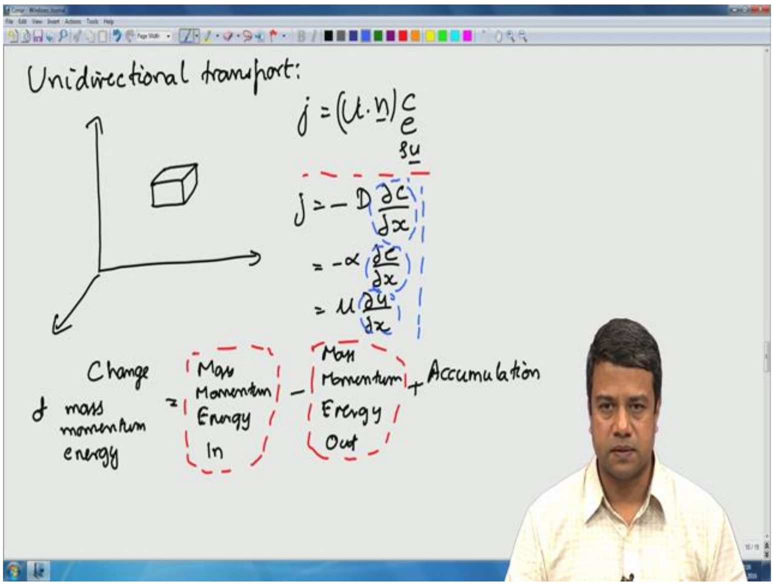
Transport Processes I: Heat and Mass Transfer
Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture- 21
Unidirectional transport: Similarity solution for infinite domain

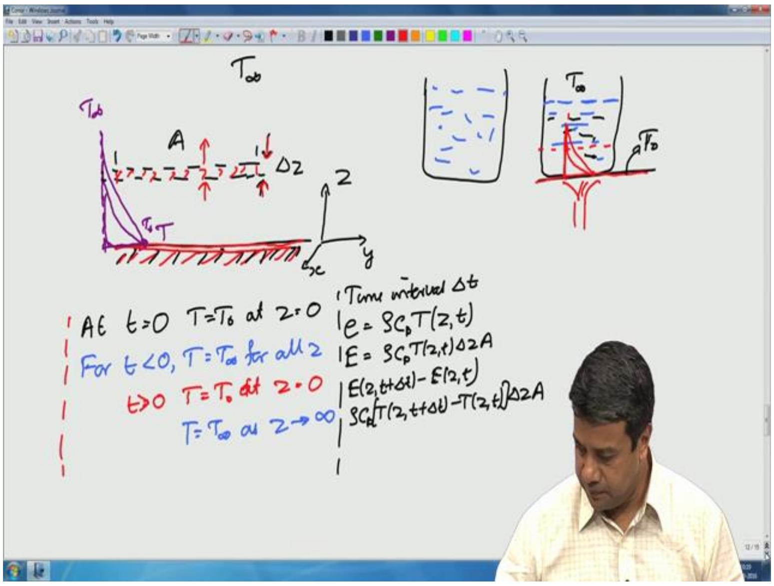
So, welcome to this is lecture number 21 of the course on fundamentals of transport processes. We had in the previous few lectures first looked at dimensional analysis and correlations and how these dimensional correlations are derived in different situations for the dependent variable that is the non dimensional flux as a function of a set of independent variables, which are ratios of convection and diffusion, different types of diffusivities and where applicable interfacial forces inertial forces gravitational forces and so on.

(Refer Slide Time: 00:33)



We had looked in some detail at the diffusion process itself. How is it that diffusion takes place? There are two broad categories of transport: one is convection which is due to the mean fluid flow and that convective flux the flux per unit area the amount transported per unit area, per unit time across the surface, it is just equal to the fluid velocity perpendicular to the surface, times the density of what is being transported the concentration energy or momentum. The diffusion flux on the other hand is related to the variation of the concentration energy or momentum across the surface and these are given by constitutive relations, they contain a diffusion coefficient with dimensions of length square per times the gradient gives the quantity of the materials of that particular point of the density of that quantity the spatial variation, difference in concentration divided by the distance in the limit as the distance goes to zero, this is just the spatial derivative of the concentration the energy or the momentum.

