Transport Processes I: Heat and Mass Transfer
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Lecture – 22
Unidirectional transport: Similarity solution for infinite domain continued

Welcome to this; this is our 22nd lecture on the fundamentals of transport processes. We had started solving problems on transport in one dimensional.

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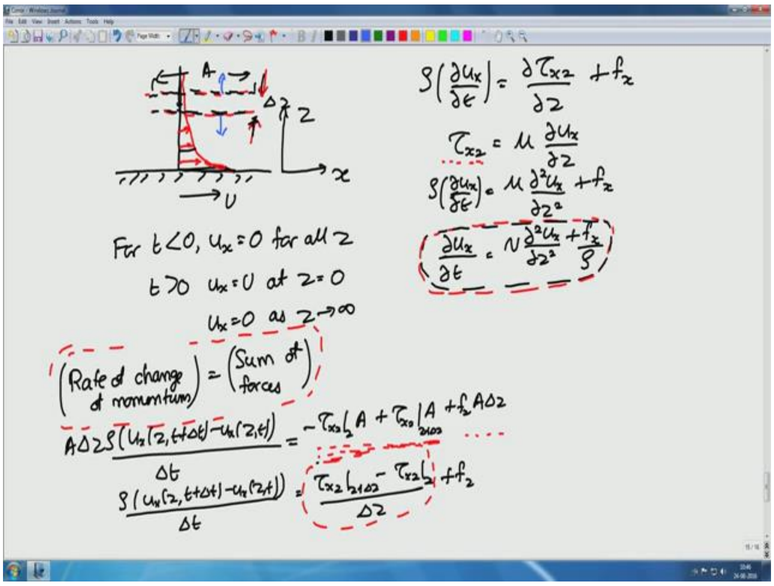

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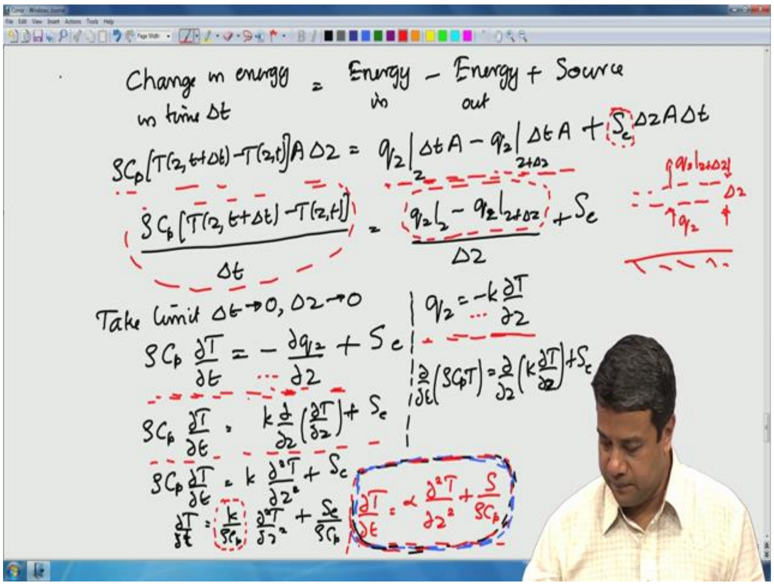

If you recall, I had in the previous case posed for you diffusion problems for heat transfer in the bottom here; for the infinite surface, infinite plane surface heated from below. We had done a balance for an internal differential volume and got the diffusion equation for heat transfer.

