Transport Processes I: Heat and Mass Transfer
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Lecture – 24
Unidirectional transport: Similarity solution for decay of a pulse

Welcome to this, this is our 24th lecture on the fundamentals of transport processes where we were solving problems in unidirectional transport. I had post for you set of problems involving either mass or heat or momentum transfer, these problems were related to transport in one direction from an infinite surface into an infinite fluid.

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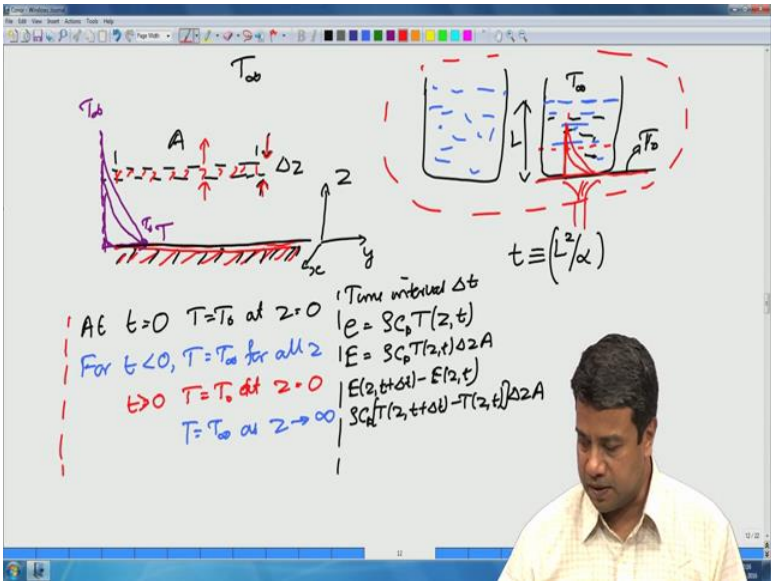

This could be post for mass transfer, heat transfer or momentum transfer. In the case of mass transfer it corresponds to having an infinite surface in which at initially the temperature everywhere is a constant and at initial time, instantaneously increase the temperature to a higher value. It corresponds to something like heating a surface that container of liquid you make the assumption of infinite fluid and that approximation is valid only when the penetration depth of the temperature field due to this heating into the fluid is small compared to the system size and we had post exactly analogous problems in heat transfer and momentum transfer, in mass transfer and momentum transfer.

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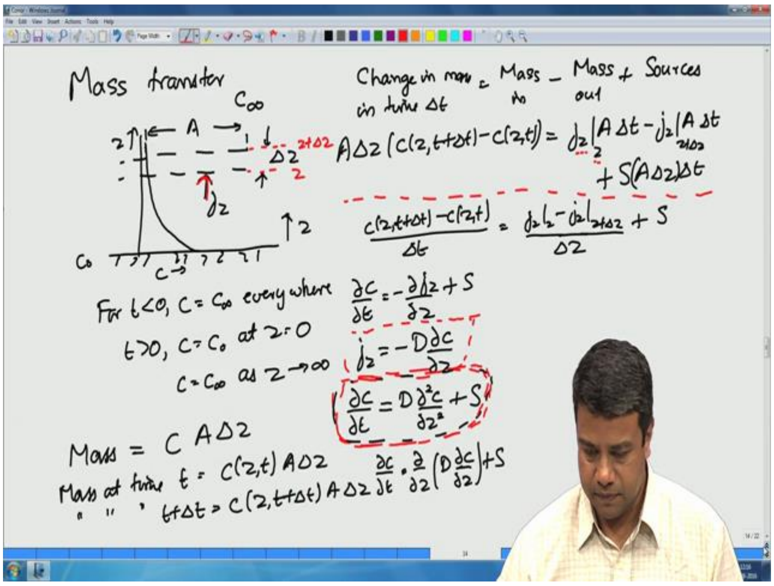

Mass transfer problem you instantaneously change the concentration at the surface and then see how the concentration varies with time and in momentum transfer you instantaneously translate the bottom plate and see how the velocity varies with time. All of these cases we get exactly the same form of the equation.

