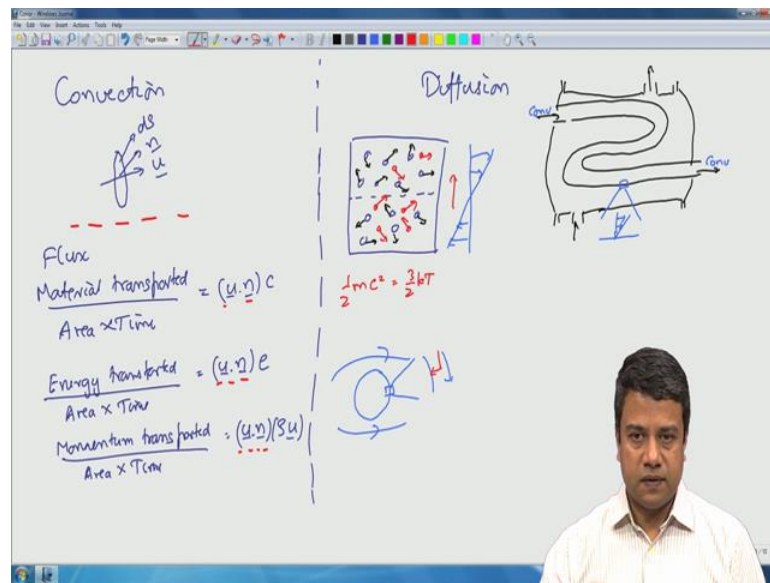


**Transport Processes I: Heat and Mass Transfer**  
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**Lecture - 19**  
**Unidirectional transport: Conservation equation for heat & mass transfer**

This is now lecture number 19, in our course on Transport Processes. We had in the last two lectures looked at diffusion coefficients and constitutive relations.

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Firstly, the two mechanisms of transport convection and diffusion; convection due to the mean velocity of the molecules, and the flux is equal to the component of the velocity perpendicular to the surface times the density of whatever is being transported with its mass momentum or energy; diffusion flux due to the fluctuating molecular velocities not the fluid mean velocity, the fluid velocity. As I had explained to you in a previous lecture the fluid velocity is the vector sum of the velocities of all of the molecules, whereas the fluid fluctuating velocity is the molecular velocity minus the mean velocity.

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Diffusion coefficients & constitutive relations:  
Gases:

Net flux  $j = BC \left|_{z=2\lambda} U_{rms} - BC \left|_{z=2\lambda} U_{rms}$

Number of red molecules coming downwards  $= BC \left|_{z=2\lambda} U_{rms} = j^-$

Number of red molecules going upwards  $= BC \left|_{z=2\lambda} U_{rms} = j^+$

Net flux  $= BU_{rms} [C|_{z=2\lambda} - C|_{z=2\lambda}] = -2AB \sqrt{U_{rms}} \left( \frac{dc}{dz} \right)_{z=0}$

The convection flux is quite easy it is equal to the velocity perpendicular to the surface times the density of whatever is being transported. Diffusion flux due to molecular velocity fluctuations only when there is a variation in the density of whatever is being transported, mass momentum or energy. If there is no variation the transport in one direction will be equal to the transport in the other direction, and there is no net flux.

The variation comes about because the molecules in the case of gases that are going across the surface; the molecules going upwards are coming from a location a little bit below the surface, whereas the molecules that are going downwards coming from a location that is slightly above the surface. The concentration of the molecules at these two locations is different; you have different concentration of molecules at these two locations when there is a concentration variation across the surface. And this results in a flux, that flux is approximately equal to the difference in concentration between these two locations times the root mean square of the fluctuating velocity. The root mean square of the fluctuating velocity in gases we know it is equal to square root of  $3 k t$  by  $m$ ; where  $t$  is the temperature. That is how the temperature is defined in gases.

Therefore, from this the flux is equal to the root mean square fluctuating velocity times the concentration difference between these two locations that concentration difference is the mean free path times the gradient of the concentration. The mean free path times the

variation across the surface. So therefore, we get expressions that end up being of the form of Fick's law.

