Transport Processes I: Heat and Mass Transfer
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Lecture – 25
Unidirectional transport: Similarity solution for decay of a pulse continued

This is lecture number 25 in our course on fundamentals of transport processes, welcome; we were looking at transport in one spatial dimension in the previous lecture, unidirectional transport where there are variations of the densities of mass momentum or energy in one spatial direction and possibly in time as well.

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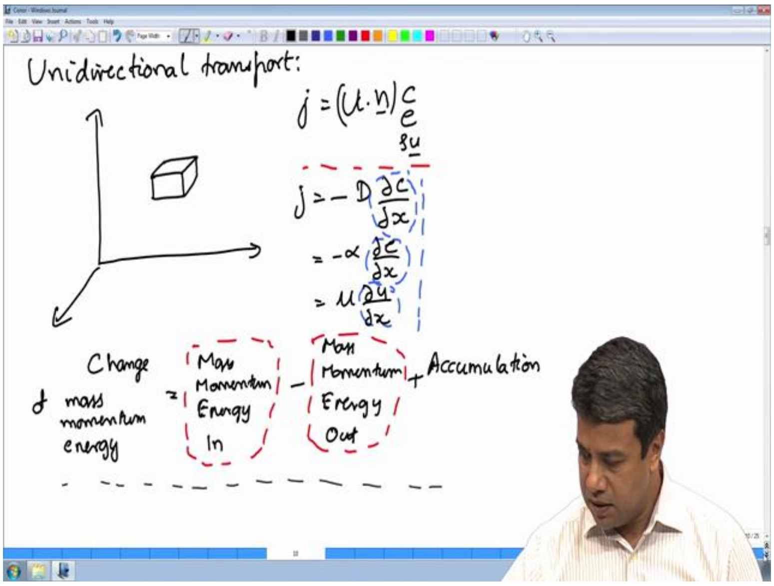

The transport the balance equation for the transport is basically of this form, the change of mass momentum or energy within a differential volume is what comes in minus what goes out plus any accumulation or depletion, in the case of mass and energy possibly due to reactions, in the case of momentum the accumulation is due to body forces.

The rates of transport are the rates of transport across the surface of this differential volume for 2 reasons, one is due to convection and the other is due to diffusion. The convective flux is just the normal velocity times the density of mass momentum or energy, the diffusive flux is given by the constitutive relations and we put these 2 together to get the differential equations.

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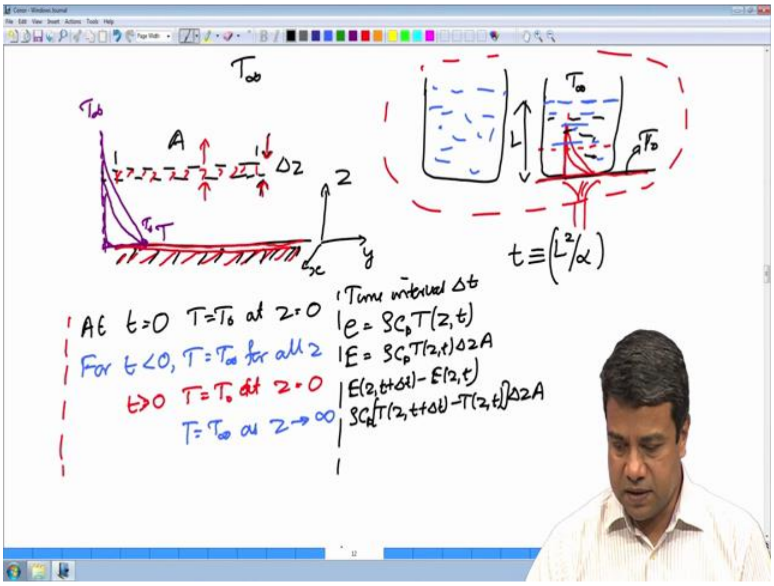

At $t=0$ $T=T_0$ at $z=0$

For $t<0$, $T=T_\infty$ for all z

$t>0$ $T=T_0$ at $z=0$

$T=T_\infty$ as $z \to \infty$

Time interval Δt

$e = \rho C_p T(z,t)$

$E = \rho C_p T(z,t) \Delta z A$

$E(z,t+\Delta t) - E(z,t)$

$\rho C_p [T(z,t+\Delta t) - T(z,t)] \Delta z A$

In the last lecture, we were looking at transport in one dimension the heat transfer problem in infinite fluid bounded by a flat surface, the entire system is at a constant temperature initially and at time T is equal to 0, you heat up the surface to a higher temperature and 1 would like to see how the temperature varies within the fluid, I had explained to you that this is an idealization of a problem where we set a container on a heated surface and we had derived the balance equations for that case for the energy density by considering a small differential volume here.

A small differential thickness in the direction of the variations so, you have to take a differential thickness or differential volume in the direction of the variation in the temperatures and write a balance for that.

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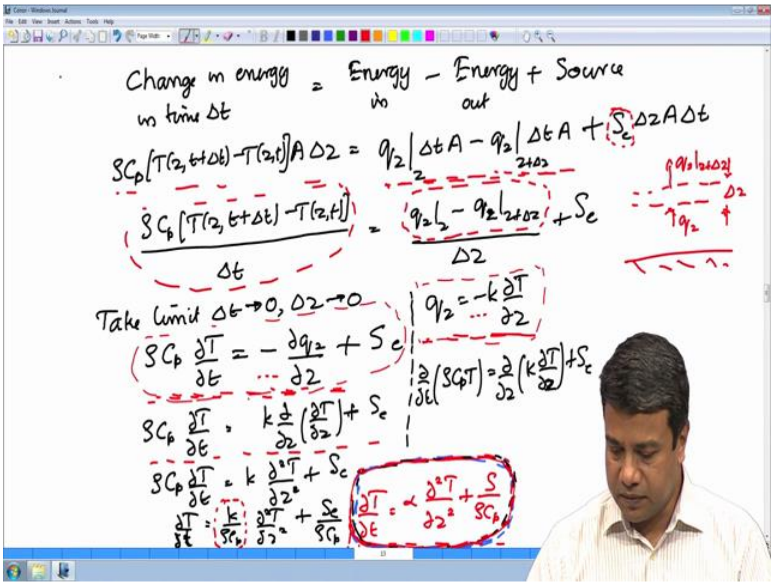

Change in energy in time Δt = Energy in − Energy out + Source

$$\rho C_p[T(z,t+\Delta t)-T(z,t)]A\Delta z = q_z|_z \Delta t A - q_z|_{z+\Delta z}\Delta t A + S_e \Delta z A \Delta t$$

$$\frac{\rho C_p[T(z,t+\Delta t)-T(z,t)]}{\Delta t} = \frac{q_z|_z - q_z|_{z+\Delta z}}{\Delta z} + S_e$$

Take limit $\Delta t \to 0, \Delta z \to 0$

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q_z}{\partial z} + S_e$$

$$q_z = -k\frac{\partial T}{\partial z}$$

$$\frac{\partial}{\partial t}(\rho C_p T) = \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + S_e$$

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) + S_e$$

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial z^2} + S_e$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial z^2} + \frac{S_e}{\rho C_p}$$

$$\boxed{\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial z^2} + \frac{S}{\rho C_p}}$$

And that balance of course, ended up being of the form, the change in energy is equal to what comes in minus what goes out and we got a relation of this kind the rate of change of energy within that volume is equal to the difference in flux divided by the thickness.