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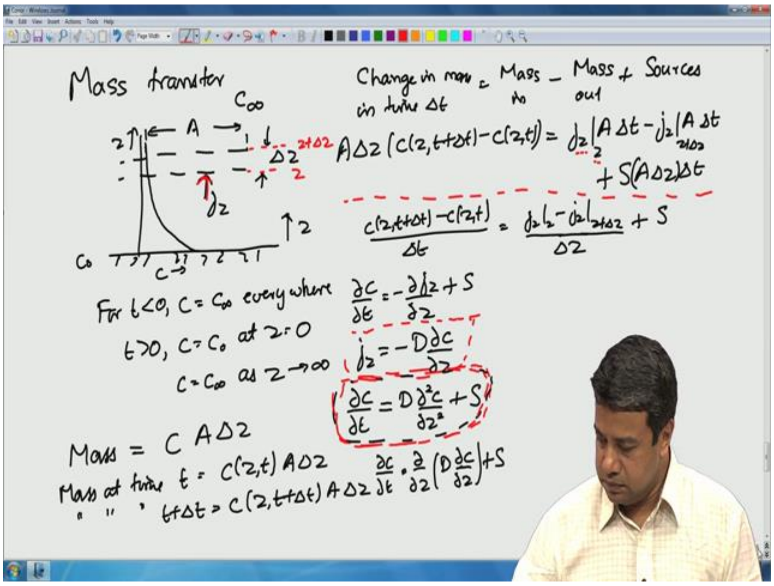

The same problem for mass transfer, we had post it as a mass transfer problem where initially the concentration is the same everywhere and at time T is equal to 0, the constant concentration is instantaneously increased at the bottom surface alone. In that case as well, we had got a diffusion equation for mass transfer and finally, for momentum transfer what you have is a stationary fluid and at time T is equal to 0, the bottom surface is moved with a constant velocity u and that was the momentum transfer problem.

The formulation was slightly different, rate of change of momentum is equal to the sum of applied forces; however, the equation that we got was exactly the same as heat and mass transfer except that the variable now was the velocity or the momentum density and the diffusion coefficient was the diffusion coefficient for momentum diffusion, the force instead of the sources we had a body force in the x direction; in the direction of momentum.

The only difference was to keep in mind is that the shear stress now contains 2 indices; one is the direction of momentum itself, after all momentum is a vector. The second is the direction across which there is transport or the direction perpendicular to the surface across which momentum is transported.

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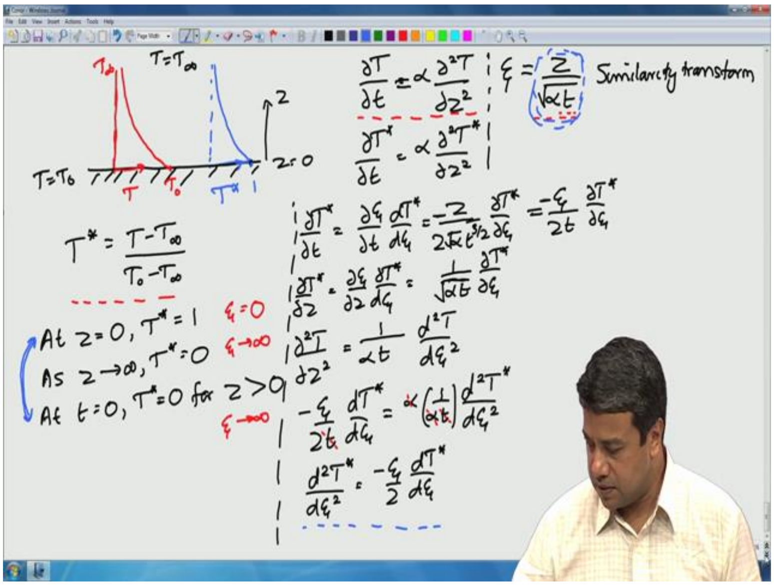

So, this heat transfer problem we had tried to get a solution. The original equation looks like this, we had neglected sources in this case; assumed that there are no sources. Let us scale the equations, using the scaling for the temperature so that in this transformed temperature, the temperature varies from 1 at the bottom to 0; far away from the surface. This is always advantageous because the temperature profile expressed in terms of the original temperature varies between T naught and T infinity, so that profile will change.

