n is equal to 3 will look something like this and so on; so expressing this function that I had here as a sum of sin functions each of which goes to 0 on both boundaries. So, you take each of these sin functions T I am writing this as the function as a sum of A n times sin of n pi z star.

I am multiplying each of these functions by a constant each of these functions is an infinite number of them by some constants so that when I add them all up I end up with the function that I had and from that I can evaluate these constants each of these functions decays at its own rate. So, for n is equal to 1, the function will be of the form sin pi z star e power minus pi square T star n is equal to 1 n pi, the whole square times T star. So, it is of the form sin of pi z star times z power n pi T star times A 1 for n is equal to 2, it is A 2 sin 2 pi. So, h star e power minus 2 pi whole square T star. So, what is the initial shape? I have decomposed into each of these sin functions each of these decays at its own rate for n is equal to 3, you will get A 3 sin 3 pi z star e power minus 3 pi the whole square T star and so on you will get higher and higher or term and each of these functions decays at its own rate.

Once from the initial shape of the function I have got all these constants I can multiply each of those functions by their own natural decay rate add them all up and I will get the solution at any instant of time. So, there is the basic principle in basically expanding these initial perturbation in the series of orthogonal functions in this case, the natural orthogonal function is the sin function with integer n, each of these decay is at its own individual rate and therefore, I can from the construction of the solution, I can get what is the temperature at any time is equal to sigma A n and you can see that for higher and higher n, these functions decay faster and faster. So, functions with n higher so, for example, function with n is equal to 2 d k is e power minus four pi square T star which is much faster than e power minus pi square T star. So, the lower functions actually decay much slower the higher functions for higher n the decay much faster.

If you are interested in only some particular resolution in the solution and if time is larger than 1, you are guaranteed that you can truncate the series at a finite set of terms in this series and neglect all the higher order terms because these higher functions decay much much faster than the lower functions. So, in that sense you can get an approximate solution for this problem there will be other problems in different configurations in which similar procedures will applied separation of variables you have to identify homogeneous boundary conditions in all directions except one in those homogeneous directions you have discrete solutions discrete Eigen values and orthogonal Eigen functions, the inhomogeneous part in terms of those orthogonal Eigen functions using orthogonality and calculate the coefficients in those.

Once you have those coefficients you can construct the entire solution as the sum of each of those orthogonal functions and you are guaranteed that any function can be written in this as a sum of these orthogonal functions because this function space is complete and orthogonal so that is the procedure that is used for solving problems by separation of variables we will see that in different configurations as we go along the course.

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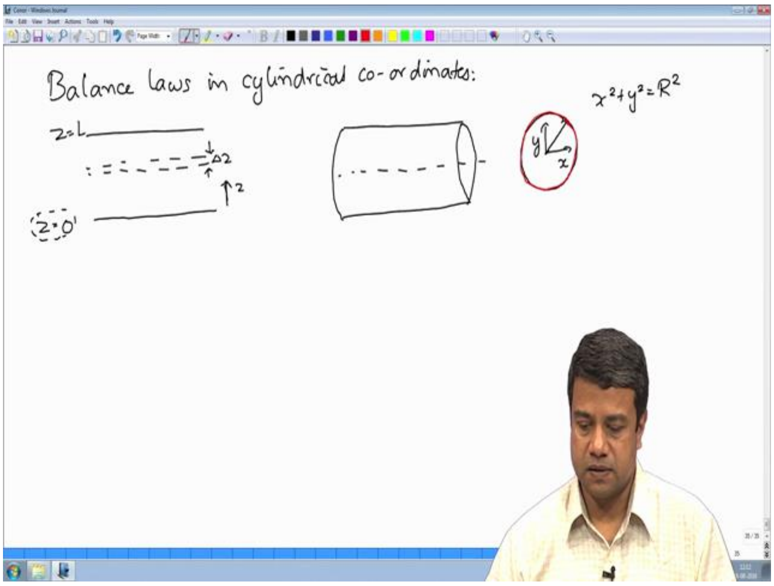

Next I would like to take up balance laws in cylindrical coordinates, the problems that we have solved so far are in what are called a Cartesian coordinate system, 2 surfaces with the coordinate perpendicular to the surface and therefore, we could write the 2 surfaces as z is equal to 0 and z is equal to L and then we could write a differential balance for a small interval delta z and then solve the problem within that small interval.

What if you are doing a solution to a problem in a cylinder for example, we often see these flows in pipes for example, these are all in cylinders and in that case the configuration has a cylindrical geometry if I could plot the cross section it looks like a circle.

Now, one could of course, use a Cartesian coordinate system and write down balance laws, in this Cartesian coordinate system and try to solve it; however, in a Cartesian coordinate system the way that I would have to describe this surface of this cylinder if I had let us say orthogonal coordinates x and y would be to say that x square plus y square is equal to r square. And then applying boundary conditions at the surface will become problematic because an expressed in terms of x and y I do not have boundaries at specific values of x and specific values of y rather I have some inconvenient combination of x and y.

It would be much easier if I could choose a coordinate which is parallel to the bounding surfaces, it could be convenient if I could have a coordinate which is parallel to the surface bounding, this geometry in other words, this surface going around should correspond to a surface of constant coordinate in that case specifying the boundary conditions because much simpler; however, specifying the balance equations becomes a little more complicated and so, I will talk about cylindrical coordinates I will start that in the next lecture and we will see how to solve problems in cylindrical coordinates. That is the program for the next series of lectures.

In Cartesian coordinates I have shown you 2 different ways of solving problems, one is the similarity solutions when there is a deficit of dimensions the second is the separation of variables, there is a third class of solutions for oscillatory flows which I have not yet covered we will see that as we go further in the context of pipe flows and so on. But these 2 procedures once again I will first frame equations in coordinate systems which are not Cartesian and then go through how these are solved. So, we will continue with cylindrical coordinates in the next lecture, we will see you then.