Transport processes I: Heat and Transfer

Prof. V. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Lecture - 20

Unidirectional transport: Conservation equation for momentum transfer

Welcome to this lecture number 20 of our course on fundamentals of transport processes.

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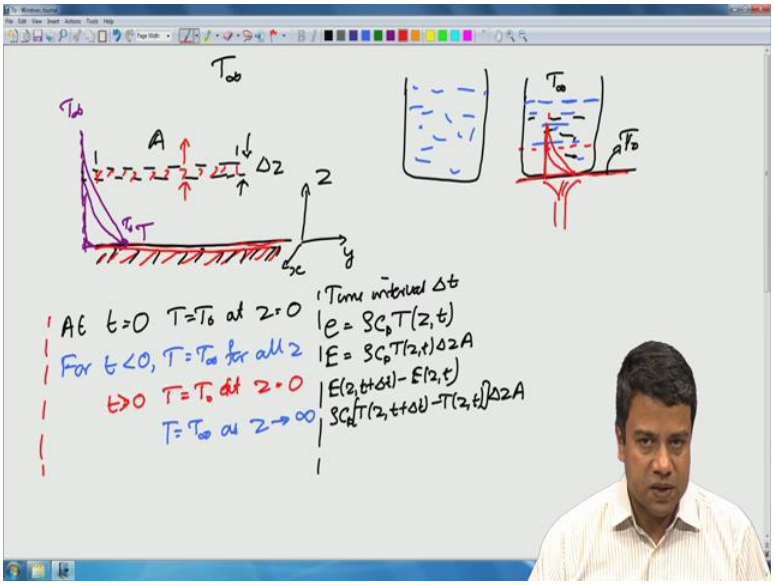

If you recall, in the previous lecture we had started solving problems involving transport in one dimension; unsteady transport in one dimension. So, we had considered the simple case of a fluid that is heated from below. Initially the system is all at constant temperature T infinity and at the initial time at time t equals 0, we had instantaneously changed the temperature at this bottom surface to T naught and we had wanted to see how the temperature varies within the fluid. We had considered the fluid to be infinite in the plane in the x y plane, the system to be infinite and we have also considered an infinite fluid in the z direction in the perpendicular z direction and I told you that this is actually an idealization of a system where if you have a container of fluid and you place it on a heat source.

So long as the penetration of the heat into the fluid, the depth of penetration is much smaller than the height of the fluid. You can consider it to be infinite in the z direction so that in that sense it is an idealization. We will come back and see under what conditions this idealization is valid. So, the conditions on this are that at t equals 0; you have a temperature T naught at this surface for t less than 0, the temperature was equal to T infinity everywhere, the entire fluid was at the temperature T infinity; that means, at t is equal to 0, the temperatures T infinity everywhere except at the surface. If you go far from the surface, so this is an infinite fluid in the limit as z goes to infinity; the temperature is once again equal to T infinity.

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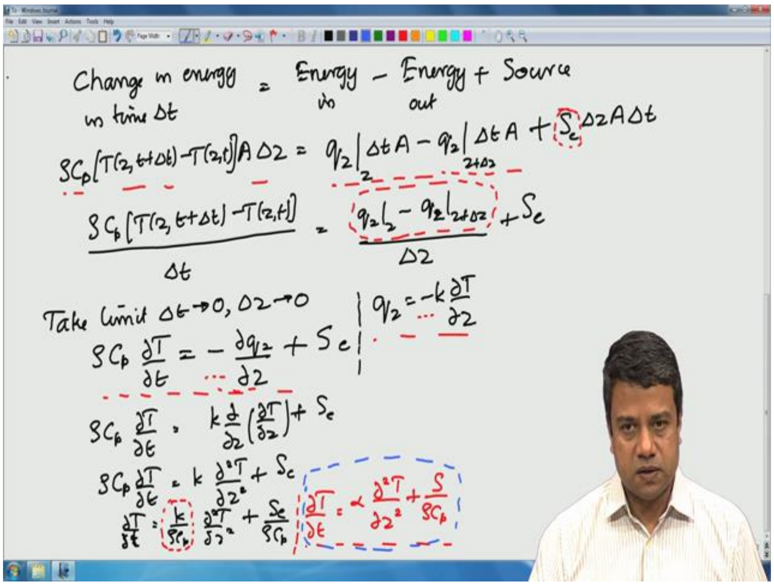

