Transport Processes I: Heat and Mass Transfer

Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture – 26
Unidirectional transport: Separation of variables for transport in a finite domain

Let us continue our discussion on unidirectional transport in our course on fundamentals of transport processes. Last lecture we were looking at transport in the infinite domains, where the thickness of the domain, the domain extends to infinity in all directions and we looked at 2 different problems - one was the conduction from flat plate which is instantaneously heated at time T is equal to 0, the second was the decay of a pulse that was instantaneously injected into a fluid at the origin.

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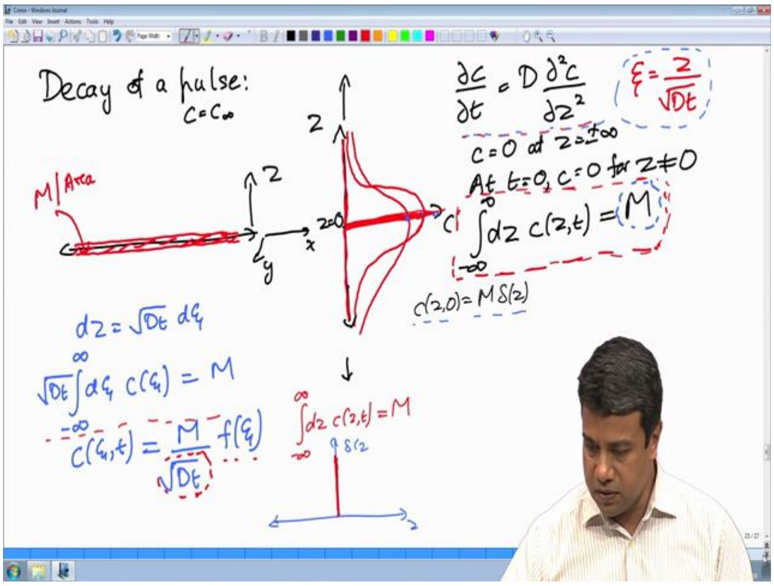

Concentration conservation equation in both cases was given by this diffusion equation d c by Dt is equal to the diffusion coefficient time T square c by d z square and for this problem of the decay of a pulse, we had considered that there was a certain mass injected within a thickness which was infinitesimally small and as I explained to you, this can effectively be written as M times a delta function in z, where M is the mass injected per unit area and the delta function it has dimensions of 1 over length.

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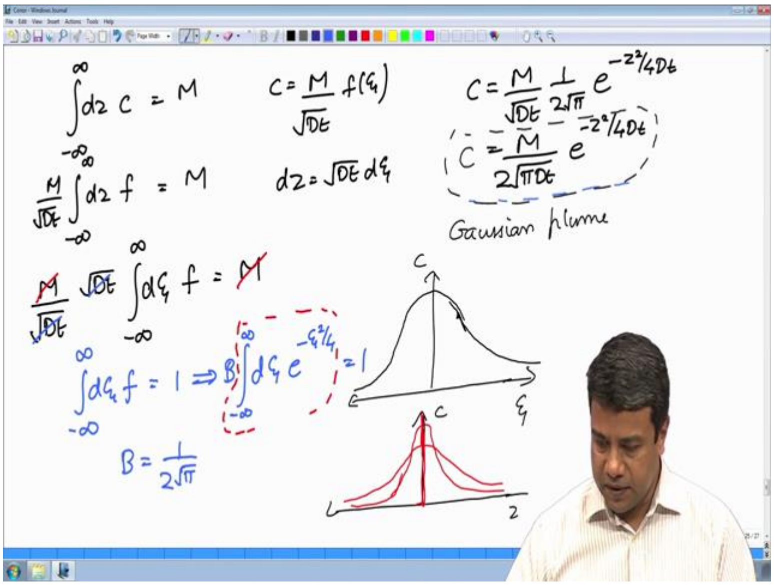

If you recall when we looked at, the definition of a delta functions.