(Refer Slide Time: 02:25)



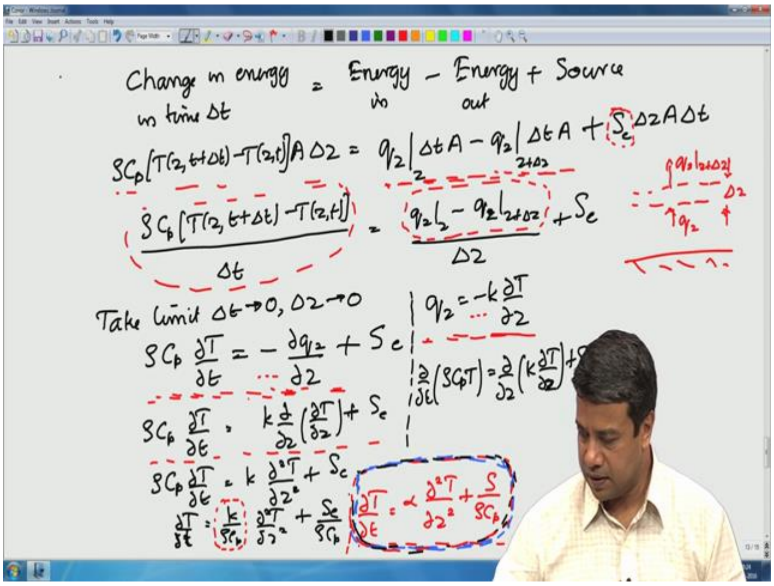
So, those are the two mechanisms and we were looking in the last lecture at how to apply this to a unidirectional transport problem? So, for definiteness what we had was a surface at z is equal to 0. So, we use a coordinate system where the x and y directions are along the plane and z is perpendicular to the plane, we had the surface with initially the temperature was a constant everywhere in the fluid and instantaneously at the time t is equal to 0 this surface alone was heated to a higher temperature. The surface alone was heated to a higher temperature and we wanted to know how this increase in temperature at the surface is going to heat up the fluid.

Note that in this case there is no convection therefore, there is no fluid flow perpendicular to the surface therefore transport has to be due to diffusion alone. So, we had written down a balanced equation, the boundary and initial conditions for this problem were initially t is equal to t infinity everywhere, the initially the temperature is equal to infinity everywhere for t less than 0, at time T is equal to 0 the surface is heated.

So, therefore, for t greater than 0 the temperature is equal to T naught at the surface; however, since we have considered this as an infinite fluid far from the surface the temperature is still equal to T infinity and I try to explain to you that this could be an idealization of a problem, where you are trying to heat a fluid in the container; initially the temperatures are constant everywhere and at time t is equal to 0 you heat up the fluid from the bottom and therefore, the heat will diffuse up the fluid, this approximation of an infinite fluid is valid when the distance to which the fluid heats up its much smaller than the total height of the container, in that case this approximation is valid and we will see a little later as we go through the course, as we go through this problem, what is the thickness of the fluid layer which is actually heated up.

So, we had forced this problem.

(Refer Slide Time: 05:00)



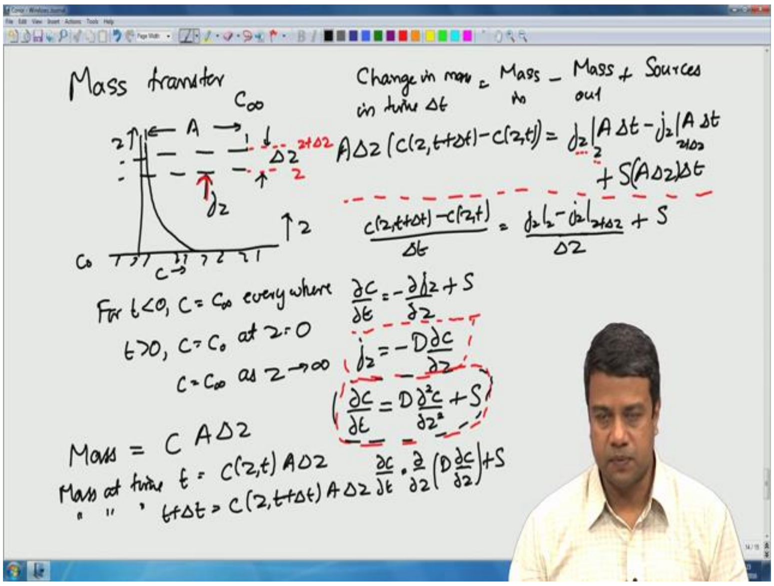
Change in energy within a time delta t is equal to energy in minus energy out, plus the sources of energy may be due to phase change, due to exothermic reactions, due to endothermic reactions and so on. The change in energy within a time t is just equal to the energy the internal energy of the fluid at time t plus delta t minus the internal energy had time t.