What comes in minus, what goes out divided by the thickness plus sources of energy and then we had used the constitutive relation for the heat flux to get this energy balance equation which contains the thermal diffusivity and the sources or sinks of energy.

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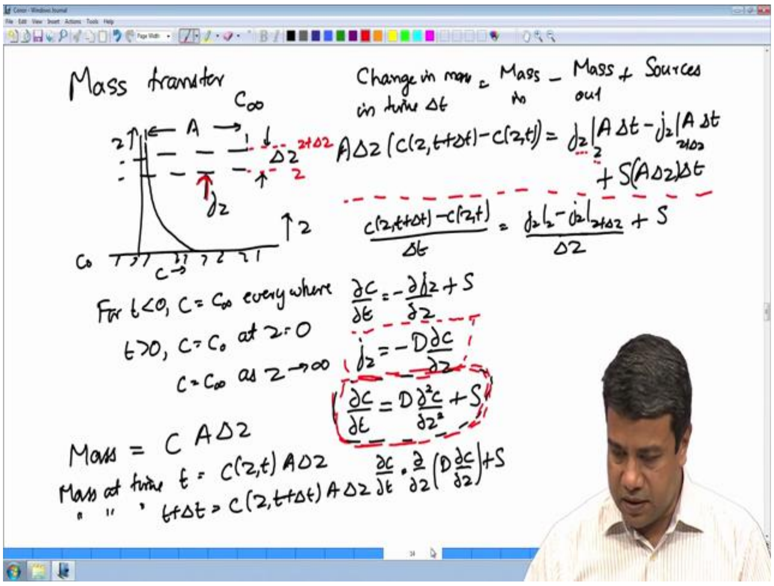

Mass transfer

Change in mass in time Δt = Mass in − Mass out + Sources

$$A\Delta z\,(c(z,t+\Delta t)-c(z,t)) = j_z|_z A\Delta t - j_z|_{z+\Delta z} A\Delta t + S(A\Delta z)\Delta t$$

$$\frac{c(z,t+\Delta t)-c(z,t)}{\Delta t} = \frac{j_z|_z - j_z|_{z+\Delta z}}{\Delta z} + S$$

For $t<0$, $C = C_\infty$ everywhere

$t>0$, $C = C_0$ at $z=0$

$C = C_\infty$ as $z \to \infty$

$$\frac{\partial c}{\partial t} = -\frac{\partial j_z}{\partial z} + S$$

$$j_z = -D\frac{\partial c}{\partial z}$$

$$\boxed{\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial z^2} + S}$$

Mass = $C A\Delta z$

Mass at time t = $C(z,t)A\Delta z$

" " " $t+\Delta t$ = $C(z,t+\Delta t)A\Delta z$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial c}{\partial z}\right) + S$$

Exactly the same formulation can be done for the concentration field and you get an equation of exactly the same form except that the instead of the temperature is the concentration variation that is being considered, instead of the thermal diffusivity it is the mass diffusivity. Instead of the thermal source of thermal energy, is the source of mass due to reactions.

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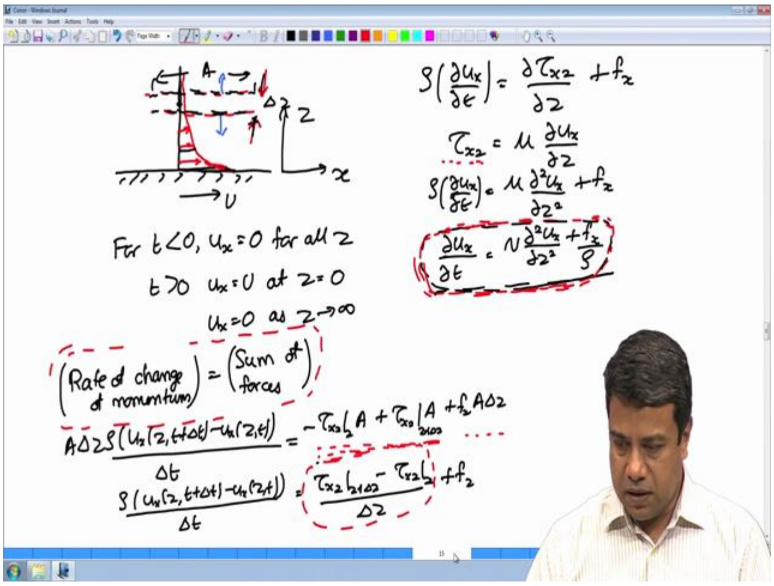

For $t<0$, $u_x=0$ for all z

$t>0$ $u_x=U$ at $z=0$

$u_x=0$ as $z\to\infty$

$$\begin{pmatrix}\text{Rate of change}\\ \text{of momentum}\end{pmatrix} = \begin{pmatrix}\text{Sum of}\\ \text{forces}\end{pmatrix}$$

$$A\Delta z\rho\frac{(u_x(z,t+\Delta t)-u_x(z,t))}{\Delta t} = -\tau_{xz}|_z A + \tau_{xz}|_{z+\Delta z}A + f_x A\Delta z$$

$$\rho\frac{(u_x(z,t+\Delta t)-u_x(z,t))}{\Delta t} = \frac{\tau_{xz}|_{z+\Delta z}-\tau_{xz}|_z}{\Delta z} + f_z$$

$$\rho\left(\frac{\partial u_x}{\partial t}\right) = \frac{\partial \tau_{xz}}{\partial z} + f_x$$

$$\tau_{xz} = \mu\frac{\partial u_x}{\partial z}$$

$$\rho\left(\frac{\partial u_x}{\partial t}\right) = \mu\frac{\partial^2 u_x}{\partial z^2} + f_x$$

$$\frac{\partial u_x}{\partial t} = \nu\frac{\partial^2 u_x}{\partial z^2} + \frac{f_x}{\rho}$$

And we had actually done the same for momentum, the formulation is slightly different, the rate of change of momentum is equal to sum of the forces, the forces are of 2 types, one is body forces and the other is surface forces. Body forces like gravitational force, centrifugal force are proportional to the volume being considered, they are proportional to the mass which is the density times the volume. Surface forces are due to momentum transfer across surfaces and in this case, we had used a slightly different sign convention, the force is considered positive for if the outward unit normal to the volume is in the plus z direction and therefore, we had caught the slight difference in the sign of the stresses, there was a difference in the sign of the constitutive relation as well.

Ultimately the final relation that we got was similar in form to the mass and energy balance equations, substitute velocity for concentration, momentum diffusivity or kinematic viscosity for mass diffusivity and the ratio of the force and the density instead of the concentration source of the mass source. So, these are partial differential equations, they contain variations both in time as well as in the spatial coordinate and we looked at how to solve these equations as I said there is no systematic procedure for solving partial differential equations the procedure that you adopt depends upon a physical insight into the problem.