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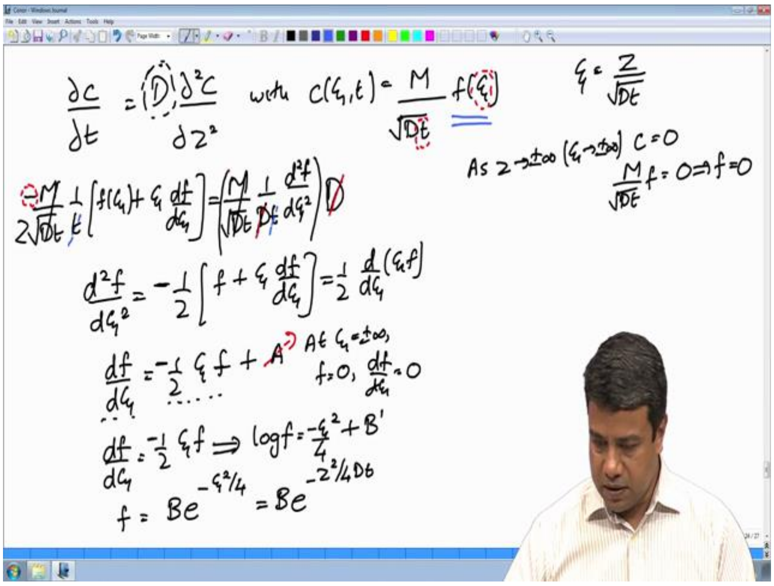

The delta function was defined as a function which was basically nonzero only at z is equal to 0, it was 0 everywhere else and the integral of the delta function was equal to 1. So, the integral of the concentration field is equal to M, so the delta function one dimensional delta function effectively has dimensions of 1 over length.

So, for this problem we had got the solution for the concentration field. The concentration field is effectively a Gaussian function in one dimensional, it is M by 2 root by Dt times e power minus h square by 4 Dt the total mass within this is a constant. So, if you look at it concentration as a function of xi, it is just equal to 1 particular value; it decays as 1 over T to the half. So, this d k is monotonically as 1 over T to the half; as time progresses the spread; however, in the xi co ordinate is the same. In the z coordinate the maximum comes down, the spread increases in such a way that the total mass is preserved at all times and it is a Gaussian function; a bell shaped function and I had given you some motivation for why this is an important function the variance of this increases proportional to t; therefore, the standard deviation goes as square root of T or the distance this width goes as 1 over goes as square root of t.

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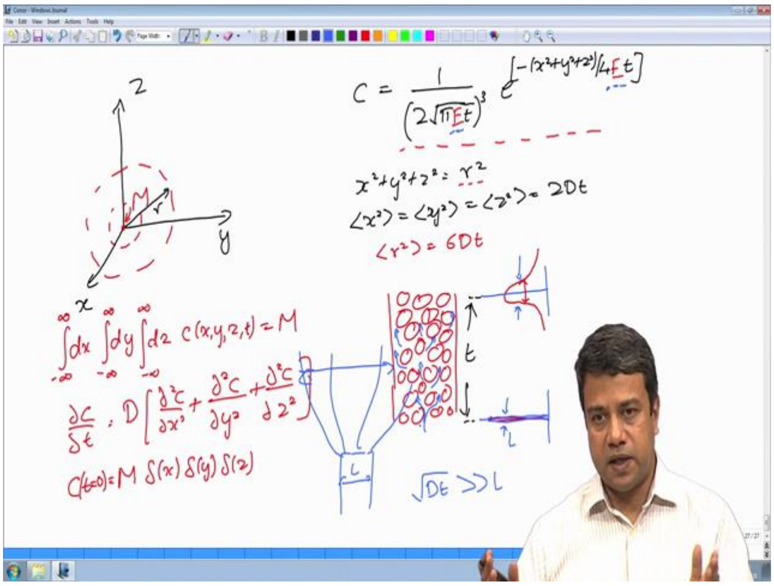

This is an important function because this can be used as a prototypical function to analyze many different kinds of flows. In cases, where the diffusion may not be due to molecular diffusion, but may be significantly enhanced due to turbulence or due to flow in porous media or other such situations; the spread will be much faster than what you would expect from molecular diffusion; however, that spread can still be analyzed using a solution of this kind; if we know how the variance of this function is increasing in time, the coefficient of that can be directly related to what is called the dispersion coefficient which accounts for both the mixing due to turbulent eddies as well as molecular diffusion or the mixing due to the flow, for flow to a porous medium in addition to molecular diffusion.