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$$j = -2AB\lambda v_{rms} \frac{dc}{dz}$$

$$j = j_x - j_y$$

$$= Bv_{rms} (c|_{z=0} - c|_{z=A})$$

$$c|_{z=0} = c_0 + \frac{dc}{dz} \Big|_{z=0} \lambda + \frac{(A\lambda)^2}{2!} \frac{d^2c}{dz^2} \Big|_{z=0} + \dots$$

$$c|_{z=A} = c_0 + \frac{dc}{dz} \Big|_{z=0} A + \frac{(A\lambda)^2}{2!} \frac{d^2c}{dz^2} \Big|_{z=0} + \dots$$

$$j = Bv_{rms} \left[ -2A\lambda \frac{dc}{dz} \Big|_{z=0} - \frac{2(A\lambda)^3}{3!} \frac{d^3c}{dz^3} \Big|_{z=0} + \dots \right]$$

$$D = 2AB\lambda v_{rms}$$

$$A \approx \frac{2}{3} \lambda \quad B \approx \frac{1}{4}$$

$$D \approx \frac{1}{3} \lambda v_{rms}$$

$$\frac{1}{2} m v_{rms}^2 = \frac{3}{2} kT \Rightarrow v_{rms} = \sqrt{\frac{3kT}{m}}$$

Hydrogen gas at STP  
 Mass of 1 mole = 2gms =  $2 \times 10^{-3}$  kg  
 Mass of 1 molecule =  $\frac{2 \times 10^{-3}}{6.023 \times 10^{23}}$  kg  
 $kT = (1.38 \times 10^{-23}) / k \times 300K = 4 \times 10^{-21} J$

$$v_{rms} = \sqrt{\frac{4 \times 10^{-21}}{3.32 \times 10^{-27}}} \approx 1200 \text{ m/s}$$

Oxygen  $m = 16 \times m_u$   
 $v_{rms} = \frac{1}{4} v_{rms} \approx 300 \text{ m/s}$

That is that the flux is equal to something times the gradient in concentration, that thing in front is the diffusion coefficient and the diffusion coefficient scales as the root mean square fluctuating velocity times the mean free path in gases. Same is true in the case of momentum diffusion and in the case of energy diffusion.

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**Momentum diffusion:**

Momentum flux  $\uparrow = Bnm v_{rms} u_x|_{z=0-A}$   
 Momentum flux  $\downarrow = Bnm v_{rms} u_x|_{z=A}$

Shear = Rate of change of momentum / Area

$$\tau_{xz} = \text{Momentum flux } \downarrow - \text{Momentum flux } \uparrow$$

$$= Bnm v_{rms} (u_x|_{z=A} - u_x|_{z=0-A})$$

$$= Bnm v_{rms} (u_0 + A \frac{du_x}{dz} \Big|_{z=0} - (u_0 - A \frac{du_x}{dz} \Big|_{z=0}))$$

$$= 2ABnm \lambda v_{rms} \frac{du_x}{dz} \Big|_{z=0}$$

$$\tau_{xz} = \mu \frac{du_x}{dz} \Big|_{z=0}$$

$$\mu = 2ABnm \lambda v_{rms}$$

$$N = \frac{\mu}{\rho} = \frac{\mu}{nm} = 2AB\lambda v_{rms}$$

In the case of momentum diffusion it is the momentum that is being transported, in gases that momentum is transported by the physical motion of molecules as in the case of mass. Similarly, energy is transported by the physical transport of molecules because there is a slight difference in the internal energy across the surface if there is a temperature variation, and that is what transports molecules across.

And therefore, in gases the mass, momentum and energy diffusion coefficients are all of the same magnitude. They all scale as the mean free path times the fluctuating velocity, because the flux is equal to the fluctuating velocity times the difference in concentration that difference is the mean free path times the gradient. So, the diffusion coefficients are all the root mean square fluctuating velocity times the mean free path. We had made some estimates of the mean free path in gases at room temperature it varies from about half a micron to our 0.6 microns. The rms fluctuating velocity is approximately the speed of sound; it is about 300 meters per second in air and oxygen and about 1200 meters per second in hydrogen.