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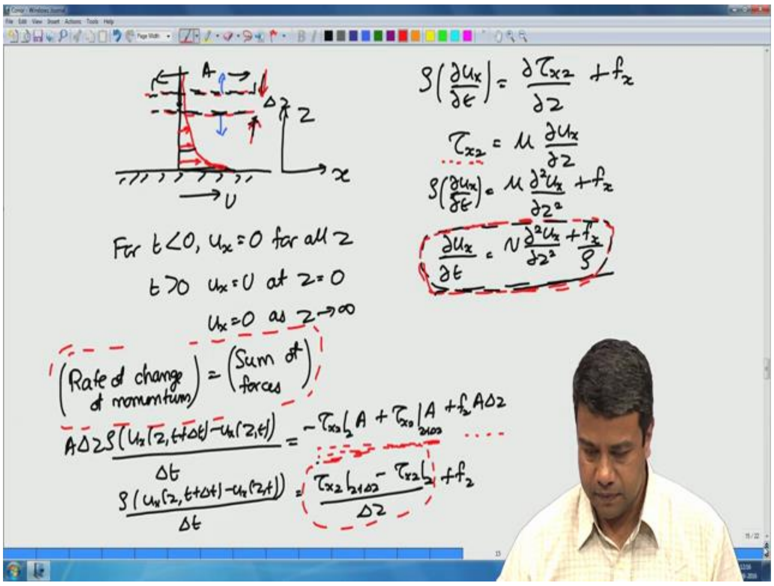

Diffusion equation which contains one derivative in time and 2 derivatives in this spatial coordinate and contains the dimensionless parameter the diffusivity whether it is for mass momentum or energy and it contains sources of mass or energy or the body forces in the case of momentum. These are all source densities the increase per unit volume per unit time for the body force is the force per unit volume, for you this is the body force and we use slightly different formulations for momentum transfer in comparison to mass and heat transfer.

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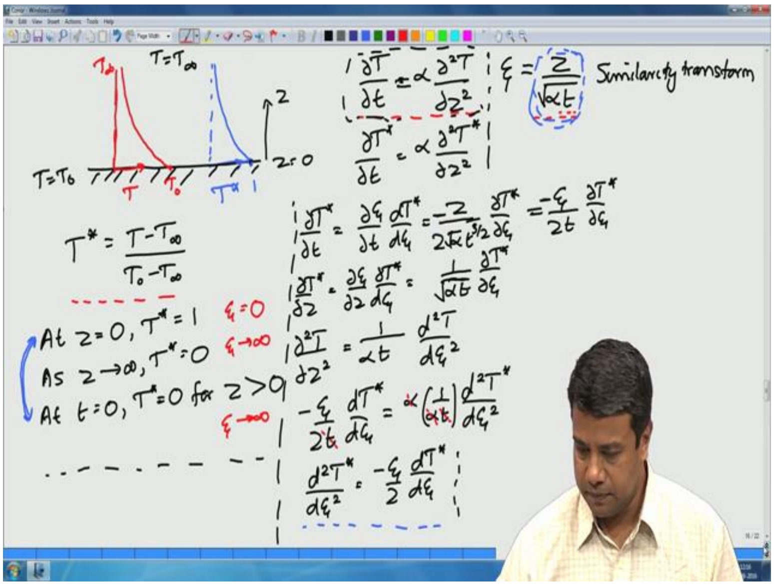

But we got the same equation and then we solve this equation using a similarity solution for this problem because there is a deficit of dimensional parameters for scaling the length and the time scales and therefore, we were able to identify a similarity parameter and we got expressions for this in terms of the similarity parameter for the temperature field the fluxes, the total transport rates.

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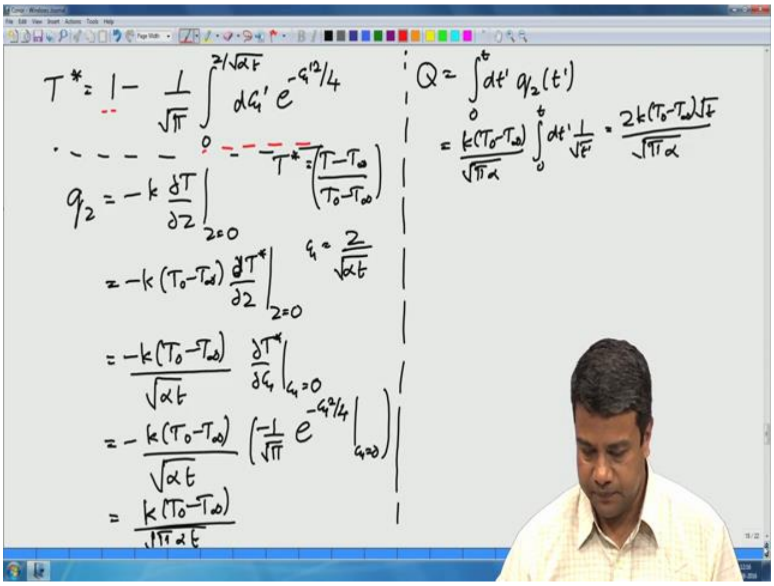

And these solutions were exactly the same whether it was for mass heat or momentum transfer.

