Transport Processes I: Heat and Mass Transfer
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Lecture - 28
Unidirectional transport: Separation of variables for transport in a finite domain cont

Welcome to this our continuing discussion on transport in 1 dimension, transport in a finite domain.

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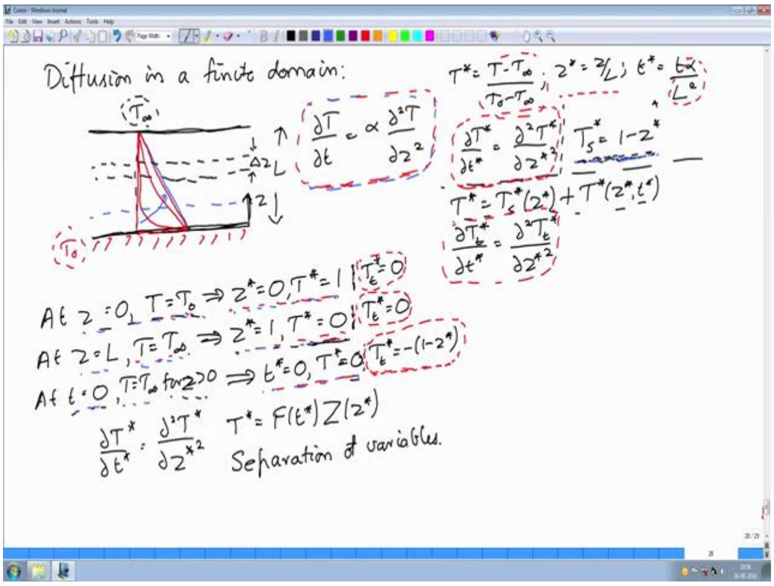

We derived the conservation equation unsteady transport in 1 dimension; the first derivative of temperature with respect to time is equal to the diffusion coefficient times the second derivative, with respect to the spatial coordinates. We had previously considered in the case of transport into an infinite fluid, presently we are considering transport in a finite domain. In this case the temperature of the surface at the bottom is T naught, the temperature of the surface at the top is T infinity and we considered an initial configuration where the entire fluid is all at constant temperature T infinity. So, at time T is equal to 0, the heater has been instantaneously switched on so that the bottom surface temperature is instantaneously increased to T naught.

We had done the scaling so that the scale temperature varies between 0 and 1 throughout the domain and when you express the equation times the scale temperature of course, the non dimensionalization cancels out on both sides and you get the exact same equation except that it is now in terms of T star, we had scaled the length by L the height you would expect Z to vary from 0 to L. So if we scale Z by L, the scaled coordinate varies between 0 and 1; that is the natural scale to scale the Z coordinate by and we had obtained a natural scaling for that time; T alpha by L square. Alternatively this particular term I can write it as T by L square; L square by alpha is a diffusion time, a time it takes for energy to diffuse over a distance comparable to L. So, that is the diffusion time and once you do that the scaled equation that results does not now depend upon the diffusion coefficient at all.

Because I have scaled the variables in such a way that I get an equation that is independent of the absolute temperatures, independent of the diffusion coefficient. In the long time limit, at steady state the temperature has to go to a profile that is independent of the time, it has to go to some steady value; that means, for the steady temperature just the right side of this equation has to be equal to 0 because there should be no time dependence. Since the second derivative is equal to 0, the temperature is a linear function of position, the steady temperature and the linear function that satisfies the boundary conditions on both surfaces is this the linear temperature that satisfies both boundary conditions that is this one.

So, then we had expressed the temperature in terms of the sum of the steady part plus a transient part. So, T was expressed as a sum of a steady part plus the transient part and you have the balance equation for the total temperature, you have the balance equation for the steady temperature subtract the two; you will get a balanced equation for the transient temperature and that equation has exactly the same form as the equation for the total temperature. So, the transient part of the temperature as the same equation as the total temperature

Similarly, you have boundary conditions for the total temperature, boundary conditions for the steady temperature. In this case both boundary conditions for the total temperature and the steady temperature are both the same therefore, the transient temperatures has to be equal to 0 at both boundaries, this is the boundary condition for the total temperature is the same as the boundary condition for the steady temperature. Therefore, the boundary condition for the transient temperature has to be equal to 0 at both boundaries it is not 0; however, at initial time. At initial time the total temperature is

0; T star is equal to 0, the steady temperature is independent of time. So, it is still equal to 1 minus Z star and that gave us an initial condition for the transient temperature; as minus of 1 minus Z star.

