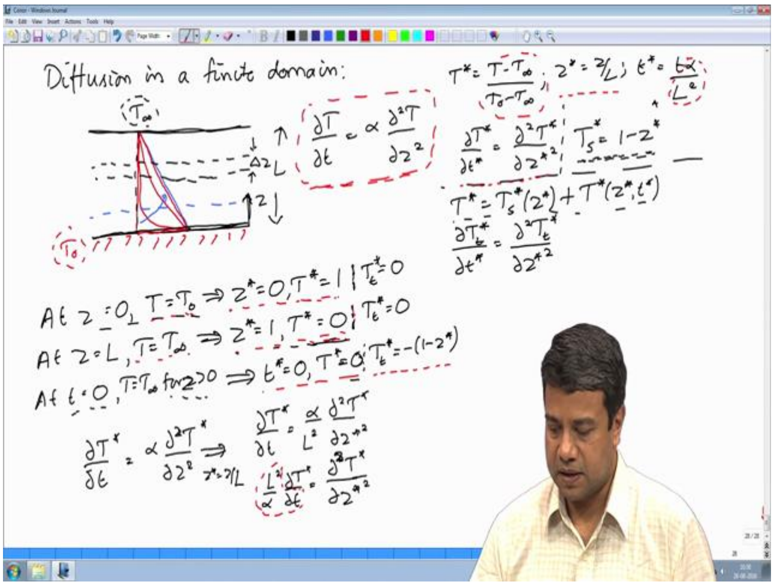
Transport Processes I: Heat and Mass Transfer
Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture – 27
Unidirectional transport: Separation of variables for transport in a finite domain continued

This is our continuing discussion on transport in 1 dimension, in our course on fundamentals of transport processes. If you recall, we had derived a balanced equation for unsteady transport in one direction in a finite domain.

(Refer Slide Time: 00:39)



In this domain of where there is variation only in the z direction perpendicular to the plates and there is no variation in the x and y directions. So, we have assumed that this is effectively an infinite domain in the x and y directions and all variations are only in the z direction.

In the previous lectures, we had solved problems in an infinite domain, in that case as well the balance equation was obtained by writing a balance across a small differential volume of thickness delta z what comes in minus what goes out plus any accumulation is equal to the net rate of change of material within this differential volume and on that basis we have got this balance equation the diffusion equation it contains one derivative in time and 2 derivatives in the spatial coordinates.

In this particular case, we assumed that there is no convective transport in the direction perpendicular to the plates. So, what we have is 2 surfaces, one surface at temperature T naught, the other surface at a lower temperature T, infinity at initial time, we assumed that the entire fluid is at temperature T infinity. So, the entire fluid is at one constant temperature and instantaneously at T is equal to 0, if the temperature is T infinity for z greater than 0; however, at z is equal to 0 I have imposed the temperature T naught on the bottom surface so that was the problem that we are considering.

The same equation, we had used earlier to get similarity solutions those solutions were in an infinite domain in that case, there was no length scale in the problem and therefore, just based upon dimensional analysis I could reduce the problem to just one similarity variable in an infinite domain and we had seen that the penetration depth in that case is proportional to square root of alpha times T just based upon dimensional analysis or the extent of spread due to a pulse input was proportional to square root of alpha times T in those cases.

In this case, we are looking at a finite domain and therefore, there is the possibility of scaling with the length by the height of the channel and that is what we had done in the previous lecture, we had scaled T by T minus T infinity divided by T naught minus T infinity so that when T is equal to T naught, we find that T star is equal to 1 and when T equals T infinity T star is equal to 0. So, this way the scale temperature varies between 0 and 1.