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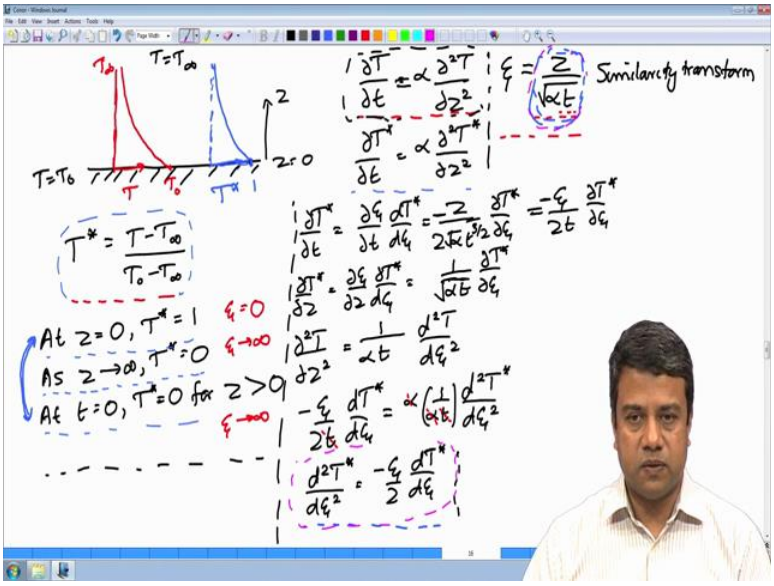

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

$$\xi = \frac{z}{\sqrt{\alpha t}}$$ Similarity transform

$$\frac{\partial T^*}{\partial t} = \alpha \frac{\partial^2 T^*}{\partial z^2}$$

$$T^* = \frac{T-T_\infty}{T_0-T_\infty}$$

At $z=0$, $T^*=1$ $\xi=0$

As $z\to\infty$, $T^*=0$ $\xi\to\infty$

At $t=0$, $T^*=0$ for $z>0$ $\xi\to\infty$

$$\frac{\partial T^*}{\partial t} = \frac{\partial \xi}{\partial t}\frac{dT^*}{d\xi} = \frac{-z}{2\sqrt{\alpha}\,t^{3/2}}\frac{\partial T^*}{\partial \xi} = \frac{-\xi}{2t}\frac{\partial T^*}{\partial \xi}$$

$$\frac{\partial T^*}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial T^*}{d\xi} = \frac{1}{\sqrt{\alpha t}}\frac{\partial T^*}{\partial \xi}$$

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha t}\frac{d^2 T}{d\xi^2}$$

$$-\frac{\xi}{2\cancel{t}}\frac{dT^*}{d\xi} = \cancel{\alpha}\left(\frac{1}{\cancel{\alpha t}}\right)\frac{d^2T^*}{d\xi^2}$$

$$\frac{d^2T^*}{d\xi^2} = -\frac{\xi}{2}\frac{dT^*}{d\xi}$$

We have done one set solution for a surface which was heated at T is equal to 0, initially the temperature is just a constant everywhere, initially the temperature is just a constant everywhere at time T is equal to 0 you heat the surface so that this surface temperature increases to a higher value and then as the heat diffuses through the medium the temperature is going to increase further and further away. Very far from the surface this is an infinite fluid. So, very far on the surface the temperature is still going to be the undisturbed temperature. Very far from the surface, the temperature is still equal to the undisturbed temperature.

At the initial time at T is equal to 0, the temperature is z equal to T naught and; however, at T is equal to 0, the heating has just started therefore, the temperature in the fluid is still equal to T infinity, we had scaled the equations by this form and got a conduction equation for the scaled variables and then we had used this additional bit of insight since there are no time or length scales in the problem by which one can scale the distance and time. The only dimensional parameter in this problem is the thermal diffusivity alpha there are no other dimensional parameters.

Out of length, time and alpha, one can get only 1 dimensionless group and that is this dimensionless group here which I had called the similarity variable in the previous lecture therefore, actually expressed the equation for the temperature in terms of this dimensionless group and if I do that I reduce the coordinates from 2 to 1 in this problem. You express it in terms of xi using chain rule for differentiation because xi depends upon T and y and you get this equation which is an equation which is only a function of xi does not contain individually z in time and this equation can be solved, we got a solution for this equation.

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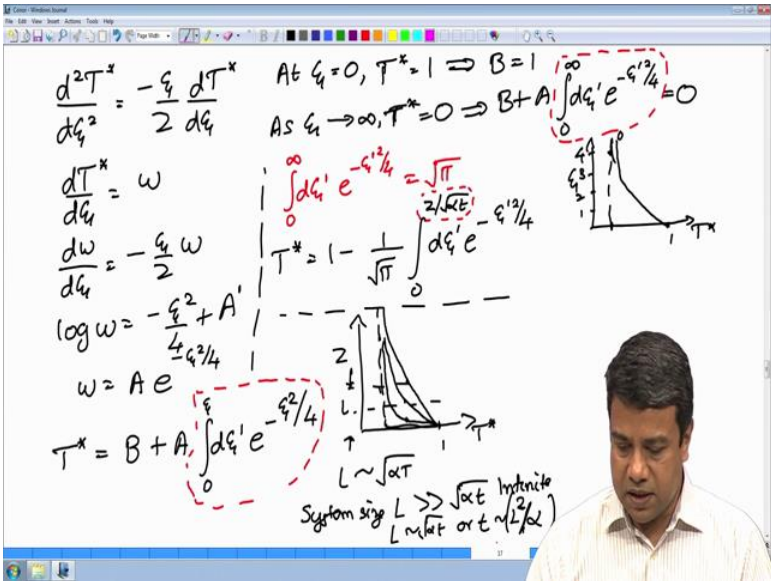

$$\frac{d^2T^*}{d\xi^2} = -\frac{\xi}{2}\frac{dT^*}{d\xi}$$

$$\frac{dT^*}{d\xi} = \omega$$

$$\frac{d\omega}{d\xi} = -\frac{\xi}{2}\omega$$

$$\log\omega = -\frac{\xi^2}{4} + A'$$

$$\omega = A e^{-\xi^2/4}$$

$$T^* = B + A\int_0^{\xi} d\xi' e^{-\xi'^2/4}$$

At $\xi = 0$, $T^* = 1 \Rightarrow B = 1$

As $\xi \to \infty$, $T^* = 0 \Rightarrow B + A\int_0^{\infty} d\xi' e^{-\xi'^2/4} = 0$

$$\int_0^{\infty} d\xi' e^{-\xi'^2/4} = \sqrt{\pi}$$

$$T^* = 1 - \frac{1}{\sqrt{\pi}}\int_0^{z/\sqrt{\alpha t}} d\xi' e^{-\xi'^2/4}$$

$L \sim \sqrt{\alpha T}$

System size $L \gg \sqrt{\alpha t}$ Infinite

$L \sim \sqrt{\alpha t}$ or $t \sim (L^2/\alpha)$

We had applied boundary conditions and this was the final solution it contains an integral, this integral cannot be evaluated analytically, but it can always be evaluated numerically and this solution is the same whether you considered mass transfer, heat transfer or momentum transfer.