Now, for this we had done a balance with the change in energy in the time delta t is equal to energy in minus energy out plus any sources of energy. This energy density we had written as rho C p times t, so the change in energy was equal to the energy density times the volume; volume is cross-sectional area, times the height and for the height we had taken a small differential distance delta z in the z direction. So, this was equal to the energy in; energy in is flux; times Area times the time interval, after all the flux is equal to the energy transported per unit area per unit time. So, energy in at the location z; z upwards is positive minus the energy out at the location z plus delta z plus any sources. You write this balance and then divide by the total volume, divide by the total time interval and then you get a diffusion equation of this form wherein the time derivative of the temperature is related to the spatial derivative of the flux, once you have this expression; now we use the constitutive relation.

If you recall in the beginning lectures I had said that there are two components to this. One is the balance law which basically states that what comes in minus what goes out plus sources has to be equal to the rate of accumulation; that is one component, the other component is the constitutive relation; that constitutive relation is required for this flux, this flux in this particular case the flux is purely diffusive because we have not enforced any velocity field, so because of that the flux is just due to diffusion alone. So, once you have that you express the flux in terms of the temperature through the constitutive relation and then you get a diffusion equation which is of this kind. The rate of tails change of temperature is equal to a diffusion coefficient times the second derivative with respect to spatial position due to diffusion plus the any source of sync of energy.

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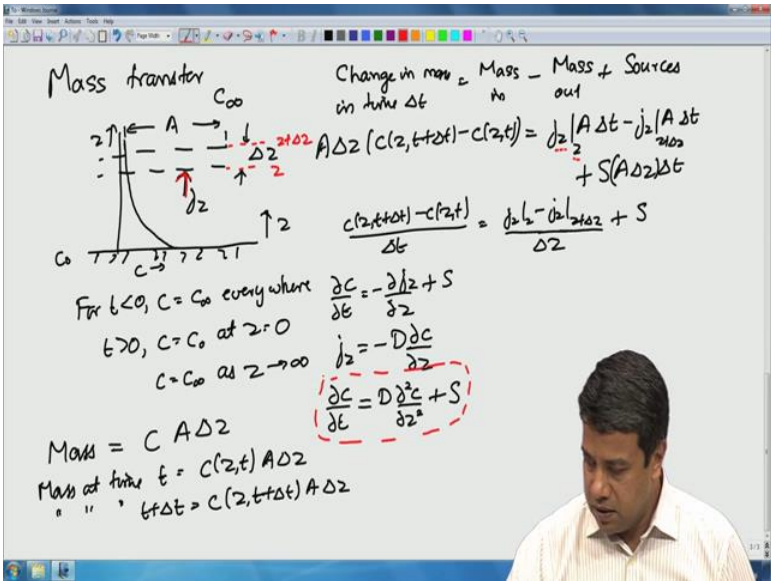

Now this problem has an exact analogy in mass transfer as well, you consider an infinite medium the z is the direction and you consider some solute in which initially the concentration is C infinity far away and for t less than 0, C is equal to C infinity everywhere, for t greater than 0, you instantaneously impose a concentration C0 at the surface C is equal to 0; C0 at z is equal to 0 and then far from the surface, the concentration is still C infinity because this is an infinite fluid and therefore, if you go further and further away, you should recover the concentrations far from the surface C is equal to C infinity as z tends to infinity and therefore, you would expect the concentration field to decay in some fashion away from the surface.

Once again this is an idealization, if I plot the concentration here as a function of distance z here give concentration decays away from the surface. This is once again an idealization; it appears a stronger idealization than in temperature case because it does not seem feasible to be able to just tune the concentration to abruptly change at a particular time interval. However, I will show you in the succeeding lectures that this kind of idealization does in fact, give you useful results.