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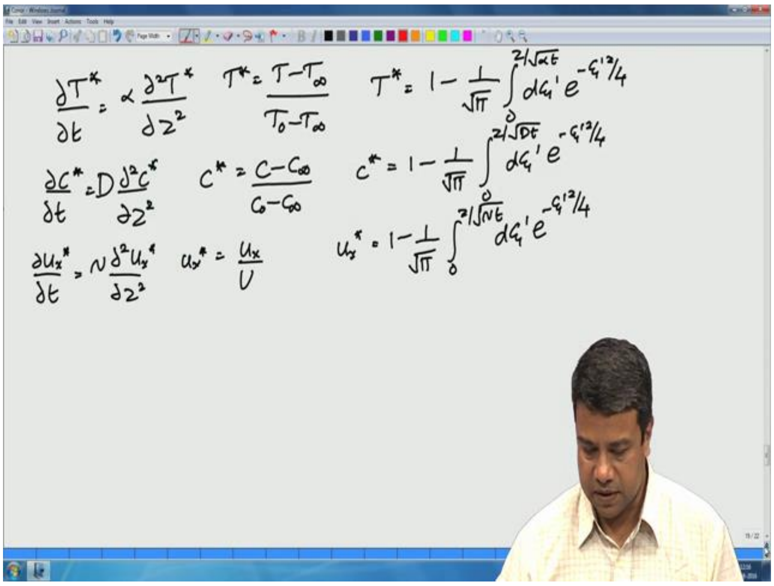

This might seem a rather idealized problem, but I had actually shown you an application where we can use this to get a correlation and that was mass transfer from a falling film.

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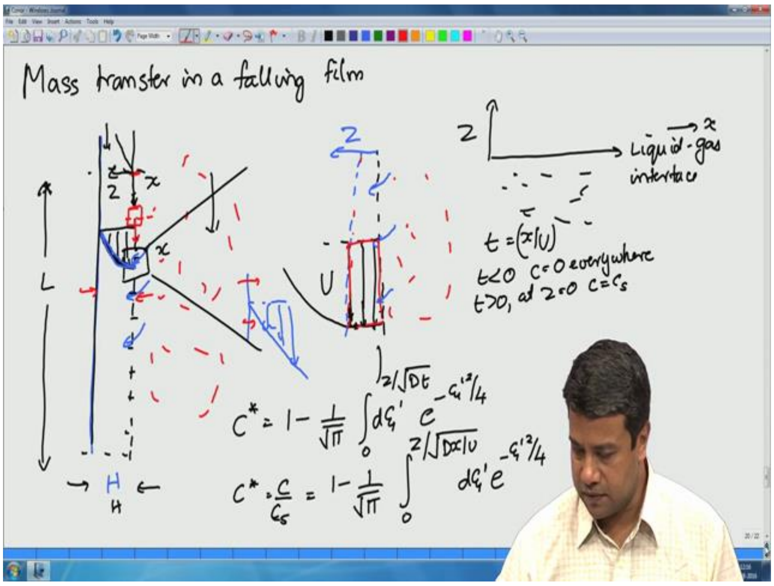

In this case rather than looking at the progression, the evolution in time, I look at the evolution for a partial of fluid as it is translating downwards. Assumptions are close to the surface, there is transport of mass penetration depth of the mass into the film is much smaller than the thickness of the film so that the film can be considered to be an infinite film. Also close to the surface since we have a 0 shear stress boundary condition, the velocity gradient is 0 therefore, the velocity close to the surface is approximately a constant because the slope is 0 therefore, I can consider this to be a fluid that is moving at a constant velocity and rather than solving it for a fluid moving past the reference frame at a constant velocity, I can set in a reference frame moving with the fluid, since it is moving with a constant velocity and in that reference frame as time progresses, it is analogous to diffusion from a surface into an infinite film.