Energy in at the surface z; if you recall we had the differential volume of thickness delta z within the fluid, the energy in at the surface z is q z there is increasing the thermal energy if q z is positive, if the flux is upwards. Energy out is at the location z plus delta z that is leaving therefore, if q z at z plus delta z is positive, the energy within this volume is being decreased and that was these two terms energy in and energy out, plus the source term.

Source is a volumetric source energy added per unit volume per unit time. So, you take this balanced equation divide throughout by volume and the time interval to get a differential equation for the temperature. Basically the states that the rate of change of energy within this volume is equal to the net flux what comes in minus what goes out a negative sign because what is coming in at z is increasing the energy, what is leaving at z plus delta z is decreasing the energy and then you have the source term.

In the constitutive relation for the heat flux q z, is equal to minus k d T by d Z, energy is transported from regions of higher temperature to regions of lower temperature therefore, the negative sign and when you put that in you get the balanced equation of this form alternatively if I divide throughout by rho C p, the balanced equation basically becomes a diffusion equation d T by d t is equal to the diffusivity times d square T by d Z square plus the dimensionless source.

(Refer Slide Time: 07:32)

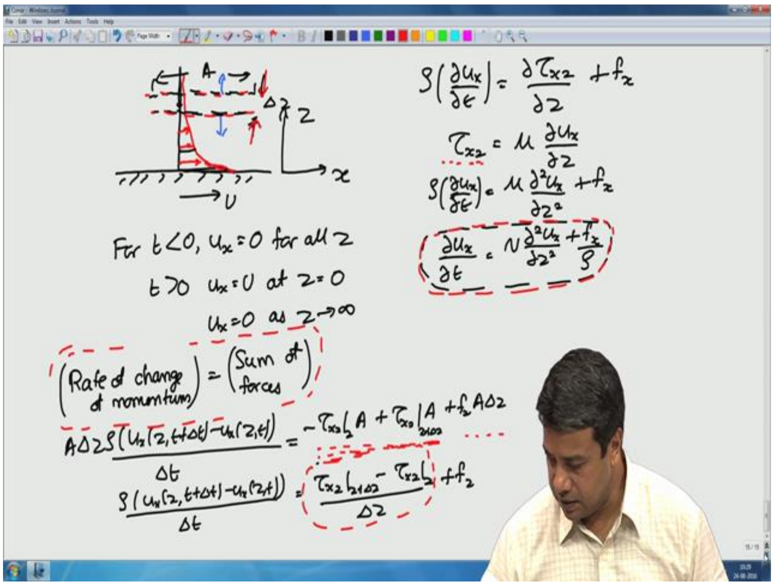


One can define an exact analogous problem for mass transfer; the surface initially the concentration of some solute is 0 everywhere or C infinity everywhere within the fluid, at time t is equal to 0 you instantaneously increase the concentration on the surface to C naught and then you want to see how the concentration evolves, the equation is the same once again change in mass instead of heat it change in mass within a time delta t is equal to mass in minus mass out plus sources. You get an equation for the concentration field in terms of the spatial derivative of the mass flux j z; j z is a mass flux in the z direction it has exactly the same form; d c by d t is equal to minus the derivative of the flux with respect to z plus any sources or sinks of mass due to reactions.