So, that is the equation that we have to solve and the boundary conditions that have to be used and we had progressed to try to solve this using the procedure called separation of variables. In the separation of variables procedure, the transient temperature it is written as the product of 2 terms; one of which is only a function of time, the other is only a function of this spatial coordinate. So, capital F is only a function of time, capital Z is only a function of the spatial coordinate; we substitute that into the balanced equation.

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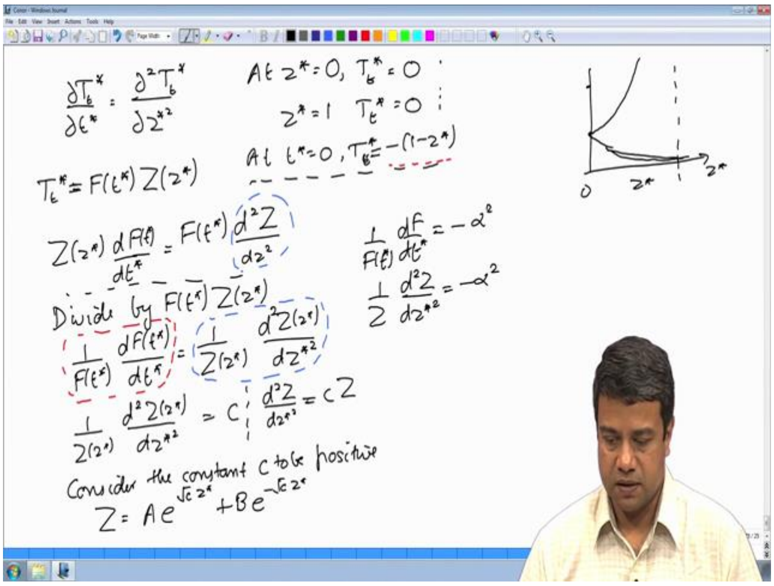

And then get equations individually for F and z, you can see that I have substitute that in and then divide throughout by the product F times Z and if you do that, you get an equation in which the left side is only a function of time, the right side is only a function of z. So, if these for any general functions if you kept time a constant and change Z, the right side would change and the left side would not in the equality will no longer be satisfied. Simultaneously I mean alternatively if you kept Z a constant and look at different instance in time, the left side would change the right side would not.

The only way that this equality will be preserved for all values of that Z and T because if both sides are equal to constants. So, first we had postulated that both sides are equal to some constant; that constant put in general be either positive or negative, we had first looked at the case where the constant is positive.

In that case the solution for Z is of this form is of the form of exponentials A e power square root of c times Z star plus B e power minus square root of c times Z star. So, it consists of an exponentially increasing and an exponentially decreasing function because the linear combination of those two.

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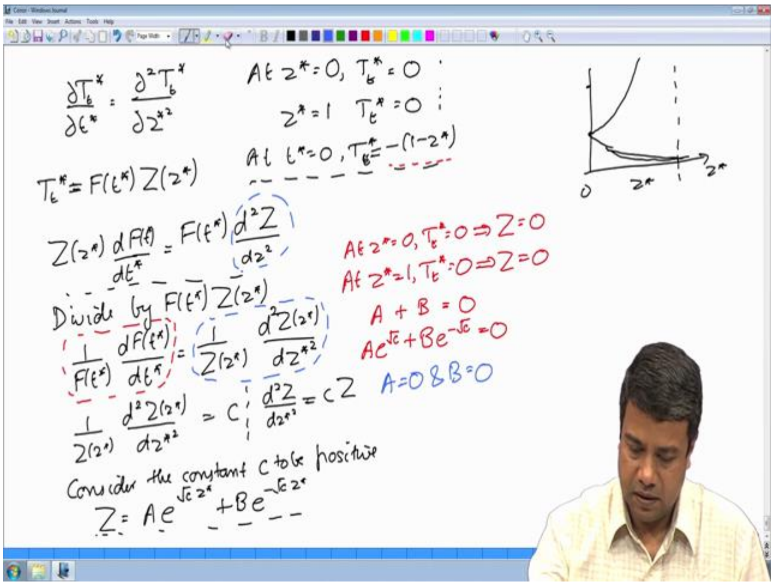