Now, these are all idealizations of course, the delta function is an idealization as in the case of if you inject the pulse; that injection is going to be over a infinite time that injection will in fact, be over infinite time and therefore, you will have a specific injection length. Similarly, when you have smoke coming out of a smokestack; there is a length for the smokestack. So, initially the concentration profiles will depend upon the details of the configuration; however as the spread becomes larger and larger; the details of the injection will not matter, we can approximate that injection as a delta function and what is that length scale; it is of course, our length scale here, the length scale the spread is approximately root of Dt; when this spread becomes much larger than the length scale the initial length scale, all dispersions will look similar, they will have a Gaussian form, it will be independent of the initial configuration that was used for injection and therefore, all of these can be treated equally as a delta function, when this width or this width it is much larger than the initial injection that length scale is square root of Dt.

So in the first case where we had looked at conduction from a surface, we said the penetration depth has to be much smaller than the system size, in this case; it is the opposite. The spread has to be much larger than the details of the injection, so this is for diffusion into infinite domains.

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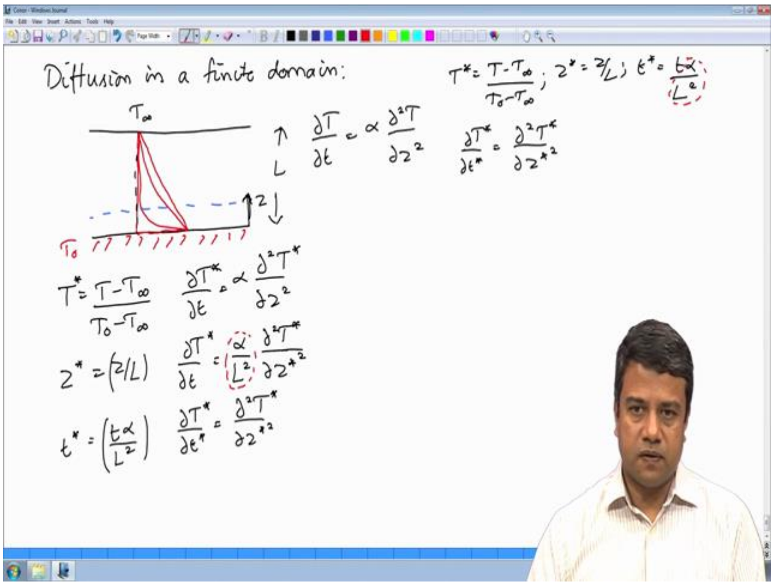

Let us now look at a problem of diffusion into a finite domain. Let us say I had a finite length L between two flat plates and initially the temperature everywhere was T infinity;

if the initial temperature everywhere was T infinity and at time T is equal to 0, I set the temperature of this bottom placed as T 0; I heated only the bottom plate and then I would like to find out how the temperature varies with time.

So initially you will have a temperature field that looks something like this, only the bottom is heated. Then as time progresses, you will have something it goes like that and then finally, the long time limit, it will reach a steady state; where you have heat conduction from the bottom to the top. What we had solved for the infinite problem case was the situation where this temperature disturbance is restricted to very close to the bottom surface. Now we will consider diffusion across a finite length, the diffusion equation of course is the same, if there are no surface or sinks is equal to alpha partial square T by partial z square, where z is this vertical port because z is if this vertical port.

However in contrast to the earlier problem, we now have a length scale L by which we can scale z. So, once again if we write; T is equal to T star is equal to T minus T infinity by T naught minus T infinity. The equation becomes partial T star by partial T is equal to alpha partial squared T star in partial z square, just scaling the temperature field since this T star is linear in the temperature, the T naught minus T infinity in the denominator on both sides will cancel out. I can now scale distance z star is equal to z by L because that is a natural length scale in the z direction; the temperature is varying over a distance L.

And if I do that; I get partial T star by partial T is equal to alpha by L square, d square T star right by T z star square. How do I scale the time coordinate in this case, we will have diffusion from the surface; this immediately gives us the scaling for time; alpha by L square, alpha is the thermal diffusivity; lens square per time. So alpha by L square gives us 1 over time. So, therefore, I can scale my time; T star is equal to T alpha by L square, it is easy to see that this is dimensionless because alpha is length square per time and if I do it that way, it is easy to verify that my equation becomes partial T star, partial T star.

So now the equation is independent of any parameters, it is does not contain any more parameters once I have done the scaling this way. So, to summarize I have defined T star is equal to T minus T infinity by T naught minus T infinity; z star is equal to z by L, T star is equal to T alpha by L square and once I do that my equation becomes partial T by partial T is equal to partial square T by partial. Note that in the z direction, the natural length scale is of course, the length L. What is the natural time scale, in this particular case what I have scaled time by is L square by alpha and L square by alpha; just from dimensional analysis is the time it takes for the diffusion of momentum across the distance L, so that is the time scale that I have scaled it by.