On that basis all diffusion coefficients are roughly about 10 power minus 5 meter square per second. In liquids the mechanisms are different.

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Mixture of two molecules A & B

$$D_{AB} = \frac{1}{3} n d_{AB}^2 \left( \frac{kT(m_A + m_B)}{2\pi m_A m_B} \right)^{1/2}$$

$m_A$  &  $m_B$  = Molecular masses  
 $d_{AB} = \frac{d_A + d_B}{2}$   
 $n$  = Total number density

Liquids:  
 Simplistically assume  $D \approx v_{rms} d$   
 Ratio of diffusion coefficients  $\frac{D_{liquid}}{D_{gas}} \approx \left( \frac{d}{\lambda} \right) \frac{1}{100} \frac{1}{1000}$

$\lambda \sim 10^{-7}$  to  $10^{-8}$  m  
 $d \sim 10^{-9}$  m -  $10^{-10}$  m  
 $v_{rms} \sim 10^3$  m/s  
 $v_{rms} \sim 10^4$  to  $10^5$  m/s  
 $\sim 10^{-9}$  m<sup>2</sup>/s or lower

Stokes-Einstein relation  
 $D = \frac{kT}{3\pi\eta d}$   
 No liquid viscosity  
 $d$  = diameter of spherical particle

Mass diffusion still requires the physical transport of molecules, because I had shown you in this case if one molecule is to move in one direction densely packed cluster of

molecules then the other molecules have to move out of the way. Therefore, a single molecule motion is not sufficient for diffusion; you require cooperative motions of many molecules. And due to that the diffusion coefficient is lower than what you would expect on the basis of the mean free path I am sorry; the molecular scales and the root mean square fluctuating velocity.

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$$\mu = \frac{5}{16d^2} \left( \frac{m k T}{\pi} \right)^{1/2}$$

$$\mu = 2AB n m \bar{v}_{rms}$$

$$= 2AB n m \frac{1}{\sqrt{2}} \sqrt{\frac{3kT}{m}}$$

$$\propto \frac{1}{d^2} \sqrt{m k T}$$

$$Sc = \frac{\text{Momentum diffusion}}{\text{Mass diffusion}} = \text{Constant} \quad Sc = \frac{\nu}{D} \gg 1$$

Momentum diffusion in liquids:  
 Water:  $\mu = 10^{-3} \text{ kg/m.s}$  ;  $\rho = 10^3 \text{ kg/m}^3$   
 $\nu \cong 10^{-6} \text{ m}^2/\text{s} \cong \nu_{mod}$   
 $D \cong 10^{-9} - 10^{-8} \text{ m}^2/\text{s}$

In liquids momentum diffusion is relatively fast, because you do not require molecules to move out of the way, momentum can be transported by intermolecular interactions. So, momentum can be transported from one molecule to the next due to molecular contacts. And therefore, momentum diffusion is much faster in liquids. The momentum diffusivity is of the order of 10 power minus 6 meter square per second, whereas the mass diffusivity (Refer Time: 06:31) 10 power minus nine meter square per second in liquids for small molecules.

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Thermal conductivity,  
Spherical gas molecules:

$$k = \frac{75}{64d^2} \left( \frac{kT}{\pi m} \right)^{3/2}$$

$$G_0 = \frac{3k}{2m}$$

$$\text{Prandtl number} = \frac{\nu}{\alpha}$$

Liquids:

Organic liquid

Liquid metal

Very high k

$$\alpha \propto v_{rms} \lambda$$

The whiteboard contains handwritten mathematical derivations for thermal conductivity and Prandtl number. It also includes diagrams of organic liquid and liquid metal structures, and a note about high thermal conductivity in liquid metals.

The thermal diffusivity in liquids depends upon the mechanism. As I had tried to explain you in the last lecture; the thermal diffusion coefficient depends upon how energy is transported. If it is due to intermolecular forces you get a relatively small thermal diffusion coefficient of the order of  $10^7$ . Whereas, in special situations like liquid metals for example, energy could be transported through the electron gas around the nuclei in a liquid metal, and that results in very fast transport.