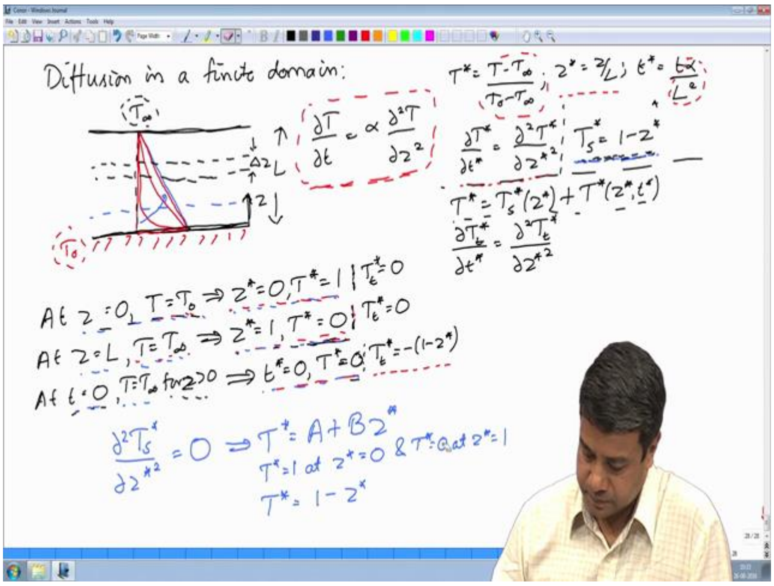
Therefore, this solution will be common for any temperature of the top and bottom plates provided and expressed it in this scaled form, we also had a length scale that is the height of the channel, in this particular case, we could scale length by the height of the channel and we insert that into the equations and we get a natural scale for time. So, if you insert these 2 into this conservation equation, what you get is that partial T star by partial T is equal to alpha partial square T star by partial z square, if I scale the z coordinate z star is equal to z by l what you get is partial T star by partial T is equal to alpha by l square partial square T star by partial z star square I can rewrite this as l square by alpha partial T star by partial T is equal to partial square T star by partial z star.

Now, the right side is entirely dimensionless on the left side therefore, this is effectively a dimensionless time therefore, I had defined T star as T times alpha by l square and once I do that I get an equation that now no longer depends on either the length in the problem l nor on the thermal diffusion coefficient alpha and all you get is an equation which basically balances the first derivative of temperature in time with the second derivative in the spatial coordinates. So, that is the problem that we have to solve and if you recall; in the last lecture we had separated, we had first also expressed the boundary conditions and the initial conditions in terms of T star at z is equal to 0 T is equal to T 0 and that implies that at z star is equal to 0 T star equals 1 at z is equal to l T is equal to T infinity which means that at z star equal to 1, the scale distance is equal to 1 T star is equal to 0, since T is 0 on the top surface, 1 at the bottom surface and at initial time T star is 0 for all z except at the bottom surface which has just been heated up to T star is equal to 1. So, T is equal to 0 T is a infinity for z greater than 0; that means, that T star is equal to 0 at time equals 0.

Physically what would you expect initially the fluid is all at temperature T star equals 0, instantaneously the bottom surface is heat to a temperature T star is equal 1 therefore, that heating progresses from bottom progressively upwards, what is the temperature profile in the long time limit? One would expect the temperature profile to tend to a constant value in the long time limit, it should be independent of time it should not be a constant in space. So, you should be independent of time profile should be independent of time in the long time limit.

Therefore the steady profile should satisfy the equation.

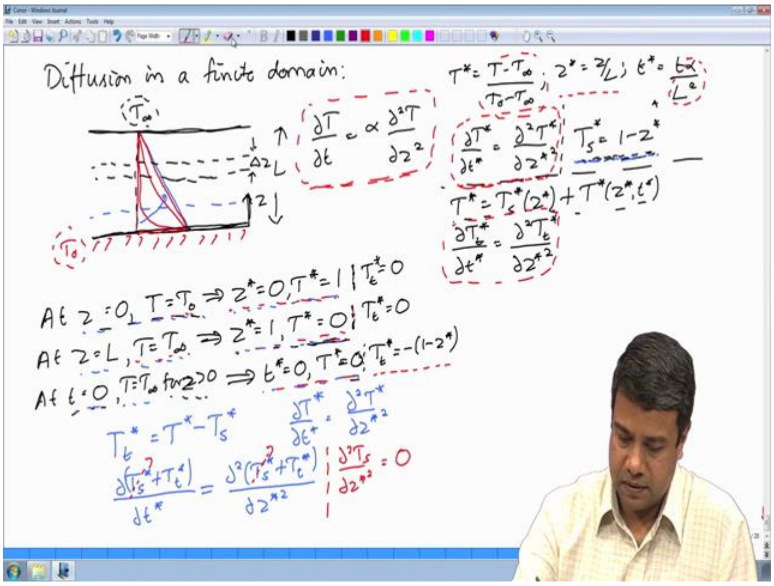
(Refer Slide Time: 07:44)