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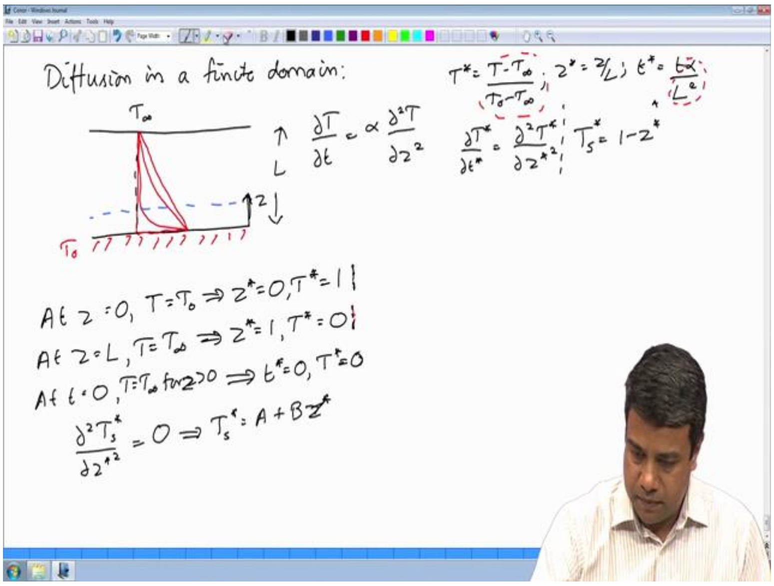

So, what are the boundary conditions; the boundary conditions are that z star is equal to 0 at or at z is equal to 0, let us start with this dimensional; at z is equal to 0, T is equal to T naught; z is equal to 0 implies that z star is equal to 0 and when T is equal to T 0; T star has to be equal to 1.

The other condition at z is equal to L; the top surface T is equal to T infinity which implies that at z star is equal to 1; the top surface is at z star is equal to 1, T star is equal to 0, so that is the solution that is the boundary condition on the two surfaces. At T is equal to 0; the temperature was uniform throughout the fluid at T is equal to 0, the temperature was changed on the bottom surface alone. So, the temperature was uniform throughout the fluid, T is equal to T infinity for all z greater than 0; this implied that at T star is equal to 0 for all z that was the instant at which the heating had started.

So, now you can see that we have a series of conditions which is inhomogeneous in space; T star is equal to 1 at 1 spatial boundary T star is equal to 0 at the other spatial boundary and T star is equal to 0 at initial time. What is it that we would expect? As time goes to infinity, as you wait longer and longer this thing has to evolve to some particular steady temperature field which becomes independent of time in the long time limit.

What is the study in the temperature field, when it once it has reached a steady temperature field; there is no longer a variation in time. Therefore, for the steady temperature field; the equation will be d square T steady by d z square is equal to 0 because there is no longer a variation in time. So, there is a steady temperature field subject to the boundary conditions that at z is equal to 0; T is equal to 1 and that z is equal to 1, T is equal to 0.

This solution for the steady temperature field is therefore of the form T steady is equal to some constant plus some constant into z; it is a linear function since the second derivative is 0 it is a linear function and these constants can be determined from the condition that for the steady temperature field, these are the boundary conditions; at z is equal to 0, T is equal to 1 and at z is equal to 1, T is equal to 0 and that is straightaway gives us the solution for the steady temperature field; T steady is equal to 1 minus z star, so we will write that a little lower.

T steady is 1 minus z star, you can verify that you see this temperature is equal to 1 at z is equal to 0 as required by this boundary condition is equal to 0 at z is equal to 1 as required by this condition. So, A is equal to 1 and T is equal to minus 1 that is a steady temperature field you would expect in the limit as time goes to infinity. Now what we can do is, we can write the temperature as the sum of a steady part plus the transient part. The initial transient which basically captures the evolution of this temperature field as time progresses, so this is the steady temperature field 1 minus z which is 0 at 1 and it is 1 and 0, if I express it in terms of the scale temperature. On top of this we can superpose the transient part of the temperature which basically captures the time evolution.