So, that in the summary is diffusion.

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Unidirectional transport:

$$j = (u \cdot n) \frac{C}{\rho u}$$

$$j = -D \frac{\partial C}{\partial x}$$

$$= -\alpha \frac{\partial C}{\partial x}$$

$$= u \frac{\partial u}{\partial x}$$

Rate of change of mass momentum energy = Mass Momentum Energy In - Mass Momentum Energy Out + Accumulation

The whiteboard shows a diagram of a cube in a 3D coordinate system. It includes mathematical equations for unidirectional transport and a mass/energy balance diagram with dashed red boxes around the terms.

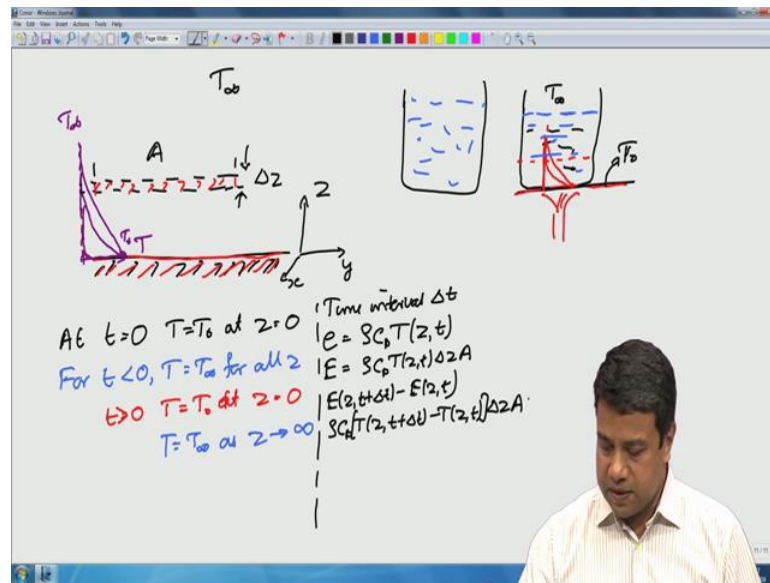
And now, these two mechanisms convection and diffusion we will use in order to solve some problems. Simple problem of unidirectional transport first. Let us start off with unidirectional transport; where transport takes place in only one direction. So, basic ingredients we have are the flux is due to convection, so what we will try to do is to write a balanced equation for differential volume where we write the rate of change of mass, momentum and energy within a volume is equal to mass momentum energy in minus mass momentum energy out plus accumulation. That is going to be the basic balance. And for these two terms what comes in and what goes out can be either due to convection or give to diffusion.

In the case of convection the fluxes  $j$  is just equal to  $u \cdot n$ , the velocity dotted with the unit normal to the bounding surfaces times concentration density energy density or momentum density. And the flux due to diffusion is given by the diffusion coefficient times the gradient across the surface; the variation across the surface there should be the variation the direction perpendicular to the surface. So, those are the basic or rather minus alpha into the variation in the energy density there is equal to I had explained the last lecture that when we consider momentum diffusion, we consider the viscosity the positive sign for the viscosity because the increase in momentum of a surfaces outward unit normal is considered to be positive and this is  $\partial u / \partial x$ ; where  $x$  is the direction of variation of the mass momentum or energy. So, those are the basic ingredients that we will use.

Consider a definite problem.



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We have a surface, let us consider the surface that is infinite extent along the plane with a unit perpendicular to the surface is in the z direction and along the plane we have the x and y coordinates. We will consider the surface to be of infinite extent so that there is no variation in the x and y direction. And we will consider the fluid above the surface to also be of infinite extent.

So, initially the fluid temperature which considers the heat transfer problem first; the fluid temperature everywhere is  $T_{\infty}$ . So, initially the system is prepared in such a way that the fluid temperature everywhere is equal to  $T_{\infty}$ , some temperature let us say room temperature. At the initial time  $t$  is equal to 0 we instantaneously heat this bottom surface and apply a different temperature to this bottom surface  $T$  is equal to  $T_0$  at  $z$  is equal to 0.