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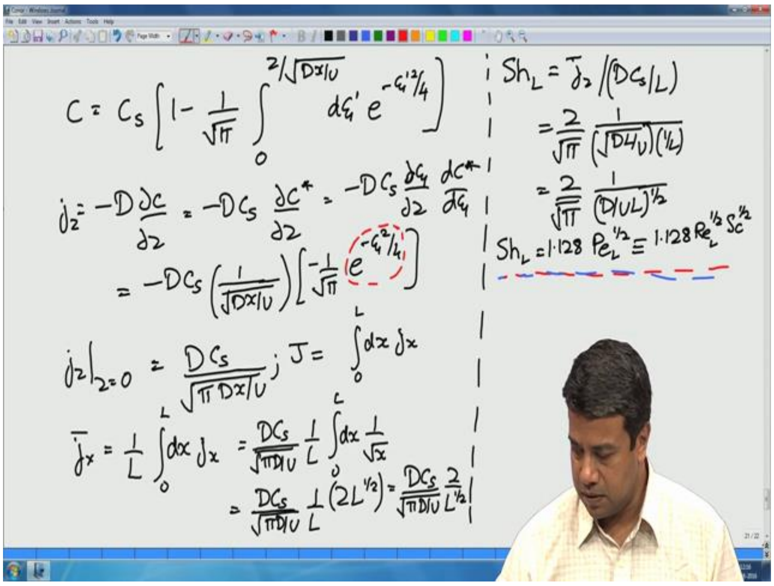

Based upon this analogy I had time is just equal to the spatial the distance moved divided by the velocity and based upon this I could I just written down and analogous equation is the concentration filed and based upon that you can calculate the flux, you can calculate the average flux, integrate the flux over the height and divide by the length of the film and from that average flux the Sherwood number and we got the correlation and I had also looked briefly at the conditions under which this approximation is valid.

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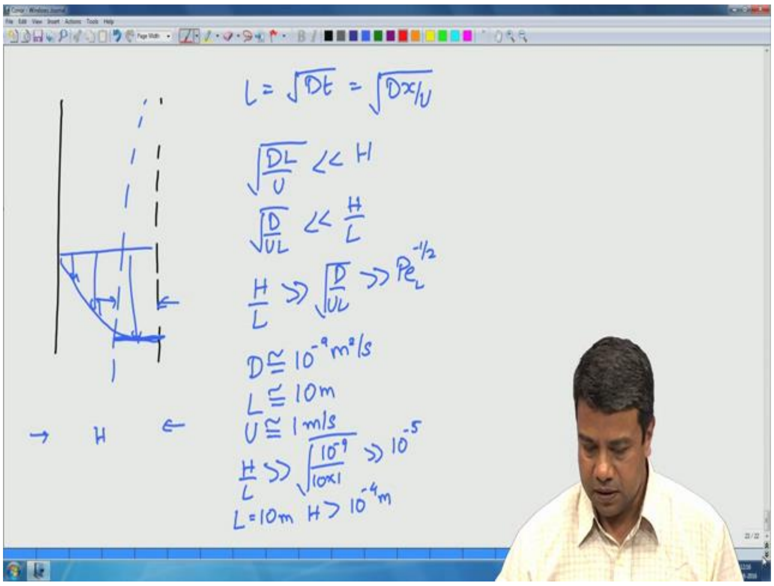

For real systems, these conditions are not very severe. So, long as the film thickness even if your length is of the order of 10 meters. So, long as your film thickness is of the other 4 millimeter or so, these approximations are still valid that is because diffusion is such a slow process the penetration depth is only of the order of 0.1 millimeters when the film has moved or a distance comparable to 10 meters for small molecule diffusion where the diffusion coefficients is of the order 10 power minus 9. So, this is a very good approximation in liquids.

Let us look at another problem here and that problem corresponds to the decay of a pulse.