D square T steady by d z stars square is equal to 0 because it should be independent of time this gives a linear profile T star is equal to A plus B z star in the long time limit, this temperature profile has to have boundary conditions T star is equal to 1 at z star is equal to 0 and T star is equal to 0 at z star is equal to 1, it goes with the boundary conditions for this temperature profile and on that basis, you just get a simple equation for T star which is 1 minus z star. You can see that this linear relation satisfies both boundary conditions that was the steady temperature that we have to obtain that was a steady temperature that we had obtained in the previous lecture.

Now, what is the correction to that steady temperature at initial times, what is the difference between the temperature and the steady temperature at initial times when time is not small?

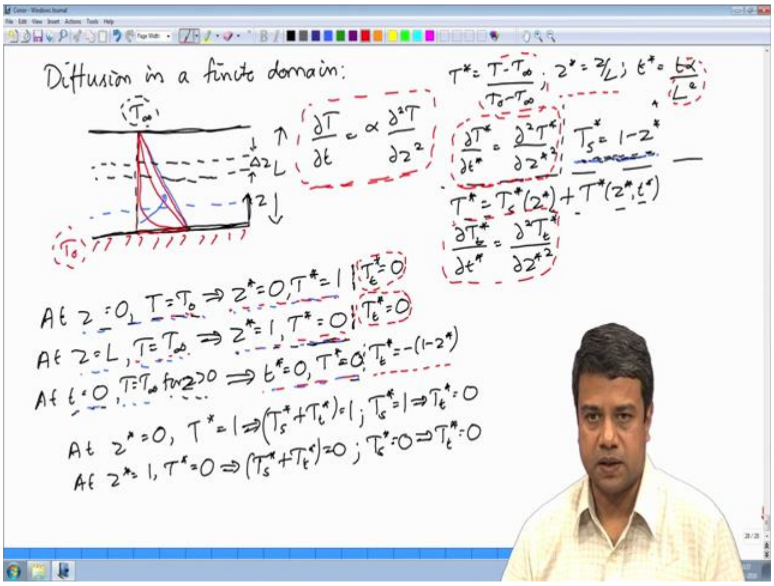
(Refer Slide Time: 09:05)



The difference is what we had called the transient temperature T, T star is equal to T star minus T s star the transient part in the limit as T going to infinity you would expect this transient part to go to 0 because the temperature should approach the steady temperatures in the long time limit. So, in that case, this transient part should go to 0 therefore, for this transient part I know what is the balanced equation for the total temperature partial T by partial T equal to d square?

If I express it the sum of the steady plus the transient part, I will get partial T steady plus T transient so that is the equation expressed in terms of the steady in the transient part, we know that the steady part has to be independent of time. So, this derivative is equal to 0 because the steady part has to be independent of time. Also we know that d square of T trans steady by d z square is equal to 0 we know that d square T steady by d z square is equal to 0, we just solve that to get the linear temperature profile therefore, this term is also equal to 0. So, the transient part satisfies the same equation as the total temperature the transient part of the temperature profile satisfies the same equation as the total temperature profile. So, this is the equation for the transient part it is identical to the equation for the total temperature profile.