So, if you look at the temperature profile as a function of position across as I said there is no variation in the x and y direction. If you look at the temperature profile initially everything is at  $T_{\infty}$ , so it takes this as the temperature across; everything is that the temperature  $T_{\infty}$  at time  $t$  equals 0 you are increasing this to  $T_0$ . And therefore, everything is at  $T_{\infty}$  and as time progresses there will be a diffusion of heat from the surface and the temperature will start to rise in some manner away from the surface. And what we would like to find out is how the temperature rises, what is the flux from the surface, how does the flux vary with time.



This is of course an idealized situation. You could consider it for example an idealization of if you have a container which contains some flow at some temperature. And you instantaneously put it onto a burner which is at a higher temperature. So, initially the fluid was at room temperature that is  $T_{\infty}$  and at time  $t$  is equal to 0 instantaneously you put this at the temperature  $T_0$  here. So, there is an idealization of the problem. Instantaneously you place the container on this. And this fluid can be considered to be infinite of infinite extent so long as, time progresses the temperature profile will of course change; initially there will be will be heated only here, as time progresses the temperature profile will change.

The infinite fluid idealization is valid so long as the distance which the temperature has increased is much smaller than the total height of the flowing. So, long as the distance for which this temperature disturbance due to the base has penetrated that distance is much smaller than the total height of the fluid. In that case you can consider it to be an infinite system. So, that is the problem.

So to be more precise; at for  $t < 0$  the temperature is equal to  $T_{\infty}$  for all  $z$ . For  $t > 0$ ,  $T$  is equal to  $T_0$  at  $z = 0$  that is the hot temperature. So,  $t > 0$ , the temperature is maintained that  $T_0$  at  $z = 0$ . However, if you go far from the surface since this is an infinite fluid if you go sufficiently far from the surface the temperature is going to be equal to  $T_{\infty}$ . So,  $T$  is equal to  $T_{\infty}$  as  $z$  goes to infinity. If you go sufficiently far from the surface the temperature is equal to  $T_{\infty}$ . So, that is the problem that we have to solve, and that we have to do by doing a balance.

Now, what do we do in a temperature balance? We take a small slice of the fluid of height  $\Delta z$  and cross sectional area  $A$ , maybe take a small slice of the fluid of height  $\Delta z$  and cross sectional area  $A$ , and we write a balance for this. The rate of change of thermal energy within this volume is equal to what comes in minus what goes out plus any sources or sinks within this differential volume. Now what is the rate of change of energy? So, let us consider a time interval  $\Delta t$ ;  $\Delta t \ll t$ . Within this time interval what is the change in energy.

The energy is equal to if we assume that it is a constant pressure process that is it is open to the atmosphere then the energy per unit volume  $e$  is equal to  $\rho C_p T$  at the

location at the height  $z$  and the time  $t$ . This is the energy density times the volume is the total energy. Therefore, the total energy is equal to  $\rho C_p T$  at  $z$  and  $t$  times the volume. The volume is equal to  $\Delta z$  into the cross sectional area  $A$ . So, that is the total energy within this differential volume.

Now the change in energy within the time  $\Delta t$  is going to be equal to  $e$  at  $z$   $t$  plus  $\Delta t$  minus  $e$  and  $z$   $t$ ; if we consider the density in the specific heat to be constants then this will just be equal to  $\rho C_p T$  at  $z$   $t$  plus  $\Delta t$  minus  $T$  at  $z$   $t$  into the volume which is  $\Delta z$  times  $A$ . So, that is the change in energy with in a time  $\Delta t$ .