So this is an exact analogy and the way that you solve it, is also an exact analogy. Consider a small interval delta z; cross sectional area A and write the balance equation; change in mass in time delta t is equal to mass in minus mass out plus any sources of mass for example, if there were a reaction which was either generating this species or consuming the species, there would be either a source or a sink of mass within this volume, so that will come in the source term.

What is the change in mass within the time delta t, the mass within this volume is going to be equal to the concentration times the volume itself A times delta z. So, therefore, the mass at time delta at time t; C equal to C at z; t right A times delta z and mass at time t plus delta t is equal to C at z t plus delta t; A times delta z. So, therefore, the change in mass is the change in concentration times the volume. So, this is going to be equal to A delta z times C at z t plus delta t, minus C at z t; that is going to be the change in mass within the time delta t is equal to mass in at the location z, the mass flux times the area, times the time interval. This is going to be equal to the mass flux at the location z; that is the mass in; times the area, times the time interval, so there is the mass in within this time interval delta t.

Minus the mass flux at z plus delta z times A times delta t plus any sources or sinks of mass these sources or sinks of mass within a time delta t will be of the form of a rate of generation of mass; a rate of generation or consumption generation will be positive, consumption will be negative. So, if we have the form the rate of generation or consumption; times the volume times the time interval the rate of; so is the rate of generation of mass per unit volume, the time rate of generation per unit volume multiplied that by the volume, multiply that by the time interval to get how much mass has been generated within this volume within the time interval.

So, this is the balance equation; now to get the differential equation, we divide throughout by delta z, throughout by the volume and the time interval. So, we have to divided throughout by A, delta z, delta t and you will get C at z t plus delta t minus C at z t by delta t is equal to now we had j z at z minus j z at z plus delta z, you divide by A delta t, delta z; you will get j z at z minus the flux at that plus delta z. Note this first subscript denotes that the flux is in the z direction, the first subscript denotes the flux is in the z direction, the direction of the flux depends upon the direction of the concentration variation. In this particular case, we are taking variations in concentration only in the z direction; therefore, the flux will also be only in the z direction.

The second one denotes the location; at which you are taking the flux. So, for the input we are taking the flux at the location z, for the output we are taking the flux at the location z plus delta z. So, this has now got to be divided by delta z and you just get the source over here, so that is the balance equation; take the limit delta z going to 0, delta t going to 0 to get a differential equation; that equation as of the form partial C by partial t is equal to minus, partial j z by partial z plus S; the negative sign because I have j z at z minus j z at z plus delta z. Normally the derivative is defined as j z at z plus delta z minus j z at z divided by delta z so that is why the negative sign; so this is the balance equation.

Now we have to go on to determining the constitutive relation, the constitutive relations of the form j z is equal to minus the diffusion coefficient times d c by d z. The negative sign because mass is transported from regions of higher concentration to regions of lower concentration and b here is the mass diffusion coefficient; put that into the equation once again using the assumption that the diffusion coefficient is independent of the concentration of position you will get partial c by partial t; is equal to D t square c by d z square plus source.

So that is the mass diffusion equation you can see on the left side you have a first derivative concentration with respect to time, on the right side you have a diffusion coefficient times the second derivative of concentration field with respect to the z coordinate plus a source. This is the mass diffusion equation and you can see that it is exactly analogous to the thermal diffusion coefficient that I have here except that the temperature has been replaced by concentration, the thermal diffusivity has been replaced by mass diffusivity and the thermal sources have been replaced by the mass source, so there is the only difference otherwise they are exactly the same. So, you could go from one to the other just by replacing the temperature with concentration and the diffusivities. There is an exact analogy for momentum transfer as well; in that case what you would consider is a surface in a fluid.