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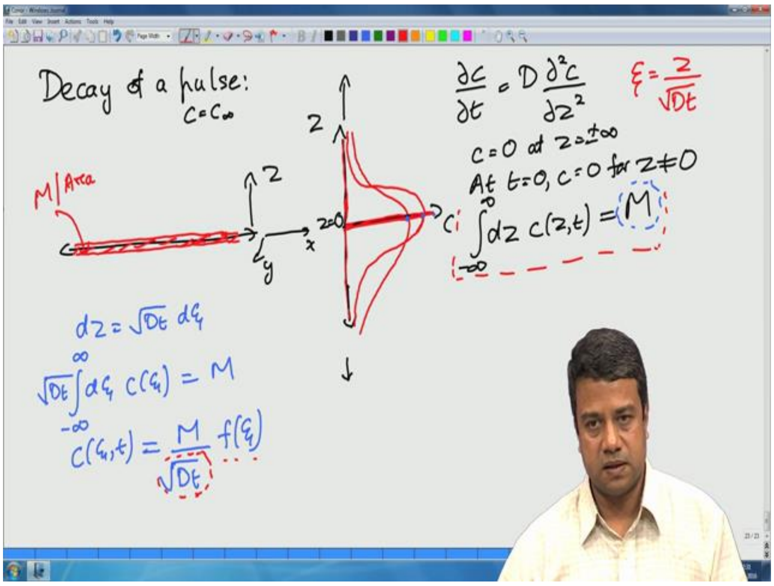

Physically, what does it represent? I will first discuss this problem in one dimension and then I will go on to 3 dimensions. So, I have this z code that perpendicular and the x y plane this is an infinite fluid the concentration c is equal to c infinity everywhere initially and at time t is equal to 0, I input a small pulse of the solute at z is equal to 0 I instantaneously put in a certain amount of mass of the solute at this location in this case the mass is per unit surface area in the x y plane.

I do not define the concentration here, but rather the total amount of mass that is input is m per unit area it is instantaneously inserted at this location it is like generating a certain amount of mass instantaneously due to a reaction at the surface. So, if I look at the progression of the concentration with respect to time I put a certain amount of mass into an infinitesimal thickness of fluid. So, if I look at the concentration variation in the z direction at different times, initially the concentration is 0 everywhere because I put the mass m at time t is equal to 0 only at this location and this location the concentration is very large if the concentration will basically go as the mass put in divided by the thickness because masses m has been put in per unit area, the thickness of this pulse is infinitesimally small, so the concentration will be very large.

As time progresses this mass that has been put in is going to diffuse out, it is going to diffuse out and as time progresses I will get that profile that looks like this initially, then I the progresses I will get a profile that goes looks like this my apologies this should be symmetric because this system as symmetry between plus and minus z right because there is nothing those distinguishing plus and minus z. So, the mass should actually concentration profile should actually be symmetric about z is equal to 0.

I should note that since the total amount of mass is conserved as it spreads the height has to come down, it was not like the previous problem where we had a constant concentration at the surface in this case the total mass is conserved. So, whatever is put in initially is what is going to diffuse. So, as time progresses as the total amount of mass is conserved the width of the concentration profile will increase with time the maximum will decrease with time.

What is the conservation equation is the same equation that we had I told you earlier while deriving the conservation equation that the conservation equation is derived for a volume within the fluid, therefore the conservation equation will just be partial c by partial t is equal to d, d square c by d z square and this has to be solved subject to the boundary conditions the concentration should be equal to 0 at z is equal to plus or minus infinity as you go further and further away from this initial pulse as z goes to plus or minus infinity the material has not defused there yet. So, the concentration has to be equal to 0.

Those are the boundary conditions for the concentration at plus and minus infinity. In this particular case due to symmetry you also know the d c by d z has to be equal to 0 and z is equal to 0 because the mass is being diffused symmetrically about z is equal to 0 therefore, d c by d z has to be equal to 0 that the slope has to be 0 at z is equal to 0 you do not have a boundary condition at a fixed boundary; however, we know that at t is equal to 0 c is equal to 0 for z not equal to 0.

Whenever for any location, apart from the location z is equal to 0 the concentration is equal to 0 at the initial time when you just inject at this pulse n at that initial time, when you inject this pulse in the concentration is equal to 0 everywhere else apart from z is equal 0. However, we also have an integral condition the total mass has to be conserved that is if I take the integral from minus infinity to infinity d z times c at any z and t at any instant of time I integrate the concentration over the entire z axis this has to be equal to the total mass put in per unit area initially this has to be equal to the total mass put in per unit area initially which is equal to m.

Those are the 3 conditions that I have in this particular case since I put in a pulse and it is diffusing in time, I have this integral condition. Now, how do we solve this problem once again as in the case of the previous similarity solution there are no length or time scales in the problem, there are no lengths or time scales in the problem therefore, I can write the equations in terms of one similarity variable that is nothing but I can scale z with because there is no length scale. So, it is an infinite fluid if the mass is diffusing in there.