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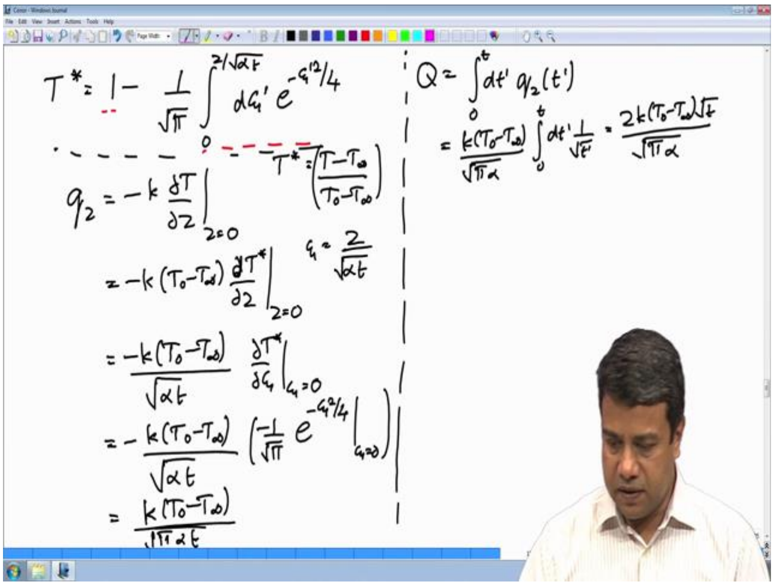

$$T^* = 1 - \frac{1}{\sqrt{\pi}} \int_0^{2/\sqrt{\alpha t}} d\xi' e^{-\xi'^2/4}$$

$$T^* = \left(\frac{T - T_\infty}{T_0 - T_\infty}\right)$$

$$q_z = -k \left.\frac{\partial T}{\partial z}\right|_{z=0}$$

$$\xi = \frac{z}{\sqrt{\alpha t}}$$

$$= -k (T_0 - T_\infty) \left.\frac{\partial T^*}{\partial z}\right|_{z=0}$$

$$= \frac{-k (T_0 - T_\infty)}{\sqrt{\alpha t}} \left.\frac{\partial T^*}{\partial \xi}\right|_{\xi=0}$$

$$= -\frac{k (T_0 - T_\infty)}{\sqrt{\alpha t}} \left(\frac{-1}{\sqrt{\pi}} \left. e^{-\xi^2/4} \right|_{\xi=0}\right)$$

$$= \frac{k (T_0 - T_\infty)}{\sqrt{\pi \alpha t}}$$

$$Q = \int_0^t dt' \, q_z(t')$$

$$= \frac{k (T_0 - T_\infty)}{\sqrt{\pi \alpha}} \int_0^t dt' \frac{1}{\sqrt{t'}} = \frac{2k (T_0 - T_\infty) \sqrt{t}}{\sqrt{\pi \alpha}}$$

You can consider the heat from those; you can calculate the heat flux and the total heat transported.

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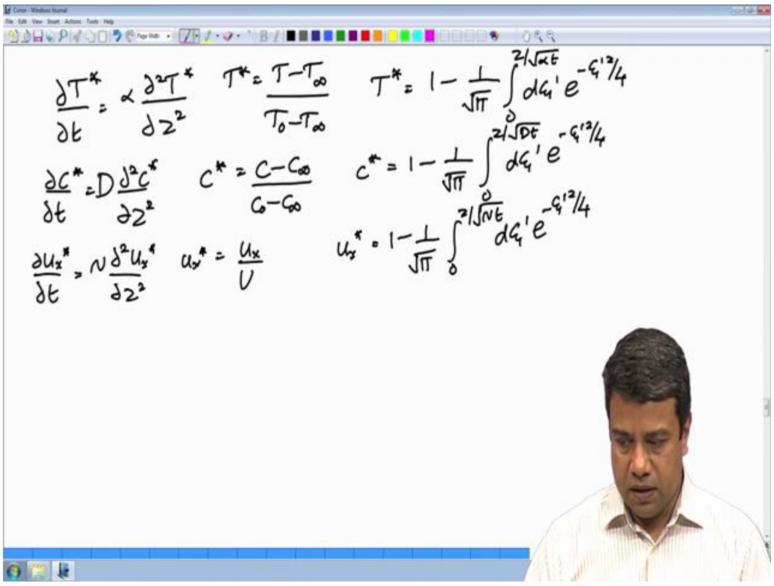

$$\frac{\partial T^*}{\partial t} = \alpha \frac{\partial^2 T^*}{\partial z^2} \qquad T^* = \frac{T - T_\infty}{T_0 - T_\infty} \qquad T^* = 1 - \frac{1}{\sqrt{\pi}} \int_0^{2/\sqrt{\alpha t}} d\xi' e^{-\xi'^2/4}$$

$$\frac{\partial c^*}{\partial t} = D \frac{\partial^2 c^*}{\partial z^2} \qquad c^* = \frac{c - c_\infty}{c_0 - c_\infty} \qquad c^* = 1 - \frac{1}{\sqrt{\pi}} \int_0^{2/\sqrt{Dt}} d\xi' e^{-\xi'^2/4}$$

$$\frac{\partial u_x^*}{\partial t} = \nu \frac{\partial^2 u_x^*}{\partial z^2} \qquad u_x^* = \frac{u_x}{U} \qquad u_x^* = 1 - \frac{1}{\sqrt{\pi}} \int_0^{2/\sqrt{\nu t}} d\xi' e^{-\xi'^2/4}$$

And therefore, you get exactly the same set of solutions suppose there is mass heat or momentum transfer and we had actually looked at one practical application where this is actually can give you a correlation and that was a falling film.