If you try to apply the boundary conditions now, at Z star is equal to 0; T; t star is equal to 0, the transient part has to be equal to 0; which implies that capital Z has to be equal to 0. So, therefore, A plus B equal to 0 both the exponentials are just equal to 1; at Z star is equal to 1; T t star is equal to 0, I told you that the transient part of the temperature is 0 on both boundaries which implies that this Z is equal to 0, This will give us A e power root C plus B e power minus root c is equal to 0 and if you solve this equation for any value of c, the only solution you get is that A equals 0 and B equals 0.

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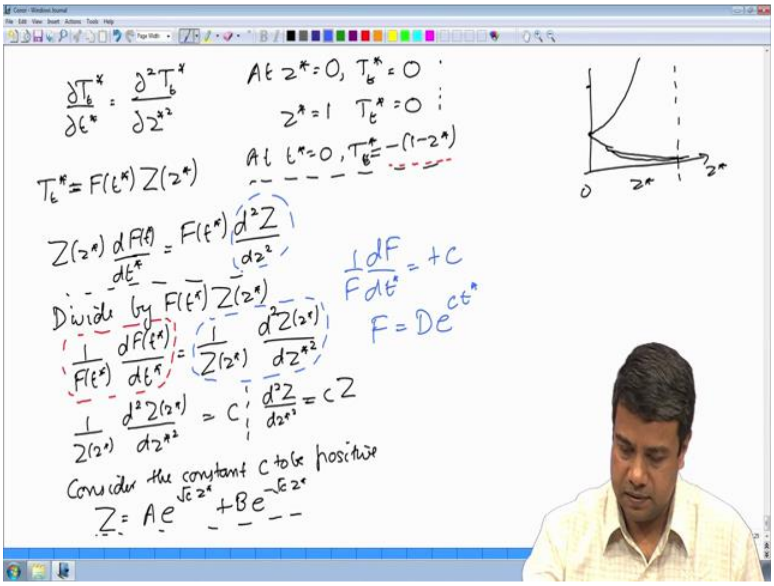

So we get only a trivial solution, the same applies for the function of time as well because if I have a function of the form D F by d T times 1 over F is equal to plus some constant a positive number, the solution is that F is equal to De power c times T; this is an exponentially increasing function in time. So, therefore this function never decreases to 0 in the long time limit, so clearly the positive constant will not give us a non trivial solution.

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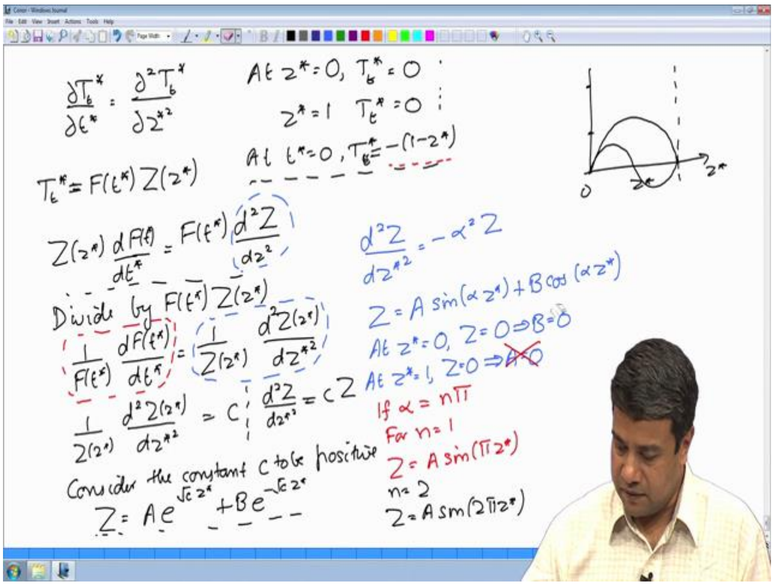