Similarly, there is no timescale therefore, I can define the similarity variable xi as before as z by root d p, how do I scale concentration? Now that is where there is a difference between this and the previous problem, the condition that we have on the concentration is this total mass condition the concentration condition does not state that the concentration is fixed at one particular location if you look at the central location for example, the constitution is actually decreasing with time the maximum concentration is decreasing with time the width of the pulse is increasing with time.

Therefore I cannot specify a concentration and get any particular location; the concentration scaling has to come out of this balance condition. So, if I postulate if the concentration field is given by this function of z and if I re express that in terms of xi you know that d z is equal to root d t times d xi therefore, in terms of xi if the concentration field is no not individually a function of z and t, but only a function of xi my condition becomes minus infinity to infinity the xi root of d t in into c of xi is equal to m. Therefore, this maximum concentration is actually decreasing with time this maximum concentration by this scaling argument, if the total mass is fixed and the length scale penetration is root of d t the maximum concentration has to decrease as one over root d t which means that I can write this concentration it is no longer just a function of xi it should also be a function of t. So, I can write this as m by root of d t times some dimensionless function of xi because the maximum concentration now has to decrease as one over root d t as the thickness of this is bloom spreads proportional to root d t.

This is the form of the expression that I have to use in the concentration balance equation in order to get a similarity solution. Note that the dimensional parameter here with dimensional parameter here is not a concentration, but rather a mass per unit area. So, to get a concentration field I have to scale that mass per unit area by a length scale to get a concentration and therefore, this concentration is explicitly a function of xi and time it can be written as m by root d t times some function of xi so that is a similarity transform that we have to use in this problem and that comes out of the fact that concentration that the total mass per unit area is concerned.

Let us go ahead and solve this problem.

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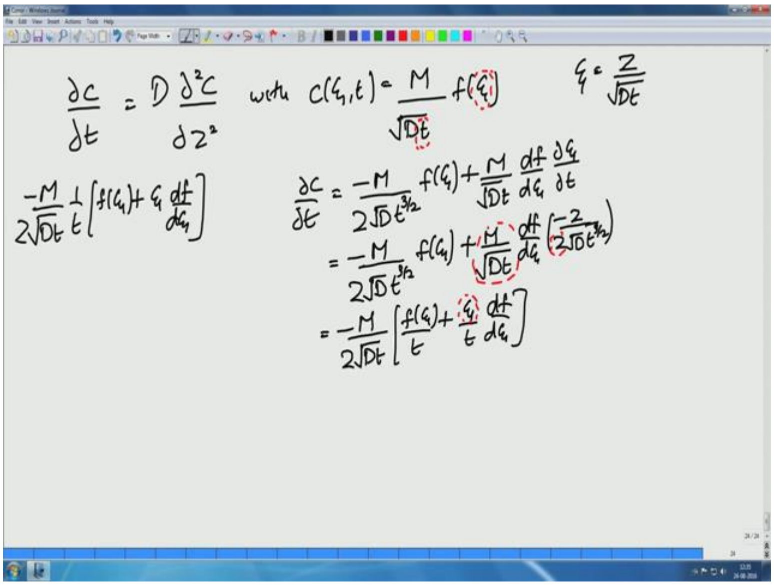

D square c, I am sorry, partial c by partial t is equal to D, d square c by d z square with c of xi t is equal to M by root D t some function xi. Therefore, partial c by partial t it has 2 terms here there is a dependence of time in the pre factor and there is a dependence of time through the function xi as well. So, the pre factor will basically give me minus M by root D t power 3 halves right f of I should have a factor of 2 here t power minus half is pre factor differentiation using chain rule.

It is equal to minus M by 2 root D, t power 3 halves f of xi plus M by root D t d f by d xi into we call that xi is equal to Z by root D t that was our similarity variable. So, this becomes minus Z by 2 root D t to the 3 halves. So, this is minus M by 2 root of D t into f of xi by t plus this term here is basically equal to xi by t you can see that I have taken this factor of m by root D t out, if I have take this factor of M by root D t out and this factor of 2 out and the negative sign also come out here therefore, I have Z by root d times D t to the 3 halves Z by root D t is xi and since I have t D t to the 3 halves in the denominator I have to include that 1 over t over there. That is the left side of this equation. Let us write that down here l over t into this factor of 2 there so that is the time derivative in this equation minus M by 2 root D t 1 over time times f of xi plus xi into d f by d xi, what about the right side.