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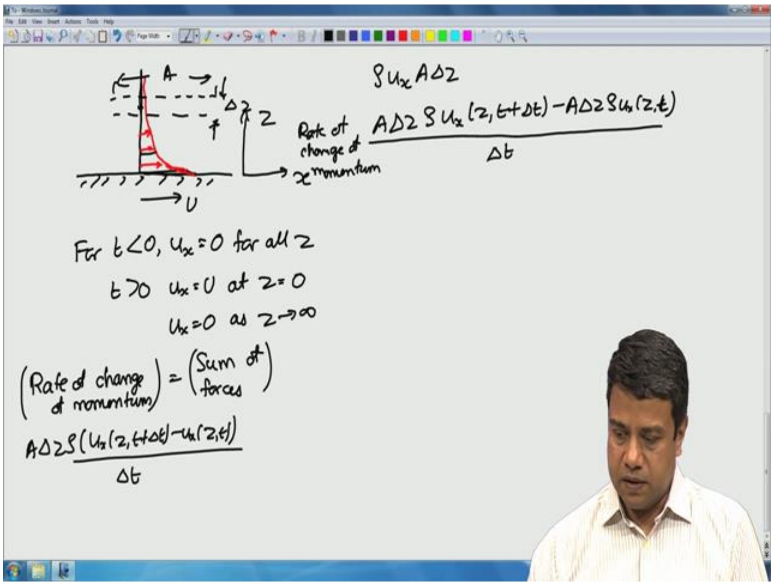

So, let us take this as the x direction, this as the z direction. Consider the fluid to be stationary fluid. So, for t less than 0; u x is equal to 0 everywhere, for t greater than 0; you instantaneously give a velocity u to the bottom surface; u x is equal to u at z is equal to 0 and because of that; due to momentum diffusion, the fluid close to the surface will start will initially you will just have velocity u here and velocity 0 everywhere, due to momentum diffusion; the surface closest will start to move and you will get some kind of a velocity profile; however, we consider this fluid to be of infinite height and therefore, far from the surface, the velocity is still equal to 0 which means that u x is equal to 0 as z goes to infinity. This would be an idealization of for example, if I had container like this and I started to move the bottom surface then there will be some fluid that is flowing within the region close to the bottom surface; however, if the height is much larger than the length scale for the penetration of momentum, you can assume that it is an infinite fluid, so that is the idealization here.

Now, the balance law in this case is usually written slightly differently. So, I will use the slightly different formulation of the balance law, which is basically Newton’s law; rate of change of momentum is equal to sum of applied forces, so that is the balance law in this case. So if I have a differential volume of height delta z and cross sectional area A, if I have a differential volume of height delta z and cross sectional area A; the balance law in this case is written as the rate of change of momentum is equal to the sum of forces, that is Newton's law; rate of change of momentum within this differential volume is equal to the sum of all the forces that are exerted on this differential volume.

What is the rate of change of momentum? The momentum itself in this case one should be careful because the momentum is a vector, in this particular problem the momentum is in the x direction. So, we are talking about the rate of change of momentum in the x direction; the momentum in the x direction can be written as the density times the velocity in the x direction, times A volume; this is a momentum density, the density times the velocity is a momentum density, times the volume is the momentum within this differential volume the x momentum within this differential volume.

So, the rate of change of momentum within a time delta t is going to be equal to the momentum rho u x at z t plus delta t minus A; delta z, rho u x at t; z t, that is the change in momentum. You have to divide that by the time interval delta t to get the rate of change of momentum, so this is the rate of change of momentum. There is a rate of change of momentum within this differential volume, so therefore, is equal to A delta z rho into u x at z t plus delta t minus u x at z t divided by delta t; that is the rate of change of momentum that has to be equal to the sum of the applied forces, so let us look at the sum of the applied forces.