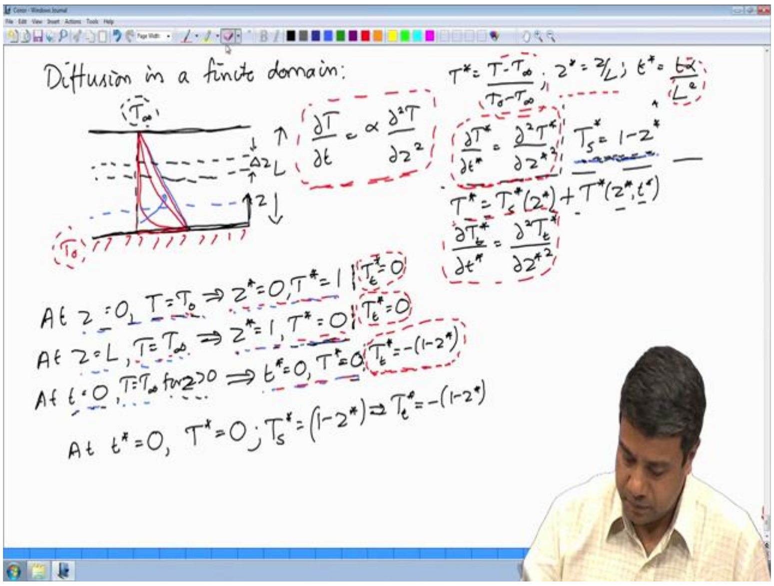
(Refer Slide Time: 11:30)



The boundary conditions are not the same the boundary conditions are not the same because at T star at I am sorry, at z star is equal to 0, the total temperature is equal to 1 which implies that T steady plus T transient is equal to 1, we know that the steady part just does satisfy this boundary condition, the steady part does satisfied this boundary condition and that implies that the transient part has to be equal to 0, so that at z is equal to 0, the transient part of the temperature is equal to 0 the transient part of the temperature is equal to 0.

Similarly, at z star is equal 1, T star is equal to 0 and that implies that T steady plus T transient is equal to 0; however, we also know that T steady is equal to 0 from this equation, here we know that T steady is equal to 0 at z is equal to 0 which means that T transient is also equal to 0. So, those are the boundary conditions as you can see both boundary conditions the transient part is equal to 0 on both the boundaries it is not 0; however, at the initial condition at the initial condition it is not equal to 0.

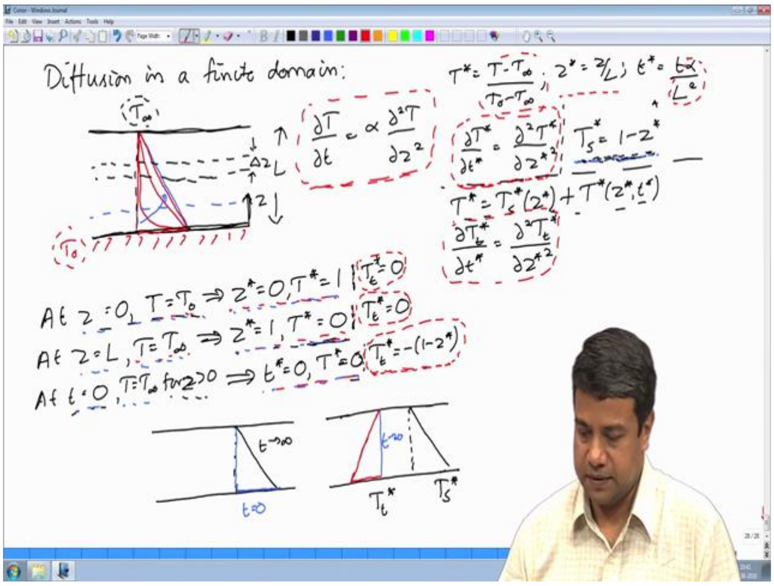
(Refer Slide Time: 13:15)



If we take for example, at T star is equal to 0, we know that the total temperature is equal to 0; however, the steady temperature is independent of time. So, T steady is equal to 1 minus z T steady is equal to 1 minus z that implies that T transient if T steady and T transient are to sum to 0; that means, that T transient has to be equal to minus of 1 minus z. So, the initial condition for the transient temperature is equal to 0 as shown here; however, the boundary conditions is am sorry, the initial condition for the transient temperature is non0; however, both the boundary conditions are identically equal to 0.

Physically what have we done here?