Temperature profile expressed in terms of the transform coordinate varies from 1 to 0 and that will not change regardless of the problem that is solved. So though we managed to scale the temperature, we could not find scales for the length or the time because as far as the spatial coordinates are concerned, the plane is infinite in extent and the fluid is also infinite, so we were not able to find coordinate, dimension to scale the length. Similarly the heating starts at; T equal to 0 and continues for all time, T equal to infinity. So, we were not able to find scaling for the time either there was a deficit of dimensional quantities with which we could scale the spatial dimension in time and the only parameter in the problem is this thermal diffusion coefficient alpha and therefore, we could find only one dimensionless independent parameter, a combination of the spatial in the time dimensions which was this parameter xi; z by square root of alpha t; this was the only dimensionless coordinate that we could find.

This is what is called a similarity transform, reducing from 2 independent variables to one independent variables because there is a deficit in the number of dimensional parameters you can use for scaling the problem. So, we had expressed the temperature in terms of xi alone, put it into this equation and the final equation that we get is only a function of xi; it does not depend individually on z and T. We had also scaled the boundary conditions, expressed those boundary conditions in terms of xi and we had found that 2 of these boundary conditions; one initial condition, the other boundary condition far from the surface when expressed in terms of xi reduce to just one condition that is required because what we have finally, is a second order differential equation in xi and we should get only 2 boundary conditions in xi.

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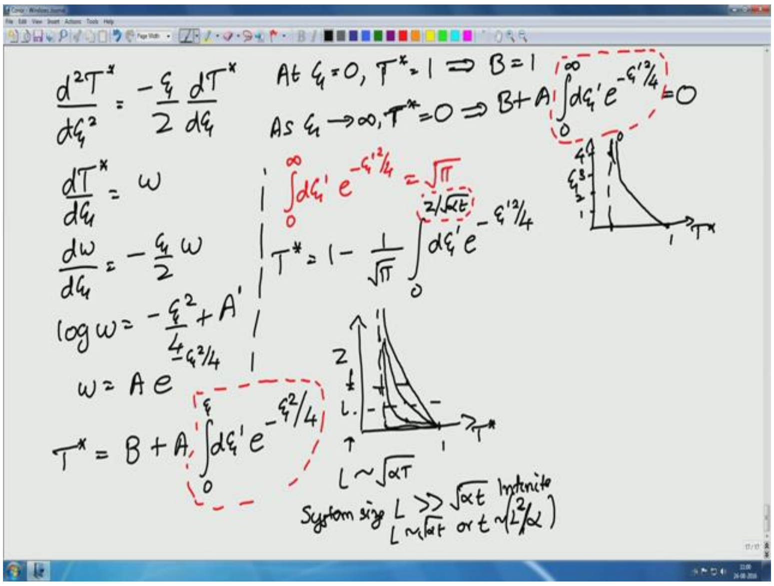

So, next let us proceed to solve this problem; the equation that I have is that d square T by d xi square is equal to; and the boundary conditions are at T star is equal to 1. So, those are the equations on the boundary conditions, this can be solved quite easily; you just write d T by d xi is equal to some parameter w the derivative therefore, d w by d xi is equal to minus xi by 2 w, if you integrate this one; you will get log w is equal 2 minus xi square by 4 plus some constant A. Therefore, w can be written as let us call this constant as A prime, w can be written as A e power minus xi square by 4 and this is equal to d T star by d xi which means the T star is equal to some constant B plus A integral; note that I have to integrate xi, so I have to use a dummy variable from 0 to xi; d xi prime e power minus xi prime square by 4.

So, this is the solution let me just remove this (Refer Time: 07:57), so that is the solution. Boundary conditions at xi is equal to 0; T star is equal to 1; therefore, this implies that B equals 1; as xi goes to infinity T star is equal to 0 this means that B plus A integral 0 to infinity is equal to 0. This integral, you cannot express it analytically further it is a Gaussian integral so you cannot express it analytically further, it is what is called an error function; however, you can get this one.