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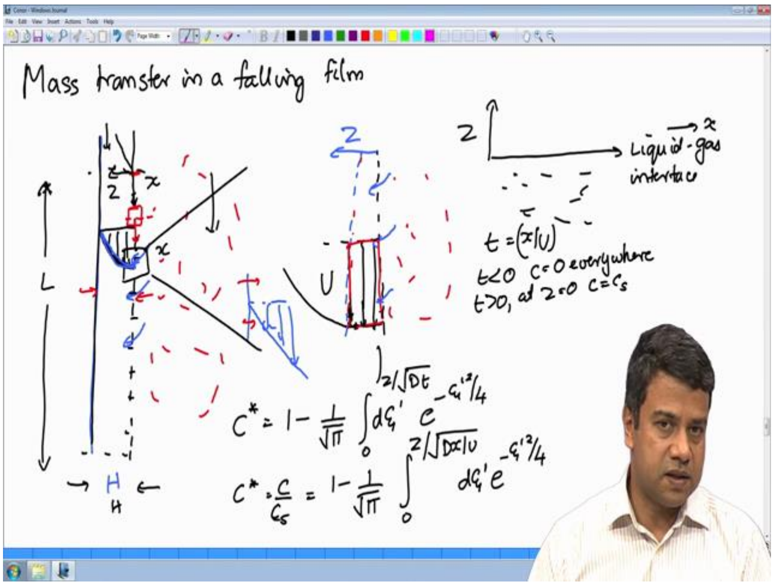
Mass transfer in a falling film

$t = (x/U)$

$t < 0$ $c = 0$ everywhere

$t > 0$, at $z = 0$ $c = c_s$

$$C^* = 1 - \frac{1}{\sqrt{\pi}} \int_0^{z/\sqrt{Dt}} d\xi' \, e^{-\xi'^2/4}$$

$$C^* = \frac{c}{c_s} = 1 - \frac{1}{\sqrt{\pi}} \int_0^{z/\sqrt{Dx/U}} d\xi' \, e^{-\xi'^2/4}$$

Rather than considering the film that is falling in space since the velocity of that falling film close to the interface was a constant due to the 0 shear stress at the convicted condition at the interface the velocity has to tend to a constant value at that interface.

So is the velocity was nearly a constant we had assumed that we were moving in a reference frame moving with the film in that reference frame the film is stationary and therefore, we can monitor the progress of the concentration field with time and the time relaxed is just equal to the distance traveled divided by the flow velocity the maximum flow velocity. So on the basis of this we had got an equation for the concentration field which now contained the flow velocity instead of time and once we had done that, we manage to get the flux at the surface as a function of downstream distance

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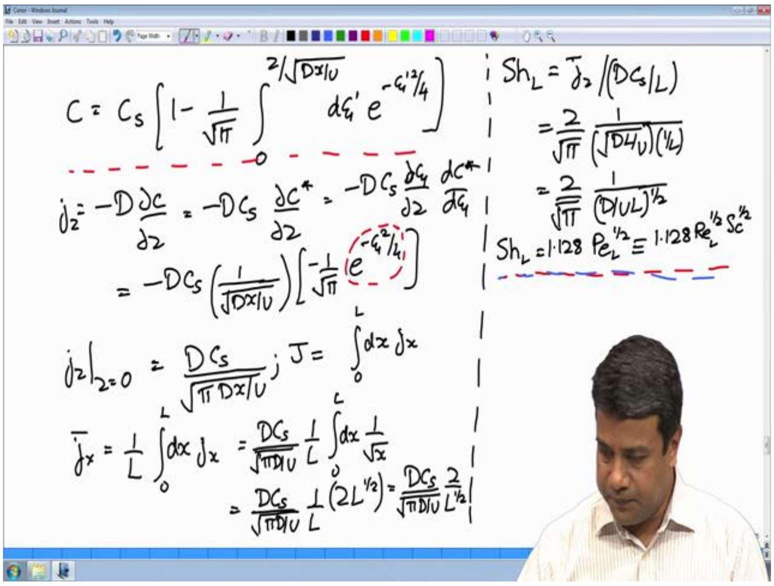

$$C = C_s\left[1 - \frac{1}{\sqrt{\pi}}\int_0^{2/\sqrt{Dx/U}} d\xi'\, e^{-\xi'^2/4}\right]$$

$$j_z = -D\frac{\partial c}{\partial z} = -DC_s\frac{\partial c^*}{\partial z} = -DC_s\frac{\partial \xi}{\partial z}\frac{dc^*}{d\xi}$$

$$= -DC_s\left(\frac{1}{\sqrt{Dx/U}}\right)\left[-\frac{1}{\sqrt{\pi}}e^{-\xi^2/4}\right]$$

$$j_z\big|_{z=0} = \frac{DC_s}{\sqrt{\pi Dx/U}};\quad J = \int_0^L dx\, j_x$$

$$\bar{j}_x = \frac{1}{L}\int_0^L dx\, j_x = \frac{DC_s}{\sqrt{\pi D/U}}\frac{1}{L}\int_0^L dx\,\frac{1}{\sqrt{x}}$$

$$= \frac{DC_s}{\sqrt{\pi D/U}}\frac{1}{L}\left(2L^{1/2}\right) = \frac{DC_s}{\sqrt{\pi D/U}}\frac{2}{L^{1/2}}$$

$$Sh_L = \bar{j}_z/(DC_s/L)$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{\left(\sqrt{D/U}\right)(1/L)}$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{(D/UL)^{1/2}}$$

$$Sh_L = 1.128\, Pe_L^{1/2} \equiv 1.128\, Re_L^{1/2} Sc^{1/2}$$

And then we have calculated the average flux averaged over the downstream distance averaged over x and from that we had actually got a correlation for the Schmidt number.

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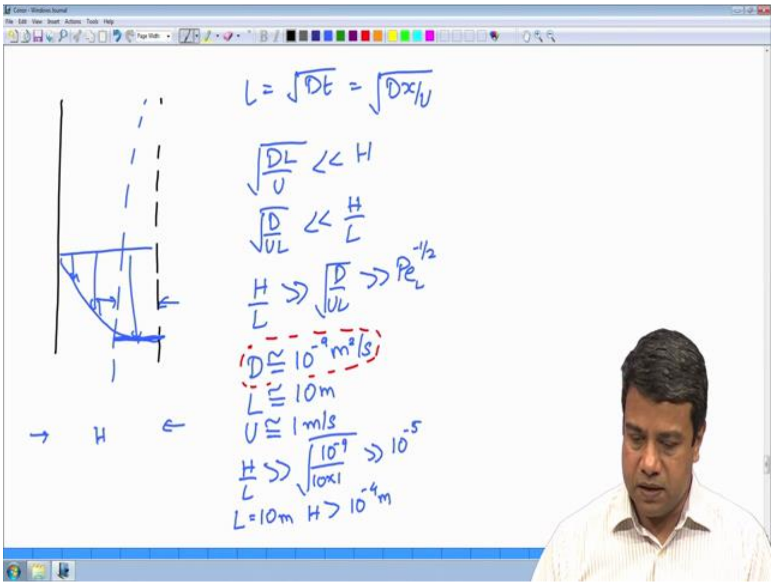

$$L = \sqrt{Dt} = \sqrt{Dx/U}$$

$$\sqrt{\frac{DL}{U}} \ll H$$

$$\sqrt{\frac{D}{UL}} \ll \frac{H}{L}$$

$$\frac{H}{L} \gg \sqrt{\frac{D}{UL}} \gg Pe_L^{-1/2}$$

$$D \cong 10^{-9}\, m^2/s$$

$$L \cong 10m$$

$$U \cong 1\, m/s$$

$$\frac{H}{L} \gg \sqrt{\frac{10^{-9}}{10\times 1}} \gg 10^{-5}$$

$$L = 10m \quad H > 10^{-4}m$$

Rather than a heated surface which is heated at T is equal to 0 we had considered; we had looked at the limitations and so on and found that even if the distance is of the order of 10s of meters, a film thickness of what a millimeter is adequate to ensure that the penetration depth is much smaller than the film thickness so that is the reason why this approximation is expected to work extremely well in real applications.