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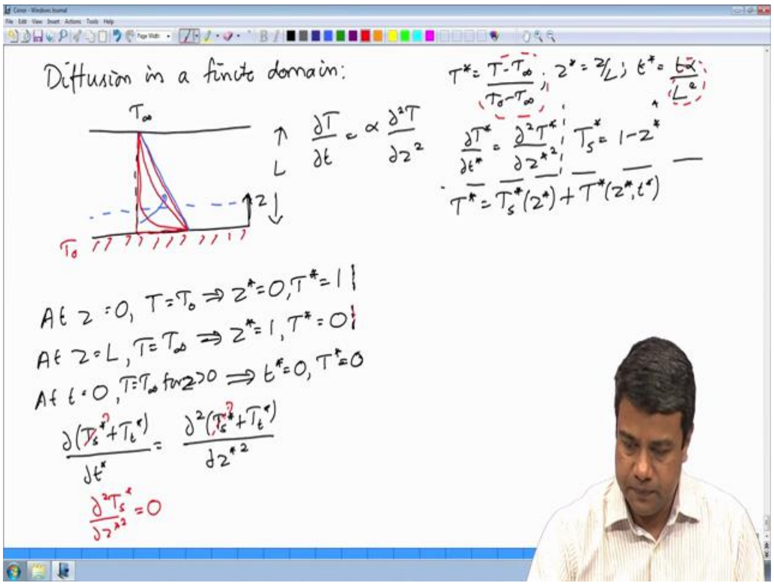

So, in such a formulation what you would do is to write down the temperature is equal to a steady temperature plus the transient temperature. This steady temperature is only a function of position and this second transient plot is a function of position and time. So, in the limit as T goes to infinity, the transient part has to decrease to 0, so that is the purpose of writing this in terms of a steady part plus a transient part. So as T goes to infinity, the transient part decreases to 0 and you are left with just the steady part.

So, let us insert this into the balanced equation, so the balanced equation basically is that the derivative of T steady plus T transient by Dt is equal to the second derivative of T steady plus the transient by partial z square. The steady part is independent of time, if the steady part is only a function of position. So the time derivative the steady part will be 0, the second spatial derivative of the steady part, and the steady part and the set has to satisfy the conservation equation; d square T steady by d z square is equal to 0. So, the second derivative the steady part once again will be equal to 0.

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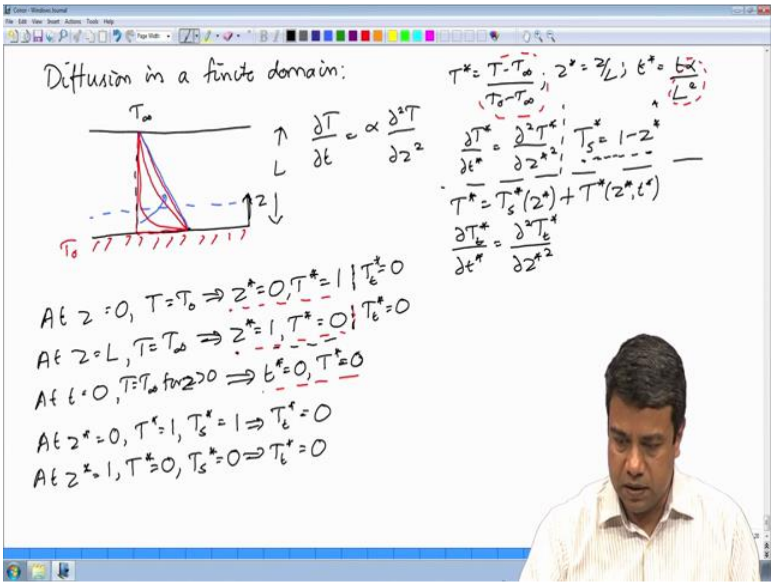

So, this gives us an equation for the transient part alone, the equation for the transient part is of the form, it has the same form as the equation for the total temperature partial T transient by partial T is equal to partial square T transient right partial z square. So, that is the equation for the transient part, it is identical to the equation for the steady part.

However I need to do one other thing and that is I need to enforce the boundary conditions for the transient part. I have boundary conditions here for the total temperature, from this I need to get boundary conditions for the transient part of the temperature alone. So, that the first condition at z star is equal to 0, T star is equal to 1; the steady part of the temperature does satisfy the boundary condition that at z star is equal to 0, T star is equal to 1. So, the steady part is equal to 1, you can see that over here this is a steady part of the temperature field; at z is equal to 0, that is equal to 1 which implies since T s plus T transient has to be equal to the total temperature, this implies that the transient temperature is equal to 0. So, the boundary condition of the transient temperature here is that T star is equal to 0.