And then you have to put in the constitutive relation, the flux is equal to minus D times d c by d z and from that you get the concentration diffusion equation d c by d t is equal to D times t square c by d z square plus the sources. All of these assume that the diffusivities are independent of position; in general if the diffusivities do depend upon position the correct equation is actually d by d z of D d c by d z plus S. If the diffusivities do depend upon position the diffusion coefficient does have to come within one of the derivatives. Similarly if the thermal diffusion coefficient was dependent upon position then the equation which will actually be rho C p or rather. If the density and the specific heat do depend upon position, the equation be of the form d by d t of rho C p T is equal to d by d z of k d T by d z plus the source.

So, that is how the equation would look if the thermal conductivity in the specific heat were dependent upon position, when these are independent if these are constants for the fluid you will get the diffusion equation.

(Refer Slide Time: 10:14)



We had solved this for momentum transfer as well, in that case we had considered the surface the fluid adjoining a surface and infinite fluid, initially everything was at rest and at time t is equal to 0 the surface was moved instantaneously with a velocity U.

So, for t less than 0 the velocity U x is 0 everywhere and 40 greater than 0, U x is equal to capital U at z is equal to 0 far from the surface the momentum, diffusion has not yet happened if you go to a large distance from the surface, how large we will see once again, but if you go to a large distance from the surface since momentum diffusion has not yet taken place, the velocity still has to be equal to zero. You had framed the problem slightly differently in this case; the rate of change of momentum is equal to the sum of the forces acting on that fluid element of thickness delta Z.

So, the rate of change of momentum is equal to sum of forces. Forces are of two kinds: surface forces which act on surfaces and body forces which act throughout the entire volume. Surface forces are the shear stress the momentum diffusion across the surface, body forces are forces like gravitational force centrifugal force and so on, which depend upon the volume of the fluid. So, the gravitational force exerted known volume is proportional to the volume itself he increase volume the forces higher.

So, you had two forces the surface forces acting on the top and bottom surface and the volume volumetric forces and on that basis once again there was an intricacy here regarding the sign convention, we had defined the stress as positive if attacked at a surface whose outward unit normal is in the plus side direction, note that the stress tau x z momentum in this case is a vector, it has the momentum itself has a direction the direction of the force and there is also a direction of the surface, the direction of the transport; the force itself is in the x direction, the transport is taking place in the z direction.

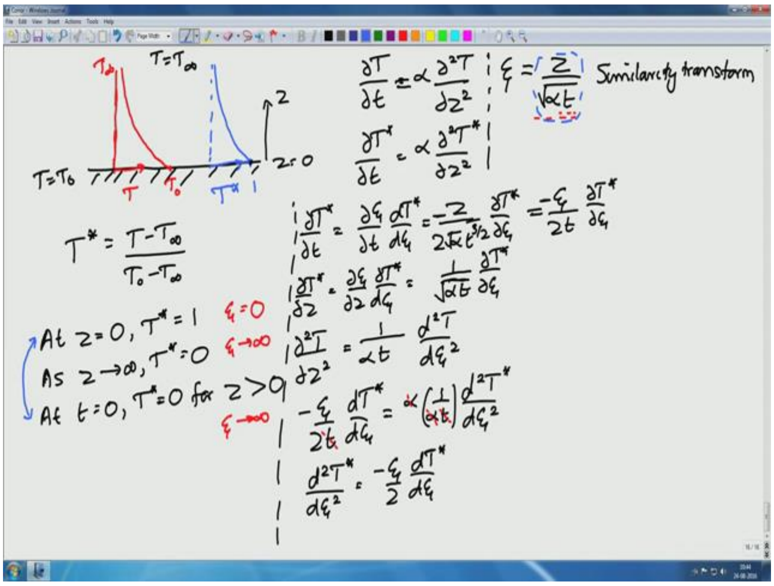
In other words the z coordinate is the direction perpendicular to the surface across which the transport is taking place. So, the stress will have two indices: the first one the direction of the momentum, the second one the direction in which the transport is taking place or the direction of the perpendicular to the surface across which the transport is taking place. In this case the velocity is in the x direction, so the direction of momentum transport is the x direction and the direction in which the transport is taking place is the z direction.