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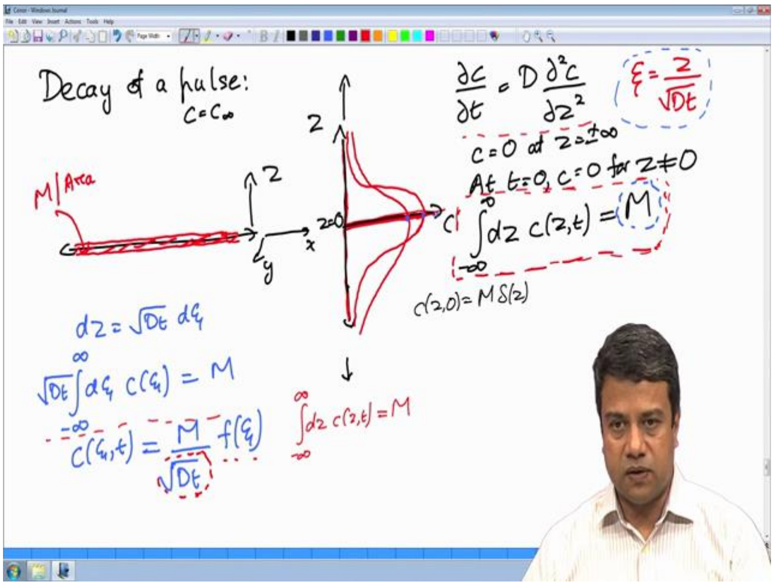

Decay of a pulse:

$c = c_\infty$

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial z^2} \qquad \xi = \frac{z}{\sqrt{Dt}}$$

$c = 0$ at $z = \pm\infty$

At $t = 0$, $c = 0$ for $z \neq 0$

$$\int_{-\infty}^{\infty} dz\, c(z,t) = M$$

$c(z,0) = M\,\delta(z)$

$dz = \sqrt{Dt}\, d\xi$

$$\sqrt{Dt}\int_{-\infty}^{\infty} d\xi\, c(\xi) = M$$

$$c(\xi,t) = \frac{M}{\sqrt{Dt}} f(\xi)$$

$$\int_{-\infty}^{\infty} dz\, c(z,t) = M$$

We will come to another problem which was the decay of a pulse, you initially inject a pulse of fluid at the claim z is equal to 0, the pulse is a mass per unit area, if you instantaneously inject the mass of fluid within an infinitesimal thickness, you inject a mass of fluid within an infinitesimal thickness over here. So, everywhere else the mass is equal to 0 at the surface itself, the mass is a finite value per unit area and their diffusion equation remains exactly the same.

However the difference is that as the mass diffuses outwards due to molecular diffusion, the total mass per unit area has to be preserved, the total mass per unit area has to be preserved so that at T is equal to 0, the concentration is nonzero, only at z is equal to 0 the concentration is nonzero only at z is equal to 0 is equal to 0, everywhere else the integral of the concentration over the entire region is equal to M.

This is an idealization this is called delta function.

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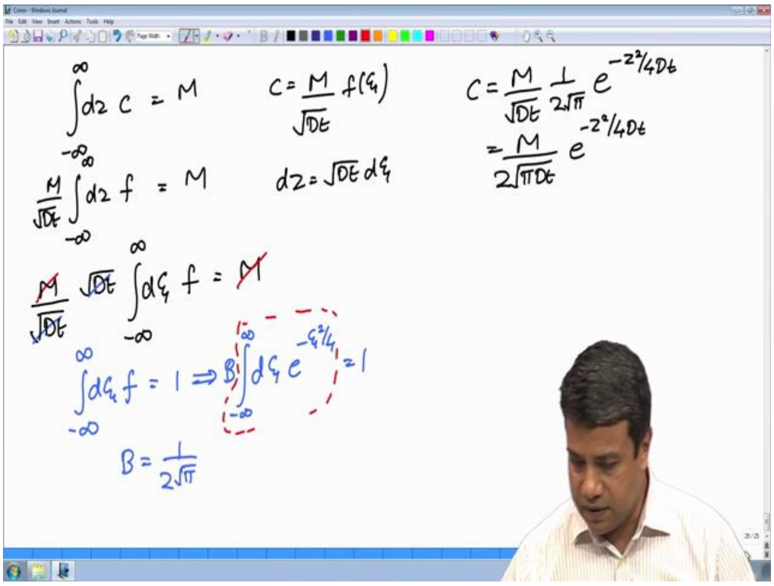

$$\int_{-\infty}^{\infty} dz\, c = M \qquad c = \frac{M}{\sqrt{Dt}} f(\xi) \qquad c = \frac{M}{\sqrt{Dt}} \frac{1}{2\sqrt{\pi}} e^{-z^2/4Dt}$$

$$= \frac{M}{2\sqrt{\pi Dt}} e^{-z^2/4Dt}$$

$$\frac{M}{\sqrt{Dt}} \int_{-\infty}^{\infty} dz\, f = M \qquad dz = \sqrt{Dt}\, d\xi$$

$$\frac{\cancel{M}}{\cancel{\sqrt{Dt}}} \cancel{\sqrt{Dt}} \int_{-\infty}^{\infty} d\xi\, f = \cancel{M}$$

$$\int_{-\infty}^{\infty} d\xi\, f = 1 \Rightarrow B \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2/4} = 1$$

$$B = \frac{1}{2\sqrt{\pi}}$$

I will just briefly introduce that.

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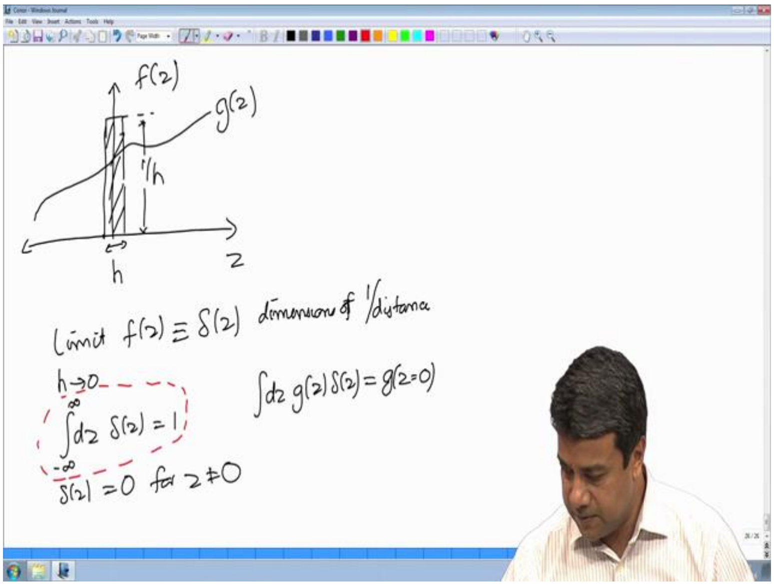

$$\underset{h \to 0}{\text{limit}}\ f(z) \equiv \delta(z) \quad \text{dimension of } 1/\text{distance}$$

$$\int_{-\infty}^{\infty} dz\, \delta(z) = 1 \qquad \int dz\, g(z)\delta(z) = g(z=0)$$

$$\delta(z) = 0 \text{ for } z \neq 0$$

If I have a function f of z as a function of this coordinate z, the delta function it is basically defined as a function about the origin z is equal to 0 which has a width h the width of this function is equal to h and the height is equal to 1 over h. If you take the limit for s h goes to 0 of this function f of z this is a delta function delta outside.