(Refer Slide Time: 14:31)

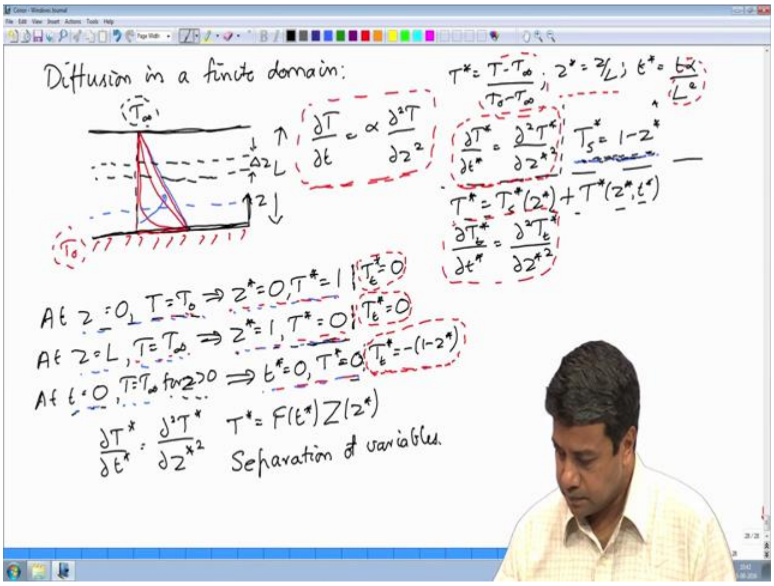


Physically what we have done is the following, the total temperature profile the total temperature profile in the long time limit is a linear temperature profile in the long time limit it is a linear temperature profile at time T is equal to 0 the temperature is 0 everywhere except at the base where is equal to one. So, that is at T is equal to 0. So, this temperature is at T is equal to 0 this temperature is at T, I am sorry, T tending to infinity this temperature profile I have written as the sum of 2 parts. One is a steady part the linear temperature profile one is the steady part the linear temperature profile which is the same as the temperature profile s T goes to infinity plus a second transient part plus a second transient part.

That transient part in the limit as T going to infinity it has to effectively b equal to 0, the transient part of the temperature as T goes to infinity this has to be equal to 0 because the sum of these 2 has to be equal to the steady temperature. So, sum of these 2 parts. So, and plot here this is T steady and T transient this transient part as T going to infinity, this has to be effectively 0 so that I recover the steady temperature and the transient part at T is equal to 0 it has to look something like this.

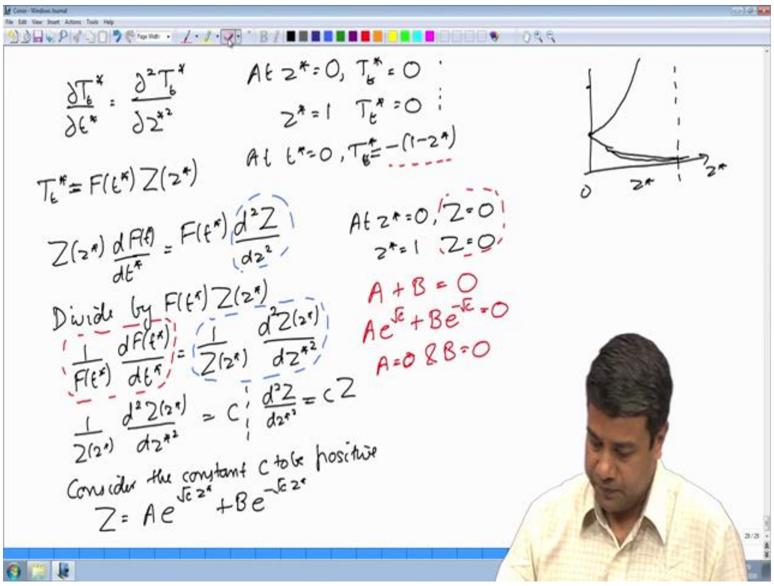
The transient part at T is equal to 0 has to look something like this so that at T is equal to 0 when I add up this red curve and the black curve I get the total temperature profile. So, that is basically the difference; the rewriting that I have done here we have already got a solution for the steady part. So, how do we get a solution for the transient part?