This integral 0 to infinity d xi prime; e power minus xi prime square by 4 is just equal to square root of pi. So, using that we know that b is equal to 1, so therefore, the final expression for T star becomes equal to 1 minus 1 over root pi; integral 0 to xi, b xi prime e power minus xi times square by 4. So, that is the final solution for T star and rather than writing it as xi, I can write this in terms of the original variables set by root d T; write in terms of the; once again I should write it in terms of the thermal diffusivity alpha z by root alpha t, so that is a final expression for T star.

Now we had earlier assumed that the fluid was infinite; you can ask the question, when does that approximation breakdown? So, let us first look at the approximations that we have made while solving this problem.

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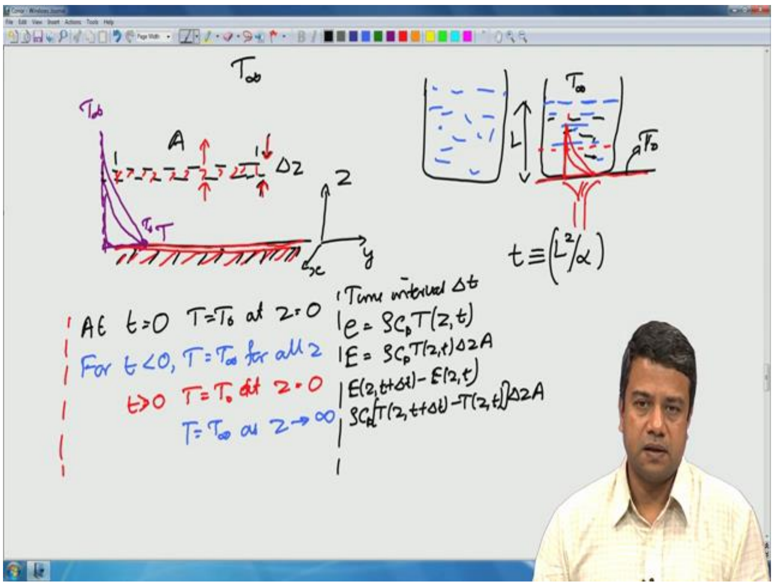

In this case a practical application will have a finite fluid; however, we have assumed in this case if the fluid is of infinite extent, so real fluids will always have some height and some area in the x direction. When we set infinite fluid approximation well; in other words what is the distance over which the fluid that the temperature due to the bottom has penetrated into the fluid; that is obvious here, the similarity solutions depends only upon this variable z. Therefore, if I were to express the similarity that the temperature in terms of the variables xi; at all times I am going to get a universal curve, T star as a function of xi. This curve is going to be independent of time, it goes from 1 to 0; this curve is going to be independent of time is given by this functional form; however, if I were to express this in terms of z I would get different curves.

This once again has to go from 1 to 0 far away; however, at early times it would basically be a step function then as time progresses this depth over which this temperature field penetrates would be small and then as time further progresses, this would increase and increase further. How does it increase? It is obvious from this functional form, this penetration depth of the temperature field, again this length scale L, over which the temperature has penetrated. L has obviously got to be proportional to square root of alpha T.

It has to be proportional to square root of alpha T, it is obvious from here. What this is saying is that, if I had different values of z and different time intervals, the temperature at each of those would be the same. So, if I had different values of z and different time intervals, so long as square z by root alpha T is the same, the temperature is the same at all of these locations even those z may change, T may change. So, long as z by root alpha T is a constant, the temperature is the constant.

Now, the temperature field actually decays pretty quickly actually; if I actually plot this temperature field the 1, 2, 3, 4, 5 and so on; this decays to very close to 0 by the time you reach a distance of about 4; when xi is equal to 4 from the surface, you reach more or less 99 percent and the temperature reduces to about 1 percent of its original value, within a box xi is equal to 4. So, the penetration depth is basically some constant times square toot of alpha T and this increases proportional to p to the half.

So, long as my system size, L is much larger than root alpha T; I can assume that the fluid is an infinite fluid. Once L becomes comparable to alpha T, then this approximation is no longer valid, this is infinite fluid when L becomes comparable to root alpha T or T scales as L square by alpha. Alpha is a thermal diffusion coefficient; it has dimensions of length square per time. So if you take L square by alpha, you get a time, so for time small compared to this thermal diffusion time across the length L, you can consider the fluid to be infinite.