Note that the height is equal to 1 over h and the width is equal to h as you take the limit of h going to 0, the width goes to 0. The height goes to infinity in such a way that the product of the 2 is equal to 1 so that is a delta function. So, if the product of the 2 is equal to 1 integral, D z delta of z from minus infinity to infinity is equal to 1, the integral the area under the curve is equal to 1; however, delta of z is equal to 0, for z is not equal to 0 this is the idealization of a function that is restricted to a very thin region about z is equal to 0 the height goes to infinity in such a way that the product of the 2 is equal to 1 and this delta function also has the property that if I take any function g of z and if I take the integral of the function D z g of z times delta of z. This delta of z is 0 when z is not equal to 0 and the area under the curve is equal to 1. So, this function is just equal to the value of the function at g at z is equal to 0.

This is the idealization what is called the delta function of course, real functions are never delta functions when you inject it there is always going to be a thickness. So, therefore, when we solve this problem we are considering the case where the spread is much larger than the initial thickness. So, the details of the initial thickness do not affect the spread at long times.

There is the delta function and this concentration field at T is equal to 0, at T is equal to 0, I said c is equal to 0, for z not equal to 0 and the integral is equal to m. So, therefore, I could very well I have written the concentration field c at z, 0 is equal to m times delta of z. So, the magnitude of the concentration is m times delta of z, if you recall when I talked about the delta function I said the height is 1 over h therefore, this delta function has dimensions of 1 by distance because the height of this function is 1 over h and h is a distance. So, the delta function has dimensions of 1 over distance that is also evident from this equation here delta of z times z is dimensionless therefore, delta z has dimensions of 1 over distance and if we go back to our formula here you can see that if I write C is equal to M times delta of z then M has dimensions of mass per unit area, it is a mass per unit area that was injected at that z is equal to 0, delta has dimensions of 1 over distance therefore, the concentration correctly has dimensions of mass per unit volume.

We inject the pulse; that means that at T is equal to 0 the concentration field is this delta function and then we follow the progression with time in this case as well there are no length or time scales in the problem therefore, I should be able to define everything in terms of this dimensionless group. However, the concentration field is evolving as time progresses initially the concentration is large as the material spreads, as the material spreads the total mass has to be preserved let me set the maximum of this concentration has to come lower and lower.

The maximum value of this concentration is actually a function of time and that comes out of this balance condition that the integral of it comes out of the balance condition that the integral of D z times c of z T at any point of time this is the total mass per unit area, in this system, if the total mass per unit area along the plane this has to be equal to M at any instant of time because the original mass is diffusing there is no accumulation or depletion of mass therefore, the original mass that was there at the initial time is present for all for their times.

This mass balance condition is the condition that I had over here the mass balance that the total mass is preserved in that sense it is slightly different from the previous problem and therefore, if I write this integral condition in terms of the similarity variables xi, I find that this concentration field has to scale as M by root D T times some function of xi. So, if the time dependence in the 3 factor in the previous case there was no time dependence because we have just fixed the concentration in this case the mass is fixed; that means, that the maximum concentration should scale as 1 over the spread in the spread the square root of D T and once we had that we have solved it we had got the concentration field.

This is the concentration field, this is what is called a Gaussian, it goes by the name of a Gaussian plume, the concentration field due to the spreading of a scalar as you can see it decreases as a bell shaped curve Gaussian curve. So, therefore, the curve C as a function of xi which is a bell shaped curve symmetric goes to 0 at plus or minus infinity. So, it is a Gaussian curve the area under the curve is always preserved, alternatively if you were to do it in terms, if you have to do it in terms of the spatial coordinate set initially I said it is going to be a delta function it is going to be a non0 only at z is equal to 0 and the thickness is going to go as 1 over the width as time progresses if there will be a spread and this spread you can actually calculate you can calculate the variance of this spread by simply taking the variance as.

Let us get rid of this delta function here since we have already completed it.

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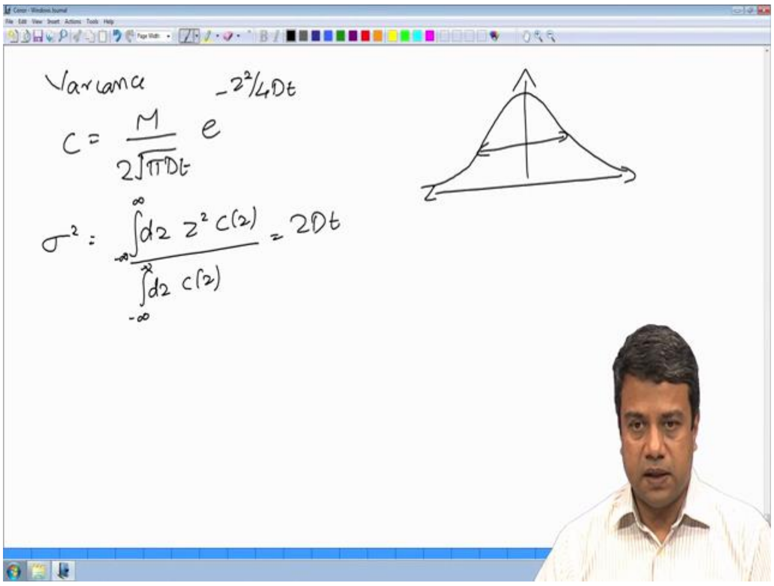

Variance

$$C = \frac{M}{2\sqrt{\pi D t}} e^{-z^2/4Dt}$$

$$\sigma^2 = \frac{\int_{-\infty}^{\infty} dz\, z^2\, c(z)}{\int_{-\infty}^{\infty} dz\, c(z)} = 2Dt$$

I know that concentration is equal to M by 2 root pi D t e power minus z square by 4 D t and that has a shape that looks something like this if I want to find out the variance of this I can take the variance the sigma square is equal to integral over z of z square times the concentration divided by integral D z of the concentration.

Integral over D z is the concentration is just the total mass m I can take integral over of z squared D z both from minus infinity to infinity in order to find out the mean square of the displacement of the mean square of the spread and this you can calculate quite easily this is just equal to 2 D t. So, the variance of this distribution function is increasing proportional to time in this case.

Now, this I did only in one dimension I assumed that there was a particular amount of material being put in at time T is equal to 0 and I was looking at the spread as a function of time. I could also do it in 3 dimensions I would not go through the details here, but in 3 dimensions what I would do is; ok.