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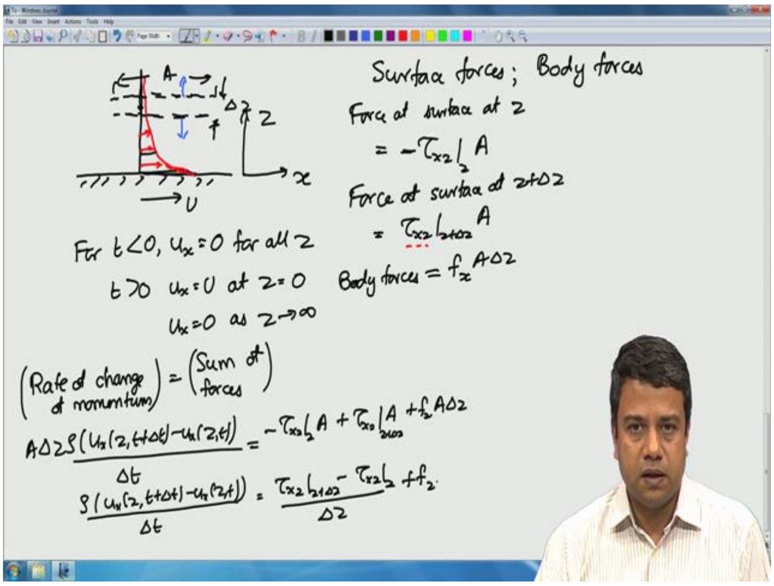

The forces can be broadly divided into two types; one is surface forces and body forces. The body forces are proportional to the volume itself, the gravitational force for example, whereas, the surface forces are proportional to the surface area. These surface forces at on two surfaces; the bottom surface here and the top surface here, so what is the surface force exerted on the bottom surface. The surface force exerted on the bottom surface is going to be equal to the shear stress on the bottom surface, times the area that is the force exerted on the bottom surface. The shear stress force per unit area multiplied by the area of the surface therefore, the sub force at surface at z; the force is given by the shear stress at the location z times the surface area.

Here is where one has to be careful about sign conventions; I told you that in the case of fluxes; the flux is positive if it is going in the plus z direction. In this particular case, the force is considered to be positive if it acts at the surface with outward unit normal in the plus z direction, look at the bottom surface; at this bottom surface to this volume, the outward unit normal is in the minus z direction. So, see outward unit normal is in the minus z direction; that means, that the force that is exerted at the surface is going to be equal to minus tau x z; at the location z. So, this is the force that is exerted tau x z is the force exerted at a surface whose outward unit normal is in the plus z direction, which tends to increase the momentum in the volume below this surface.

In this case the outward unit normal is in the minus z direction therefore, the shear stress at the surface is equal to minus tau x z times A. So, one has to be careful about the sign convention that is adopted. So, long as you adopt a consistent sign convention, the result will end up being the same. What about the force at the surface at the location z plus delta z, at surface at z plus delta z; in this case the outward unit normal is in the plus z direction. Therefore, the force is equal to tau x z at z plus delta z times A, those are the surface forces. The surface forces due to the shear stress at the surface, you do have body forces also those body forces will basically be proportional to the volume.

So, they can be written as the force density in the x direction, times the volume, so that is the general expression for the body forces. So these two have to be put into the conservation equation, so this is equal to minus tau x z at z times A plus tau x z at z plus delta z times A plus f z A delta z. So, that is the total equation. Just to reiterate tau x z force per unit area stress, it has two subscripts; the first one is x that represents the direction of the force or the direction of the momentum, I had told you earlier that in this case you are writing an equation for the balance in momentum in the x direction, that is what the first subscript means the x corresponds to the direction of the force.

The second is the direction of transport that x momentum is being transported in the z direction. Just as we had mass and heat transport in the z direction in the previous problems that we solved, in this case the x momentum is being transported in the z direction. The second index represents the direction of transport or the direction perpendicular to the surface across which momentum is being transported. So, that reference the direction perpendicular to the surface across which the transport takes place. So the direction of the unit normal to the surface, there is a second subscript, in the previous cases we had just one subscript because mass was a scalar, energy was a scalar. In this case momentum is a vector, so we need one subscript for the direction of the momentum itself another subscript for the direction in which transport takes place.