What is the constant is negative? I have a function of the form d square Z by d Z star square is equal to minus alpha square Z, where alpha is some positive number. The solution of this is Z is equal to A sin alpha Z star plus B cos alpha Z star; that is a solution for this. At Z is equal to 0; capital Z has to be equal to 0 these are the transient temperature has to be equal to 0. So, at Z star is equal to 0, the sin is 0 cos is 1, so this basically tells me that B has to be equal to 0, at Z star is equal to 1; capital Z has to be equal to 0 that implies that A equals 0 or does it.

Can I get the solution which is 0 at Z star is equal to 1 even if A is not equal to 0 and I can get it only if this constant alpha has a special value; if alpha. So, you do not require this condition that A has to be equal to 0 because if A is equal to 0, you end up with a trivial solution once again. It is also possible to get Z is equal to 0; at Z star is equal to 1 if right this factor alpha is equal to n pi where n is an integer. So, you take the case of n is equal to 1 right; that means, that Z for n is equal to 1, Z is equal to a sin; pi Z star and Z star is equal to 1, sin function is equal to 0. So, you get a function that looks like this the sin function it goes like this.

For n is equal to 2; Z is equal to A sin 2 pi Z star that is equal to 0 at Z star is equal to 1; you get a function that goes like this, n is equal to 3, 4 etcetera. For any integer value of n this solution is 0 at Z star is equal to 1 even when the constant k is not equal to 0. So, you require special values of this constant in order to ensure that you get a solution that satisfies both the boundary conditions. So, these are what are called the eigen values.

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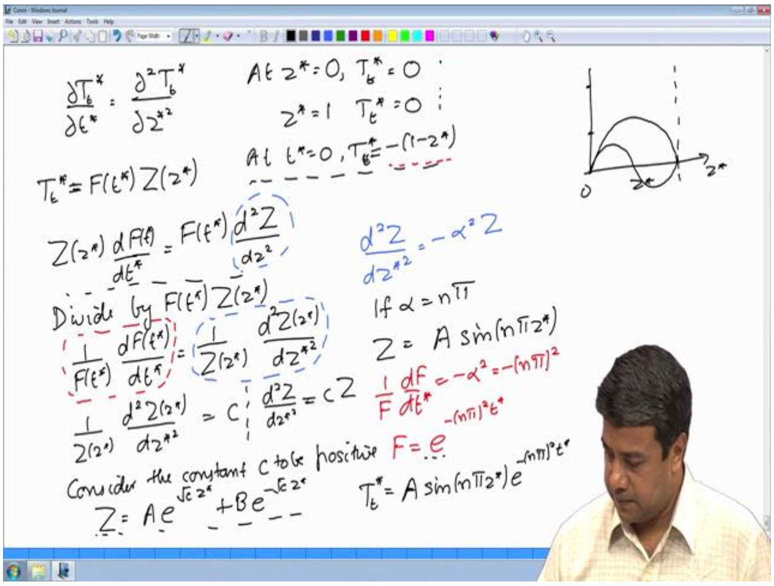

So, if alpha is equal to n pi; Z is equal to A sin n pi Z star; this satisfies both the boundary conditions, this satisfies both the co ordinate conditions if n is an integer, the condition at Z star is equal to 0 and the condition at Z star is equal to 1, but in order to solve the other equation; 1 by F; dF by dt star is equal to minus alpha square which is equal to minus n pi the whole square. This solution is F is equal to e power minus n pi the whole square t star.

So, therefore, the temperature solution T transient star is equal to the constant I have not use the constant here because that constant can be absorbed into this constant, if the constant A sin; n pi Z star e power minus n pi to the whole square T star. So, that is the solution for the transient part of the temperature. What should be the value of n, you can see that both the boundary conditions are satisfied for any value of n because sin of n pi Z star will be 0 at both Z start is 0 and 1; for any integer value of n, so n can have any integer value.