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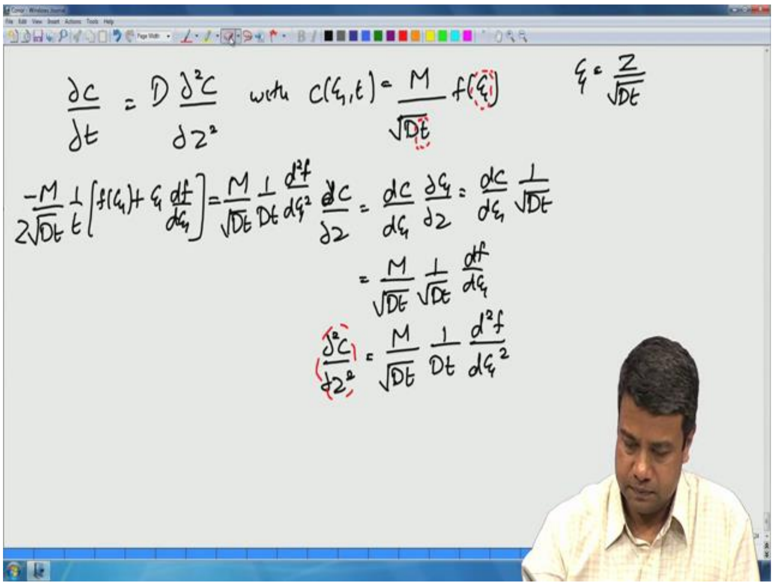

D c partial c by partial z is equal to d c by d xi partial xi by partial z is equal to 1 by root of D t. So, this will be equal to m by root d t, 1 by root D t, d f by d xi and when I take a second derivative, I will get one more factor of 1 over root D t. So, we take a second derivative partial square c by partial z square is equal to so let us put that as well on.

So, is equal to by D t my apology, it should be f here d square f d xi square so that is the second derivative here so that is the second derivative and that derivative of course, has to be multiplied by the diffusion coefficient, there is the diffusion coefficient that has to be multiplied by the diffusion coefficient so that is the equation that we have derived in terms of the similarity transform, the transform variable and you can very well see that in this case as well this M by root D t will cancel out on both sides, this diffusion coefficient cancels out and the time 1 over time cancels out on both sides and I end up with an equation which is a function of xi alone this equation is now in terms of the function f here, but the resulting equation depends only upon xi.

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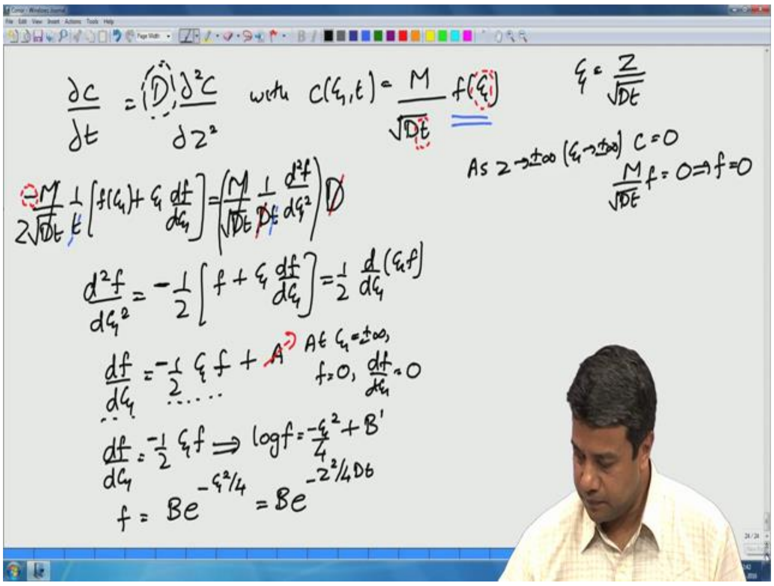

I can write this as d square f by d xi square is equal to 1 by 2 into f plus xi d f by d xi. So, that is the equation for the function f, it is actually quite easy to solve this equation because this can be written as half d by d xi of xi f integrate, it once you will get d f by d xi is equal to half xi f plus a constant a now this constant. You can determine from the boundary conditions if you recall here if I write down the boundary conditions in terms of the function f as z goes to plus or minus infinity which corresponds to xi going to plus or minus infinity c is equal to 0 which means that M by root d t f is equal to 0 because c is equal to M by root D t into this function f of xi.