Similarly, at z star is equal to 1; T star is equal to 0, the temperature T star is equal to 0 and z star is equal to 1; however, the steady solution also satisfies the same; T steady is also equal to 0, you can see that when you insert z equal to 1; the steady part is also equal to 0 which means that the transient part is equal to 0.

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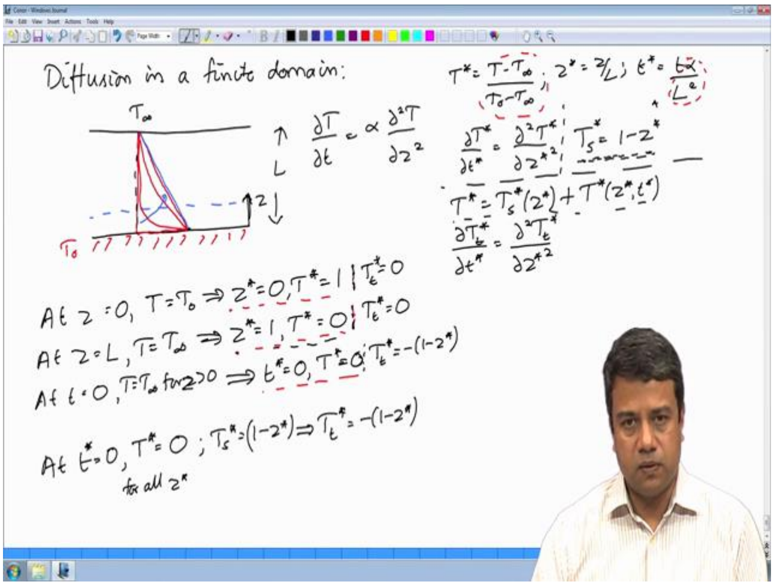

So, at this location the transient part is equal to 0 and I have a third initial condition; at T is equal to 0, at T star is equal to 0; T star is equal to 0 for all z the temperature T star is equal to 0 for all z star.

However the steady part is independent of time, you can see here that the steady part is in the independent of time. Therefore, T steady is still going to be equal to 1 minus z star, since the steady part plus the transient part has to sum to 0, therefore this implies that the transient part, so the temperature field has to be equal to minus of 1 minus z star.

So, therefore at initial time the transient part it is going to be equal to minus of 1 minus z star. So, those are the equations that I have to solve for the transient part, the total temperature is the sum of the steady part plus the transient part over here; total temperature is the sum of these two. I have already got a solution for the steady part, now I have to solve for the transient part and once I get a solution, I can add up to and get the total solution.

Just a word about why we are doing all this, you might be wondering why we are separating it out into a steady part under a transient part. The initial problem that I had, if you can see in terms of the temperature field, the temperature was 0 at 0, 0 at 1, it was 1 at 0. So, if the temperature is 0; that means, it is a homogeneous boundary condition, the scale temperature is equal to 0. So, this was 0 at time T equal to 0, it was 0 at z is equal to 1 and it was nonzero only at the surface z is equal to 0; the only in homogeneity was the spatial coordinate and z is equal to 0 that was what was driving the temperature field.

Now, I found out what was the steady part; the steady part satisfied exactly the same spatial boundary conditions that we said the transient part has to be 0 on both boundaries, it has 0 boundary conditions on both spatial boundaries; the in homogeneity is appearing in the initial condition. The only nonzero value of the transient part of the temperature is in the initial condition. Therefore, you have homogenous boundary conditions for the transient temperature in the spatial coordinates; the only homogeneity is at initial time. So, it is being forced at initial time rather than in space that is because there is a difference between the actual temperature and the steady temperature at initial time and that difference is decaying as time progresses or the transient part of the temperature is decaying as time progresses.

Therefore, I have shifted the homogeneity from the spatial co-ordinate to the time co-ordinate and that is going to be of critical importance when I solve this problem by separation of variables as I will show you in the next lecture. I will show you how to solve this problem using separation of variables, the equation for the transient part is still a partial differential equation, we still do not have standardized methods for solving the partial differential equation. The solution has to be based upon physical insight and in the next lecture I will show you how we do that using a procedure called the separation of variables.

So, I will continue this solution of this problem in a finite domain in the next lecture; I will see you then.