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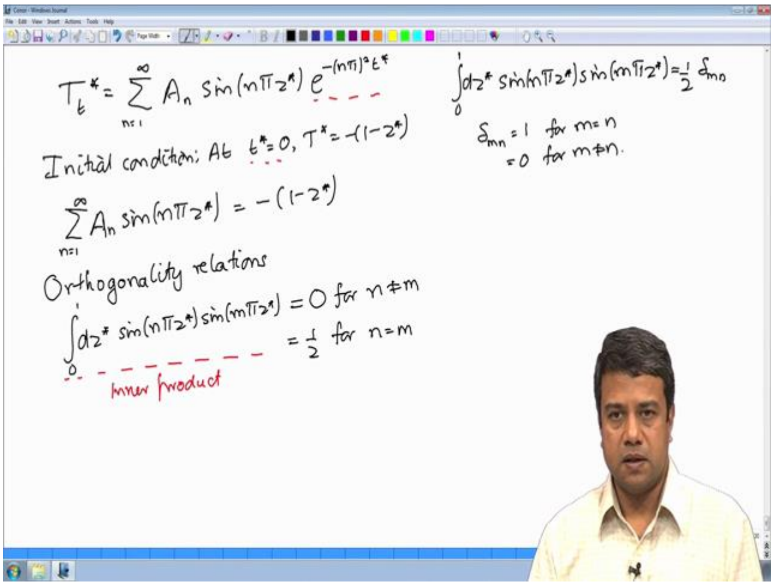

So, the most general solution for this transient part is going to be equal to a summation over all the solutions; of some constant times sin n pi Z star; e power minus n by the whole square T star and can go from 1 to infinity. It could also be 0, but for n is equal to 0 the function which was 0, so the summation has to go from n is equal to 1 to infinity. How do we determine the constants in the series; I said it can be any integer value of n, so the most general solution is one in which you sum all of these functions; pre-multiplied by some factor.

What should be the value of these constants, the solution satisfies the 2 boundary conditions at Z star is equal to 0 and Z star is equal to 1. We have not yet satisfied the initial condition at T star is equal to 0, the initial condition that we had was that at t star is equal to 0; capital T star is equal to minus of 1 minus Z star.

So, therefore I have this summation A n sin n pi Z star, at time T is equal to 0; this term this becomes 1 because when T star is equal to 0; the exponential is just 1. So, I just get this summation n is equal to 1 to infinity of n pi Z star is equal to minus of 1 minus Z star.

This is one equation which contains an infinite number of coefficients, how do I obtain a solution for this equation. It turns out that these functions sin of n pi Z star are special functions in this domain between 0 and 1, they are called orthogonal functions they satisfy relations which are called orthogonality relations and these can be used in order to find these coefficients.

If I take the integral of 2 such functions sin n; pi Z star, sin of m pi Z star where n and m are 2 integers let take the integral from 0 to 1 I take this integral of sin n pi Z star sin n pi Z star from 0 to 1 you can easily verify that this will be equal to 0 for n not equal to m. The way you have to do it is to express this as the sum of 2 cos functions and once you do that, you will see that those functions have the same values at both the upper and lower limits. So, for n is not equal to m this summation is identically equal to 0, it is not 0 for n is equal to m; the reason is because then I get sin square of n pi x star and the sin square function has to be positive throughout the domain and therefore, the integral has to be nonzero.

You can get that integral by writing this sin square function as half of 1 minus cos 2, so in the end you will get a value of half for n is equal to m. So, these are orthogonal functions in the sense that you should multiply one function by some other function and integrate it, the result is equal to 0.

This nomenclature derives from what is used in vectors' 2 vectors are orthogonal if the dot product of those 2 vectors is equal to 0. Similarly in this case 2 functions are orthogonal if this thing is what is called the inner product of these 2 function is nonzero, so in that case this goes by the name of an orthogonal function. So let us just put the orthogonally relation here; I will write it as integral 0 to 1, T Z star sin n pi Z star, sin m pi Z star I will write this as 1 by 2 times delta m n.

We can write this as 1 by 2 times delta m n, where the meaning of delta m n is that delta m n is equal to 1 for m is equal to n; n is equal to 0 for m not equal to n. So, this is the meaning of a delta function; an orthogonality relation. So now I can use this in this case, so in order to find the constants, I multiply both sides of the equation by sin of m pi Z star integrate from 0 to 1.