(Refer Slide Time: 17:11)



The equation for the transient temperature profile is of the form - D square T I am sorry, DT this is a partial differential equation and there is no straightforward method to solve it there is no systematic method to solve it. However, if I can write this temperature as some function some function of time times some function are called capital Z of Z star and insert into to this equation I can try to find what are the individual functions f and z this method goes by the name of separation of variables.

(Refer Slide Time: 18:16)



The way it works as follows my equation is partial T by partial T is equal to partial square T by partial z square with the conditions at z star equal to 0 T star take only the transient part here is equal to 0 at z star is equal to 1 T transient is equal to 0 at T star is equal to 0 T transient is equal to minus 1 minus z. So, we have homogeneous boundary conditions in this spatial coordinate and the forcing is coming in the temporal coordinate in time, the forcing is at initial time, I should take this as T star 0 minus the forcing is at initial time.

Now, I write those transient parts as f of T times another function of z and I insert it into this equation. So, one function is only a function of time the other function is only a function of z. So, if I put it in I will get z of z times d f by d T, note that f is only function of T is equal to f of T times d square capital z by d z square and now we divide both sides by f of T times z of z divide both sides by this product. So, you will get one over f of T d f yes that is the equation in this equation the left side if the left side depends only upon time in this equation, the left side depends only upon time and the right side depends only upon z coordinate, we have a situation where the left side depends only upon time and the right side depends only upon the z coordinate.

Therefore, if I keep time a constant and vary z the right side will change and the left side will remain a constant and the equality is no longer valid in general, if the right side depends upon z, the left side depends only upon time, if I keep time a constant look at 1 time instant and change z the right side will change in the left side will not alternatively if I keep the location a constant and look at different instants of time the left side will change and the right side will not and the equality will no longer be valid.

This equality is valid for any d and any z only if both the left and right are equal to constants so that even if I vary z or if I very time this function is equal to a constant and I should have a second derivative here my apologies d square z square there is a second derivative from yeah this. So, this separation of variables procedure basically tells us that both the left and the right sides have to be constants what constant should there be?

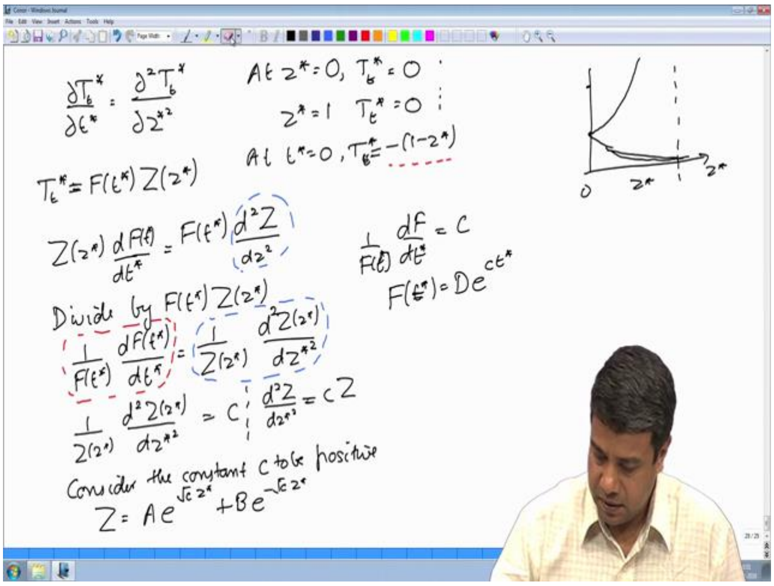
Let us take the right side first is equal to some constant c, let us assume for a moment consider the constant c to be positive if I consider the constant c to be positive this equation effectively becomes d square z by d z star square is equal to c z if this is positive we know the solutions it has to be of the form of exponential functions the solution that I get is z is equal to a e power root c z star plus b a power minus root c z star.