Once the time becomes comparable to the thermal diffusion time, you can no longer consider the fluid to be infinite and the fact that there is an upper boundary does matter and that thermal diffusion time just from dimensional analysis has to scale us then square by the thermal diffusion coefficient correctly; this is just a simple scaling good. So, now, we looked at under what conditions this approximation is valid; obviously, this proclamation is valid so long as the length scale is the time of heating a small compared to the thermal diffusion time across the height of the container, once it becomes comparable you no longer can assume that the fluid is infinite.

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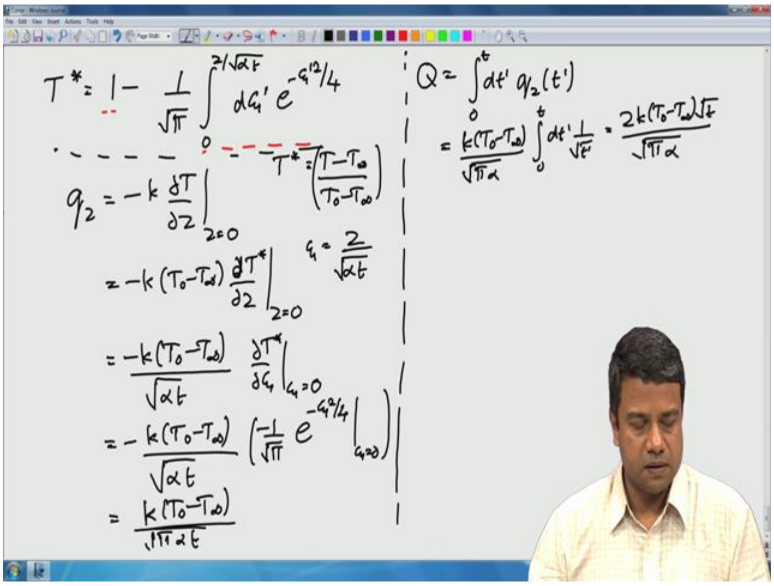

So, next thing how do we calculate the fluxes in this case for example, so I have the temperature field T star is equal to 1 minus integral 0 to z by root alpha T; recycle time. So as for final expression of temperature, what is the heat flux coming from the surface; q z is equal to minus k; d T by T z at z is equal to 0, there is a thermal flux, total heat energy coming out from the surface.

I can express this in terms of T star; after all we know that T star is equal to T minus T infinity by T naught minus T infinity. So, d T by d z will just be equal to minus k, T naught minus T infinity; d T star by d z. Now I have to use the substitution that is xi is equal to z by root alpha t. So, if I use that substitution this is at z is equal 0, I can write this as minus k into T naught minus T infinity by root of alpha T; partial T star by partial xi at xi is equal to 0 because when I take the derivative with respect to xi, I just get one factor of 1 over root alpha T coming out and what is the value of d T star by T xi at xi is equal to 0; this is the value of T star.

If I take one derivative, the derivative of the first term here 1 is just equal to 0. The derivative of the second term, second term is an integral from 0 to z. Therefore, the derivative of the second term is just equal to the value of the integrand, what is within the integral you are taking the derivative of an integral; you just get the integrand alone. So, that is equal to times minus 1 by root pi into the value of the integrand in this case is just equal to e power minus xi square by 4; at xi is equal to 0. This is just equal to 1; it is xi square it x I is equal to 0 exponential of minus xi square by 4 is just 1.

So, I basically get k by root of pi alpha t, so that is going to be the heat flux coming from the surface. So, the heat flux decreases with time as 1 over square root of T that is no surprise; the penetration depth increases proportional to square root of T, the heat flux is the derivative so derivative will scale as the temperature difference divided by the characteristic length scale; therefore it decreases 1 over square root of T from the surface.