Once we have this, we just have to divide throughout by the volume and take the limit as delta t, delta z go to 0. So, if I divide throughout by the volume I will get rho into u x at z, p plus delta t minus u x at z t divided by delta t is equal to tau x z at z plus delta z minus tau x z at z divided by delta z plus the force density. So, that is the balance equation and to from this to a differential equation, all you need to do is take the limit as delta t going to 0 delta z going to 0.

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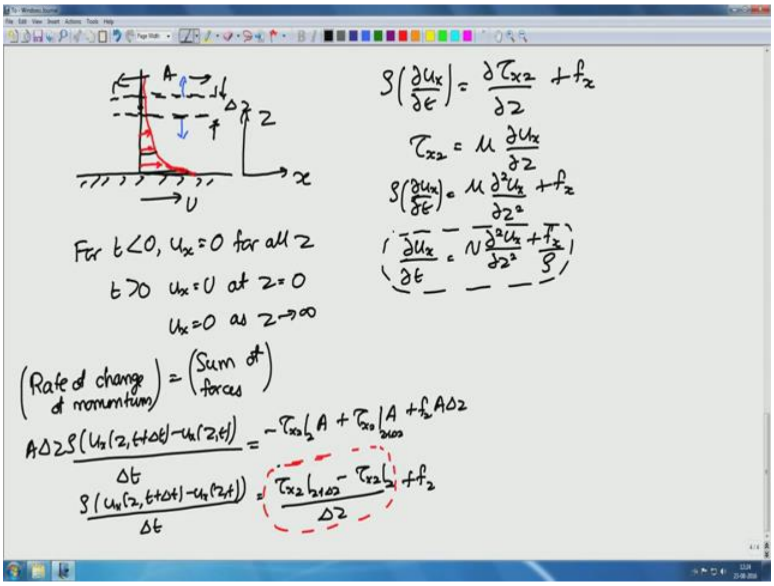

So let us do that over here, you will get rho into partial u x by partial p is equal to; in this case you have tau x z at z plus delta z minus tau x z at z divided by delta z. So, this will be equal to partial tau x z by partial z plus f z.

Now, we use Newton's law of viscosity constitutive relation of the shear stress. If you recall I said that in the case of Newton's law, we have to write the stress as plus mu times partial u x by partial z because the force is considered positive if it increases the momentum of the volume below; that was the reason. So, if you substitute that, you will get rho into d u x by d t is equal to, once again if the viscosity is considered to be a constant this becomes mu d square u x by d z square plus f x, I am sorry note that these are all forces in the x direction because the direction of the force is p in x direction, so this is the conservation equation.

If I divide throughout by density, I will get partial u x by partial t is equal to the kinematic viscosity; mu by rho is the kinematic viscosity times partial square u x by partial x square plus x. Once again you can see that you can get this equation from the concentration or temperature equations just by substituting for the x velocity instead of temperature or concentration. The momentum diffusivity or the kinematic viscosity instead of the mass or the thermal diffusivity and the force there should be a density there as well direct. So, these equations all three for mass, momentum and energy in this simple configuration; I have exactly the same form and therefore, you can solve them using exactly the same framework, for either mass momentum or energy

So, this is a one dimensional convection diffusion equation, regardless of the problem that you consider. If you have variation only in one direction and the variation in time, the equation is going to be of this form. They all three of the form of diffusion equations the only thing that changes is the density whether it is mass, momentum or energy and the diffusivity whether it is mass, momentum or energy; the sources whether they are sources of mass, momentum or energy; however, the forms of the equations are exactly the same and therefore, the method of solution in this case will also be exactly the same.

So, I have provided you a unified framework for considering diffusion of mass momentum and energy from a surface. In this case there is no convection; there is only a time rate of change of mass, momentum or energy and the diffusion from the surface and any sources or sinks. Simplest problem that you are considering, how to solve this problem; we will continue that in the next lecture, in the meantime just recall that in all of these three cases; you get exactly the same form of the equation, only the variable changes. Whether it is density of mass momentum or energy and the diffusion coefficient changes those are the only two differences, how to solve these. We will continue in the next lecture, I will see you there.