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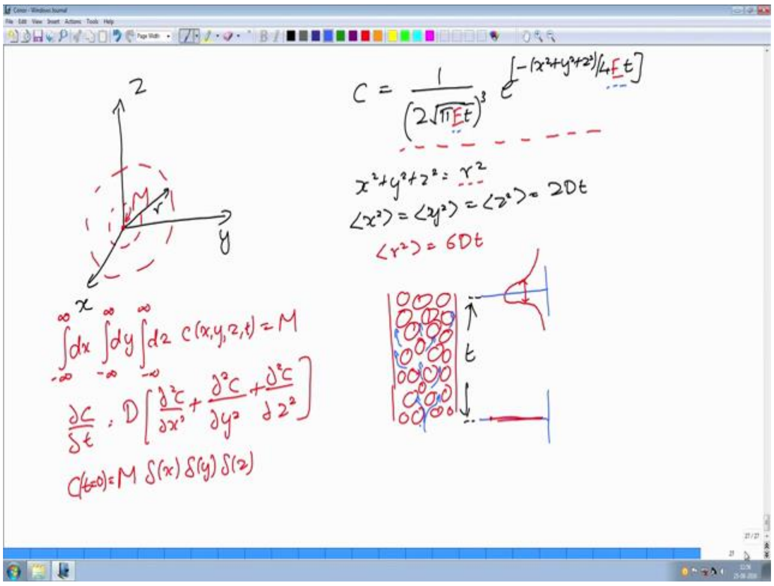

$$C = \frac{1}{(2\sqrt{\pi E t})^3} e^{[-(x^2+y^2+z^2)/4Et]}$$

$$x^2+y^2+z^2 = r^2$$

$$\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle = 2Dt$$

$$\langle r^2\rangle = 6Dt$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; c(x,y,z,t) = M$$

$$\frac{\partial c}{\delta t} = D\left[\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right]$$

$$C(t=0) = M\,\delta(x)\,\delta(y)\,\delta(z)$$

I have a 3 dimensional space x y z at time T is equal to 0, I would put a spot at the center put in a certain amount of mass M put in a certain amount of mass M at the center. So, this mass M is finite it has being put into an infinitesimal volume therefore, this is now a delta function in the x y and z directions therefore, I would have to solve the problem in the x y and the z directions subject to the condition that integral D x minus infinity to infinity of c of x y z T is equal to M.

The initial mass that was injected at that particular point, it is spreading with time as time progresses, it will spread further and further I can solve that problem I would not go through the details, but it is exactly analogous to this. The total amount of mass is M therefore, it is solving the differential equations in 3 dimensions now it turns out the differential equation is of the form d c by d t is equal to D into d square c by d x square plus d square c by d y square plus d square c by d z square I would not go into the details of how you get this because we will see later in the course, but I will just give you the result here. The concentration is equal to mass times the delta function of x delta function of y delta function of z because the spot is confined in all 3 directions; the x y and z directions.

Therefore, you have 3 delta functions each delta function I had told you has dimensions of one over length therefore, mass times one over length cube is a concentration. So, this is the concentration at T equals 0 and the solution for this concentration field is exactly analogous C is equal to 1 by 2 root pi D t whole cubed e power minus x square plus y square plus z square by 4 D t this is a Gaussian function in 3 dimensions and this. So, it is spreading symmetrically, you can see that it is symmetric with respect to x y and z and x square plus y square plus z square is equal to r square where r is the distance from the origin. So, it is spreading symmetrically in all 3 directions in this case the concentration goes as t to the minus 3 halves in this case the concentration goes as t to the minus 3 halves; you can see it over here 1 over T to the 3 halves because it is spreading in 3 directions in the previous case it was spreading only in one direction therefore, it was going as one over t to the half.

The variance in the x direction is equal to the variance in the y direction is equal to the variance in the z direction each of these is equal to 2 D t therefore, the variance in the radial spread will be equal to 6 D t in this case so that is for a symmetric flow which is spreading.

Now, why did I talk about this because this is actually of practical application? If you had for example, a gas coming out of a smokestack if gas coming out of a smokestack smoke, this smoke will spread out it will have fluctuating velocities and so on, in this case the spreading out is actually due to turbulent fluctuations it is not due to a molecular fluctuations, it is due to turbulent eddies and therefore, this spreading is not due to molecular diffusion, but due to turbulent diffusion; however, in even in that case you can use a Gaussian approximation for this plume to examine the rate at which it is spreading.

If I had a solution of this kind, if I had a concentration solution of this kind, I could use it even when there is turbulent dispersion, this is not molecular diffusion, but rather the turbulent eddies, only difference is that instead of the diffusion coefficient here I will have what is called a dispersion coefficient e, I will have a dispersion coefficient e which tells me the rate at which the turbulent eddies are spreading the material within this plume because it is actually a turbulent plume and this dispersion coefficient can actually be calculated back from the rate at which it is spreading. So, if I know after a certain time how much it has spread using this model I can find out what is the dispersion coefficient in this turbulent plume.

Similarly, in cases of laminar flows where I have dispersion rather than diffusion if I have a packed bed for example, which consists of densely packed particles which consists of densely packed particles a porous medium visible as the fluid flows through this medium, it undergoes numerous changes in direction and the fact that it has to go through the interspecies of this medium generates velocity fluctuations and that can once again enhance mixing that can be once again enhance mixing because this fluid is constantly changing direction as it is going through the medium therefore, the rate at which the spreading takes place in this case, will not be molecular diffusion alone, but it contains an convective effect because the material if the fluid is changing directions constantly.

If I want to find out the rate at which it is spreading, I can put in the pulse at this location, put in a pulse of solute at this location and then see a little while, later I can put in a pulse of solute at this location and then see a little while later how much it has spread what is the spread in this pulse a little while later. From knowing this spread and the time that has elapsed between these 2 the time that has elapsed between these 2 and knowing the spread I can find out what is the dispersion coefficient in this case as well. If it is spreading in one direction then I will use my previous expression, if it is spreading only in one direction then this expression can be used, it is spreading in 2 or 3 directions you can use the 2 dimensional or 3 dimensional delta functions.

This model is actually very useful for calculating what are dispersion coefficients due to turbulence or due to flow through porous media and so on, they are not molecular diffusion, but since the equation is similar in this case except that you substitute the molecular diffusion coefficient with the turbulent diffusion coefficient, solutions are the same. From the solutions knowing how much it has spread in a certain interval of time you can calculate what are these dispersion coefficients which are enhanced due to either turbulence or due to the flow through porous medium where the fluid velocity is constantly changing direction and that is right this is a fundamental solution the Gaussian plume is a fundamental solution which can be used to deduce what are dispersion coefficients it is useful not just in molecular diffusion, but in other cases as well where dispersion could be enhanced due to fluid velocity fluctuations generated in porous media or due to turbulence or due to other reasons.

These were all transport problems, phenomena problems in infinite domains. What does one do if we are to solve a problem in a finite domain; that I will start in the next lecture. I will see you then.