So, those who are the two indices in the expression for the momentum transfer in the expression for the stress, we had assumed the positive sign when the z direction is in the positive direction for the top surface and negative when it is in the minus z direction, the shear stress itself has a positive sign, so the equation that you get for the velocity field turns out to be exactly the same form as the equation that we had for concentration and energy diffusion except that instead of the mass is the concentration or the temperature, we have the velocity; instead of mass diffusion or energy diffusion we have momentum diffusion of the kinematic viscosity. And this equation will be the same for all unidirectional transport problems, this equation was a balance that was done for a particular differential volume within the fluid therefore, the form of the equation does not change if you change the configuration, if you had an infinite fluid instead of a semi infinite fluid or if you had a fluid between two plates so that it was infinite domain.

This equation was written for an internal volume and therefore, that is going to remain the same independent of configuration so long as transport is taking place only in one direction. I should also note here that momentum transport and fluids contains the additional complication of pressure; in this case we have not assumed any pressure differences of pressure gradients, we assume the pressure is a constant everywhere, how does one treat situations whether the pressure gradient is something that I will come to a little later, when we do the effect of pressure on fluid flows.

So, we have these equations and how do we solve them?

(Refer Slide Time: 15:28)



I will solve it for the case of the temperature field and by analogy; I will just write down the solutions for the concentration and momentum fields. So, the equation I have situation where this is the z direction, z is equal to 0 is at the surface, at the surface T is equal to T naught and far from the surface T is equal to T infinity.

So, if you plot the temperature field, depending upon how long time has elapsed, since I have heated the bottom surface, the temperature field will d k from this value of t naught 2 into 3 infinity; in some manner if I plot the temperature is functional cross stream position perpendicular to the surface. So, my equation is let us assumed for the present that there are no sources or things that the heating is entirely due to the heating of the bottom surface.

Now, this temperature changes approximately from T naught to T infinity; rather than using a dimensional temperature it is more convenient to use a scale temperature, rather than the temperature going from T naught to T infinity, I could scale the temperature so that the temperature far from the surface is just equal to 0 that is easy to do. I could write down a scale temperatures as T minus T infinity divided by T naught minus T infinity.

So, whereas, I have in this picture I have T naught here and T infinity here, if I were to plot the scale temperature you can see that far away it is equal to 0, at the surface itself T is equal to T naught and the scale temperature is equal to 1. So, if I were to plot on this axis T star instead of T, it goes from approximately 1 to 0 far away. So, I am just scaling the temperature that is all and this is convenient because even though the solution for the temperature T thus depend upon the values of T naught and T infinity, once I have scaled it the solution for T star is independent of the values of T 0 and T infinity.

And expressed in scaled form you can see that the temperature dimensions on the left and the right side of the equation are the same, this is a linear equation in the temperature and therefore, the equation just becomes d T star by partial t is equal to alpha partial square T star by partial z square and the boundary conditions at z is equal to 0, T star is equal to 1, I showed you that z is equal to 0 T star is equal to 1, as z tends to infinity, T star is equal to 0 and at t is equal to 0 that is when we have just increased the temperature of this bottom plate, T star is equal to 0 for all z which is not on this bottom surface at t equal to 0 we will just initiated the heating, T star is equal to 0 for all z greater than 0 at the surface itself T star has become equal to 1, because we have instantaneously heated this surface.

So, now we have scaled the temperature field, can we scale the time and the length, what are the time scales in the problem? What are the length scales in the problem? If you recall if we had a finite channel we would have had a length with which we could have scaled the z coordinate, you have a channel or a pipe or something or finite dimension we could have scaled the length in this coordinate, this configuration is of infinite extent this configuration is of infinite extent and therefore it is not possible to find either a length scale or a time scale time goes from 0 to infinity, length goes from 0 to infinity as well. So, is not possible to find the length scale or a time scale.

So, there is basically a deficit of dimensional quantities in this particular problem, can one get a dimensionless group involving the length and time, what are the parameters we have in this problem? We have z coordinate, we have time and we have the thermal diffusivity. So, out of these 3 can we get a dimensionless group? Since the diffusivity has dimensions of length square per time, there is only one dimensionless group that you can get from these 3 is equal to Z by root D t; because of a deficit of dimensional parameters we cannot find separate dimensionless groups involving length and time.