You can also find out what is the total heat that is coming out up to a certain time. How much heat has come out of the fluid? Up to a certain time due to this heating that should be the amount of heat that you have to supply at the bottom due to this heating. So, how much heat as come out; q will be equal to integral from 0 to T; d T prime q z at T prime and you can do this integral quite easily; you will get basically k into T naught minus T infinity by root of pi alpha; integral d T prime 1 by root T prime from 0 to t. So, this will turn out to be equal to k when I do the integral of 1 over root T; I get square root of T divided by half, so if I get a factor of 2 there.

So, I will get 2 k; T naught minus T infinity, square root of T; divided by root of phi times alpha. So, that is the total amount of energy that has emerged from the surface up to a time T and as time progresses that scales us the square root of T. The other issue here is that in all of these cases I had solved the problem for heat transfer.

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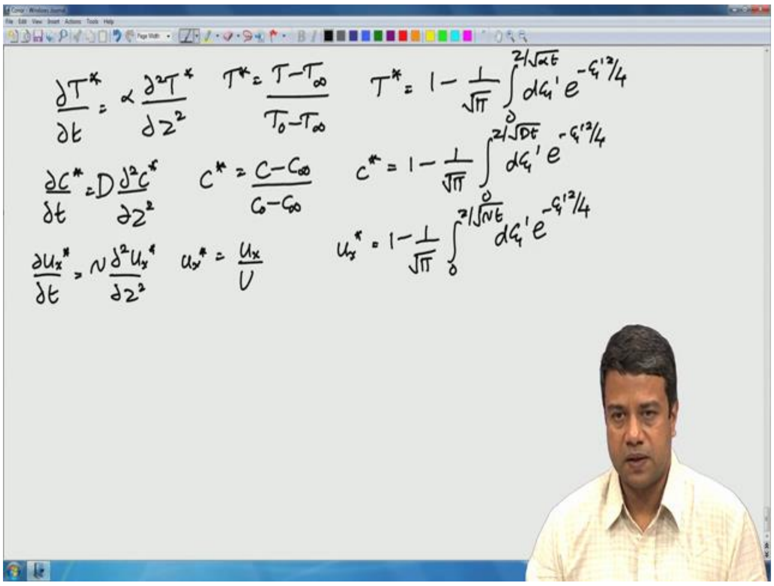

Partial z square; this equation where T star is equal to T minus T infinity by T naught minus T and the solution was T star is equal to 1 minus 1 by root pi integral 0 to z by root alpha T, the xi prime; e power minus xi times square by 4 that was the solution, I had also post for you a problem of mass transfer.

This was a problem of mass transfer; C is equal to C naught at the surface, C infinity far away. At time T is equal to 0, you impose the concentration is at constant C; infinity everywhere and you imposed a different concentration at the surface C naught therefore, diffusion takes place. Equation was exactly the same therefore, the solution that I get will also the exactly of the same form.

Partial C star by partial T is now thermal mass diffusivity partial square C star by partial z square with C star is equal to C minus C infinity by C naught minus C infinity. Suppose at this way, C star is equal to 1 minus 1 by root pi integral 0 to z by now root dt and substitute the mass diffusivity for the thermal diffusivity; d xi prime e power minus xi prime square by 4 exact same solution.

Similarly I could solve the momentum transport problem as well; in this case we are looking at the transport of momentum; x momentum in the z direction. The equation that

I ended up which was exactly the same, the equation that I ended up is exactly the same in this therefore, for this equation the solution will once again be exactly the same. My equation will be of the form partial u x star by partial T is equal to nu partial square u x star by partial z square, u x star will just be equal to u x by u. Note that the velocity u infinity was equal to 0 because before it was stationary and the solution for that will be equal to 1 minus 1 over root pi integral 0 to z my root of now kinematic viscosity times T.