At z is equal to plus or minus infinity this is equal to 0 which means that the function f has to be equal to 0 there is no surprise, I mean as you go to plus or minus infinity the concentration field has to go to 0 therefore, this function f of xi has to go to 0. As you go to plus or minus infinity this function f is equal to 0 the slope also has to be equal to 0 because if the function f goes to 0 the slope also has to be equal to 0; that means, that at xi is equal to plus or minus infinity f is equal to 0 and d f by d xi is equal to 0 because we have 0 concentration the slope is also 0; that means, that this constant a has to be equal to 0 in this particular case, from the boundary conditions at plus or minus infinity.

Therefore, you can now integrate this once again xi f which implies that log f is equal to xi square I missed a negative sign here, my apologies; if you recall, there is this negative sign here. So this has to come in this equation. So, this negative sign has to come everywhere in this equation, it does not of course, affect the constant k. So, this case will log f is equal to minus xi square by 4 plus some constant b prime therefore, f is equal to Be power minus xi square by 4 is equal to Be power minus z square by 4 d t so that is the expression for the concentration field.

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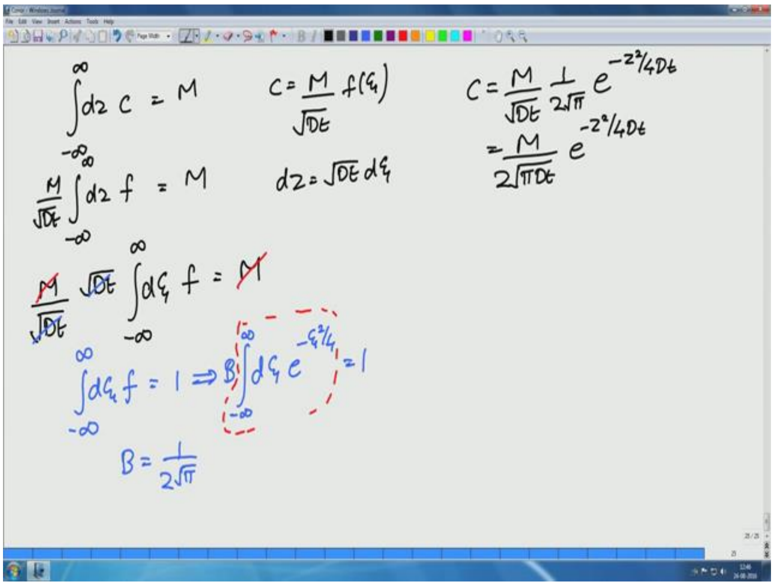

This constant f is of course, determined from the condition that integral minus infinity to infinity d z c is equal to m and we had written C is equal to M by root D t times some function f of xi that was our similarity transform therefore, if I convert now I will get integral minus infinity to infinity d z c m by root d t is equal to m converting d z is equal to root of d t times d xi I will get M by root D t, I am sorry there should be f over here times root D t integral minus infinity to infinity d xi times f is equal to M and you can see that the M cancels out on both sides and root d t cancels out and this condition basically states that this function has to be normalized integral minus infinity to infinity t xi f is equal to 1 and that determines the function b because ok and from this you will find that do this integral b will be equal to 1 by 2 root pi so that is the normalization condition for this function b is equal to 1 over 2 root pi because the integral from minus infinity to infinity, this term is just equal to 2 times root pi this term you should do the integral, it is a Gaussian integral you cannot do it analytically, but if you do numerically you will find that this is equal to 2 times root pi therefore, expression for the concentration field C is equal to M by root D t times f, f is 1 by 2 root pi e power minus z square by 4 D t I can also write this as M by 2 root of pi D t.

This is the concentration field for a pulse input in which the initial amount of material put in per unit area was equal to M. In this that as I said the maximum concentration decreases is 1 over square root of D t and the width of the pulse increases proportional to a root D t, we will look in a little more detail of the physics of this problem I have got for you the mathematical solution I will discuss a little bit about the physical solution of this problem, in the next lecture and where it is applicable, where this kind of a solution can be used it turns out that it is very important both in turbulent dispersion as well as in dispersion in porous media for example, and the dispersion coefficients are often calculated from profiles such as these. So, we will continue the discussion on the decay of a pulse in the next lecture, I will see you then.