There is only one dimensionless group that we can create, I put the mass diffusivity here my apologies it should be in this problem the thermal diffusivity; we can get only that one dimensionless group that is the ratio of the z coordinate and square root of alpha t;

square root of alpha t you can see easily as dimensions of length. So, in this problem if I were to express both time and z in terms of this parameters si which is dimensionless, I should get an equation which contains nothing else, a dimensionless equation which contains nothing else, this thing is a procedure that is called the similarity transform. Just based upon dimensional analysis I can reduce the partial differential equation in two variables time and z into an ordinary differential equation which is only a function of the scaled co ordinates si alone.

So, this basically reduces the partial differential equation to an ordinary differential equation; you would recall earlier when I had discussed the formulation of equations I had said that in the case of partial differential equations if there is no systematic way, there is no algorithm which will enable you to find the solution, a solution may exist the solution may not exist, in most of our solution procedures what we will try to do is to use physical approximations to reduce this partial differential equation to one or more ordinary differential equations. This physical insight in the present case is the realization that because there is a deficit of dimensional variables, we can reduce the length and the time coordinates to just one dimensionless variable and if what I have done is correct, the equation that I get should be only a function of si, it should not individually be a function of 70.

So, how do I do this reduction? I can write partial T by partial t is equal to using chain rule for differentiation and my posture let at this point is that t is only a function of si it does not depend individually on z and time, but only through this combination. So, therefore, in this case I can write this as just the total derivative and I can integrate and I can differentiate this I will get Z by 2 route D t to the 3 halves, partial T by partial z and z by root d T is si itself. So, I will get si by 2 t, I should have a negative sign here because in this case t is in the denominator. So, when I take the derivative I get minus 1by 2. So, if it have negative sign here.

So, that is for the emperor derivative; the spatial derivative is going to be equal to I should is alpha no negative sign here and the square root. So, I take the derivative of si with respect to z here, I just get 1 over root alpha t, when I take the delta si with respect to z, I just get one over root alpha t. Taking the second derivative, you will just get one more factor of 1 over root alpha t. So, I just get one over alpha t times partial square T by this is the total derivative because T is only a function of si.

Now, we put that into the conservation equation and what I get on the left side is minus si by 2 t, d T by d si is equal to the thermal diffusivity into 1 over alpha t, d square T by d si square and now you can see that the thermal diffusivity cancels out on the right side and the factors of 1 over t cancel out on the left and the right side and indeed I do get an equation which is only a function of si alone.

So, the equation that I get is d square T by t si square is equal to minus si by2 d T by d si. So, that is what my equation reduces to in terms of the variables si. It is not enough to write down the equations in terms of the similarity variable si one has also got to express the boundary condition in terms of the similarity variables si. So, z is equal to 0 implies that si is equal to 0 and z is the expressed in terms of si; z is equal to infinity implies that si tends to infinity because si z by root alpha t.

The initial condition at t is equal to 0; the similarity variable is equal to z by root alpha t. So, for all z greater than 0 at t is equal to 0 this similarity variable is si goes to infinity because of t is equal to 0 and z is greater than 0, si tends to infinity and you can sort of see that I had two boundary conditions here z is equal to 0 and z tends to infinity and one initial condition t is equal to 0 in the original problem.

In terms of the transformed problem, these two have reduced to identical conditions in terms of si that is required for consistency. Initially I had a first order differential equation in t and second order in z. So, I needed two conditions in z and one condition in t. The transformed equation is only second orders in si and therefore, there are only two conditions in si, there for two of the conditions that I originally had should reduce to the same condition when expressed in terms of si and that has happened correctly, this means that my similarity transform is correct.

So, we have post the problem in terms of a similarity variable; the next task is to solve this equation and that is always possible because in this case this is an ordinary differential equation and we can actually get a solution and then find out how the temperature varies with time, how the flux is very time is gone. So, how do we solve this we will continue that in the next lecture I will see you later.