So, this shows the analogy between heat mass and momentum transfer in the simple case, where we use similarity transforms. I tried to convey to you that there is a diffusion time scale or a diffusion length scale; the diffusion time scale is basically comes out of the kinematic viscosity with mass diffusivity or thermal diffusivity and when this diffusion time scale is small, you can actually assume that the fluid is infinite, but it becomes when the length scale becomes comparable to the system size, you can no longer assume that this is an infinite fluid and you have to actually solve the problem that takes into account the upper surface.

This has seemed an idealized problem, but; however, it does have practical applications. I told you earlier that I will show you one particular practical application where this can actually give us a correlation for the (Refer Time: 26:25) number for the concentration field, I solve this problem in terms of the mass concentration, it seemed the rather idealized problem, but I told you that I will solve one problem where I will actually calculate the correlation for the mass transfer and that problem is mass transfer in a falling film; this is an industrial process.

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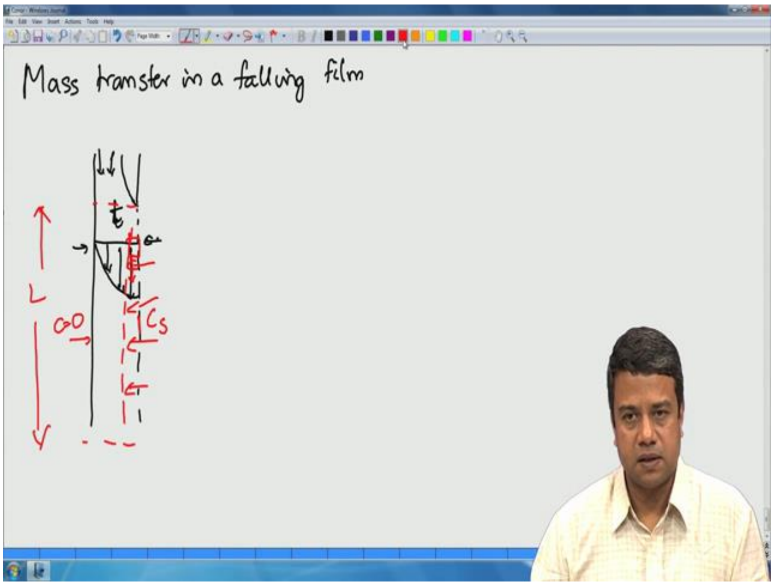

So, what happens is that you have some wall along which you have a fluid that is flowing downwards. The fluid is flowing downwards with one some particular velocity field, these are operations called scrubbers which try to reduce the poisonous gases and exhausts. So, you have a gas flow upward here and you have a fluid flow flowing downwards. So let me just draw this a little bit better for you; you will basically have this wall here and the fluid will actually come in through; we are on top. So, it will actually come in and then it will flow as a film down; a film of some thickness T and as the fluid is flowing downwards, there is going to be mass diffusion of the species that you want to remove; across this interface and we can assume that the concentration in the gas is a constant whereas, the concentration in the liquid is less than the solubility and because of that you have a mass flux across.

So, therefore, we have a situation where the concentration at the surface of this gas is some value is CS and into the liquid this concentration C is equal to 0 and that is mass diffusion across. This mass diffusion takes place over some length scale L, this is a length scale for which there is diffusion that is taking place across the interface and the fluid is moving downwards with some velocity profile u.

Now this problem, we can idealize it as the flow in an infinite film of fluid provided the penetration depth of the gas within this fluid, it is much smaller than the total thickness. In that case, it looks like the flow through an infinite fluid and we can also idealize it as a time dependent problem, if we consider the partial of fluid as it moves downwards. If you consider each individual differential volume of fluid as it is moving downwards and gas is diffusing it. So, rather than following the progression in the coordinate, we can just follow the progression of this particular element in time and so that is the idealization.

So, I will show you in the next lecture how this can give us an actual correlation for the mass transfer coefficient or the (Refer Time: 30:07) number. Using the similarity transform that we had used earlier in solving the flow infinite fluid, so this little bit we will continue in the next lecture. I will see you then and after this, we will look at some other problems where we can use similarity solution, I will see you in the next lecture.