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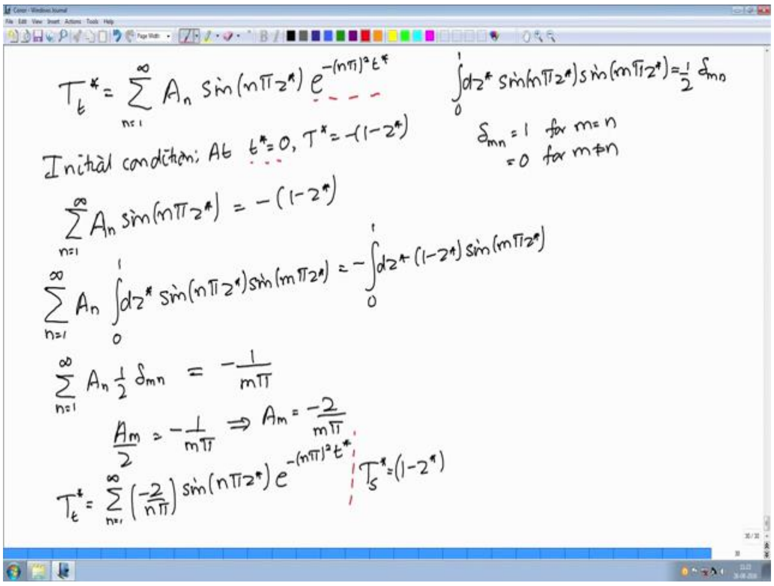

So, I have here summation n is equal to 1 to infinity, An integral 0 to 1 d Z star sin is equal to minus integral 0 to 1 take multiply both sides of the equation integrate by sin of m pi Z and integrate from 0 to 1 where m is any integer the left side we have just calculated sigma n is equal to 1 to infinity A n into 1 by 2 delta m n.

This is non 0 only when n is equal to m and on the right side you can carry out the integral I would not go into the details, but what you will basically get is minus 1 by m pi should do this integral it is not too difficult you get minus 1 over n pi. If you look at the left hand side, I have a summation of A n times delta m n; delta m n is nonzero only when n is equal to m all other terms in the series are multiplied by 0. So, the left side effectively reduces to Am by 2; which implies that Am is equal to minus 2 by; so those are the constants in the series. So, the transient temperature profile T transient is equal to sigma n is equal to 1 to infinity ok minus 2 pi; m pi.

I am sorry we should use n here because I have used an as the index that is the solution for the transient temperature and of course, the steady temperature is just 1 minus Z star. So, the total temperature is equal to the sum of the steady temperature plus the transient temperature, so that is the total temperature. So that is the solution for this problem, it is the formal mathematical solution.

What is the physical reasoning behind the solution of this kind; these orthogonal functions I am emphasizing on this because this is something that we will see repeatedly through this course. Whenever we are solving problems in finite domains, we will inevitably end up with a partial differential equation; that partial differential equation contains derivatives in multiple directions; in order to simplify it you use a separation of variables method and when you do that, you will get these kinds of ordinary differential equations in each direction.

Those ordinary differential equations can be solved only if you have homogeneous boundary conditions in all directions except 1. In this problem we saw that we had homogeneous boundary conditions in the Z direction, initial at time T is equal to 0 was the forcing. When you have homogeneous boundary conditions, the constants in the equations reduced to Eigen values; you have specific values. In this case it was n pi which are necessary for satisfying these boundary conditions and the solutions in the form of eigen functions which satisfy these homogenous boundary conditions and if you have that, if you have these eigen function solutions then you can find an orthogonality relation for these which will enable you to determine all the constants and thereby solve the problem.

I will just give you a little bit more physical insight into this process in the next lecture before we proceed because this method of solving; the first one we saw infinite domains was a similarity solution. This one was in a finite domain; it involved a separation of variables procedure and determining Eigen function solutions in the different directions. So this procedure; I will once again revise in the next lecture before we go on to looking at domains in of other types; cylindrical domains and spherical domains in the next lecture. I will continue this in the next class, we will see you then.