These are exponential functions if I plot these functions z start going from 0 to 1 if I plot these exponential functions one of these functions will decay exponentially. So, both of these functions will be 1, at one of these will decay exponentially and the other will actually grow exponentially the other function will effectively grow exponentially ok. So, these are the functions we have the boundary conditions at z star is equal to 0 z is equal to 0 and at z star is equal to one z has to be equal to 0 you recall that at both at T c z is equal to 0 and z is equal to one t. T has to be equal to 0.

The part of T, the transient temperature that depends upon the z is capital Z and therefore, if it has to be 0 at both z is equal to 1 and 0; that means, that this capitals Z has to be equal to 0 in both boundaries. So, if you put this n. So, what you will get from these boundary conditions is that A plus B equal to 0 and A e power root c plus B e power minus root c is equal to 0 the only solution that I get is a trivial solution A is equal to 0 and B equals 0. Effectively Z is equal to 0; the transient temperature is equal to 0.

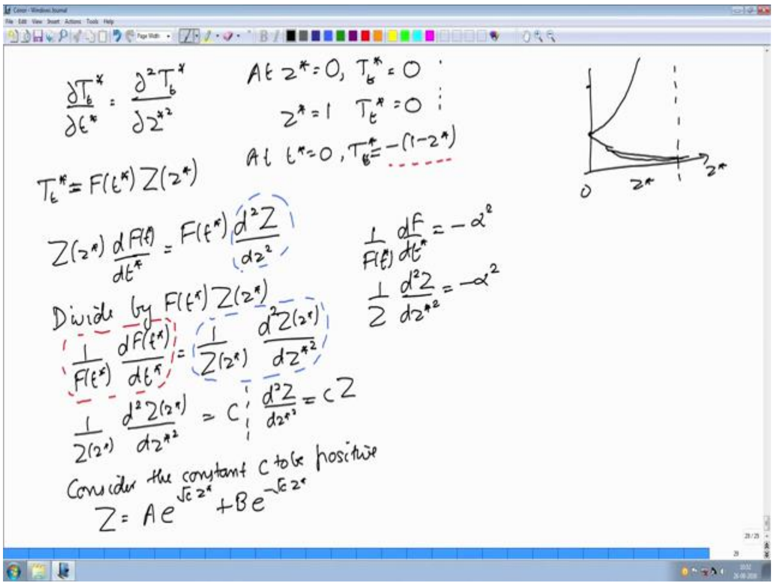
So, assuming the positive value for c does not give me a non trivial solution for the transient part of the temperature. Assuming the positive value for c does not give me a non trivial solution for the transient part of the temperature. It does not also give me a non trivial solution for the temporal part because if c was positive if c was positive, on the left hand side I have 1 over F of T, d F by d t is equal to T star is equal to c.

(Refer Slide Time: 26:16)



If I have an equation of this kind then f has to be equal to some constant, again f of T has to be equal to some constant in order to see here D times e power plus c t star. So, therefore, the function F also does not decay to 0 as t goes to infinity. We said that the transient part of the temperature should decrease to 0 as t goes to infinity for a valid solution because the temperature should reach the steady state value and the positive value of c does not give me that solution, it predicts that the temperature is constantly increasing therefore, this requirement itself tells us that the constant c should in fact, be negative if the constant c should. In fact, be negative; if it is negative what are the kinds of solutions that you will get?

(Refer Slide Time: 27:44)



We have now established, we have now established that each of these functions 1 by F of t d F by d t should be negative we will call it as some number alpha square where alpha is the positive number and 1 by Z, d square Z by t z square is scaled variables here should be also equal to minus alpha square where alpha is the positive number because if the requirement that both of these functions should be negative. What should be the value so that we can get a non trivial result for the temperature field? That I will continue in the next lecture.

Please keep in mind all of the reasoning that has gone on. So, far this is important, we had started off with a total temperature profile separated out into a transient and a steady profile and the steady profile was linear the transient part had 0 boundary conditions, in the spatial direction on both the boundaries the in homogeneity was at the initial time T is equal to 0. We had used separation of variables and got an equation where the left side was only a function of time, the right side was only a functional of z, we had said that on that basis both have to be constants and we are now discussing what the value of that constant should be, we will continue this in the next lecture, I will see you then.