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The image shows a whiteboard with handwritten mathematical derivations. The text is as follows:

Change in energy in time  $\Delta t$  = Energy in - Energy out + Source

$$\rho C_p [T(z, t + \Delta t) - T(z, t)] A \Delta z = q_z|_{z} \Delta t A - q_z|_{z + \Delta z} \Delta t A + S_c \Delta z A \Delta t$$

$$\frac{\rho C_p [T(z, t + \Delta t) - T(z, t)]}{\Delta t} = \frac{q_z|_z - q_z|_{z + \Delta z}}{\Delta z} + S_c$$

Take limit  $\Delta t \rightarrow 0, \Delta z \rightarrow 0$

$$\rho C_p \frac{\partial T}{\partial t} = - \frac{\partial q_z}{\partial z} + S_c \quad \left| \quad q_z = -k \frac{\partial T}{\partial z} \right.$$

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial}{\partial z} \left( \frac{\partial T}{\partial z} \right) + S_c$$

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} + S_c$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial z^2} + \frac{S_c}{\rho C_p} \quad \left| \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} + \frac{S}{\rho C_p} \right.$$

So, change in energy in time  $\Delta t$  is equal to energy in minus energy out plus any accumulation; that is the basic balance. The change in energy in a time  $\Delta t$  I had said was equal to  $\rho C_p$  into  $T$  at  $z$   $t$  plus  $\Delta t$  minus  $T$  at  $z$   $t$  into the volume that is the area into the height.

Now, what is the energy coming in at the location  $z$ ? The energy in at the location  $z$  is going to be equal to the thermal diffusion coefficient times the temperature gradient at that location. Therefore, I will get; this will be equal to the flux at the locations  $z$ . Energy in is the flux, flux is energy per unit area per unit time. Therefore, the flux at the location  $z$  times the area times the time interval is going to be equal to the energy that is coming

in at the location  $z$ . So, it is going to be equal to the flux times the time interval times the area; that is the energy that are coming in at the locations  $z$ .

The energy leaving at the location  $z + \Delta z$  this once again equal to the flux at that location  $z + \Delta z$  times the area times the time interval, because the flux is equal to energy transported per unit area per unit time. So, this is going to be the energy out; the flux in the  $z$  direction at the location  $z + \Delta z$  times  $\Delta t$  times the area. And this accumulation term if there is any source of energy within that element let us say because there is a reaction within that element or there is some either phase change within that element. The source of energy within that element, so I should write this as source accumulation; this will in general be source density a source per unit volume, a source density times the volume itself  $\Delta z$  in to  $A$ . So, this  $S$  is energy accumulated. So, this energy source  $S$  there is an energy generated per unit volume per unit time due to reaction. For example, if you have a certain reaction taking place there are certain amount of energy that is generated per unit volume of the fluid depending upon the concentration of the reactants per unit time. So, that has to be multiplied by the volume and by the time  $\Delta t$ .

So, that is a fundamental balance equation. And once we have this we can divide throughout by the volume  $A \Delta z$  divided throughout by the time  $\Delta t$  to get an equation. So, you write down for each differential volume for each interval of time divided by the total volume and divide by the time interval. If we divide by the volume and the time interval you will get  $\rho C_p \frac{\partial T}{\partial t}$  at  $z + \Delta z$  minus  $\rho C_p \frac{\partial T}{\partial t}$  at  $z$  when I divide by  $A \Delta z \Delta t$ , I will get a  $\Delta t$  here I will get  $q_z$  at  $z + \Delta z$  minus  $q_z$  at  $z$  plus  $S \Delta z$  divided by  $\Delta z$  plus the source of energy.

Now we go from to a differential volume, and the differential time interval. Take the limit  $\Delta t$  going to 0 and  $\Delta z$  going to 0, when you take that limit you get a derivative with respect to temperature of with respect to time on the left and the derivative of the flux with respect to  $z$  on the right. So, you will get  $\rho C_p \frac{\partial T}{\partial t}$  when you take the limit of time interval going to 0. This on the right side is  $q_z$  at  $z + \Delta z$  minus  $q_z$  at  $z$  divided by  $\Delta z$ . Normally we write the derivative as  $q_z$  at  $z + \Delta z$  minus  $q_z$  at  $z$  divided by  $\Delta z$ .

So, this is basically the negative of the derivative with respect to  $z$ . So, this is equal to  $-\frac{dq}{dz}$  at  $z + \Delta z$  minus  $-\frac{dq}{dz}$  at  $z$  divided by  $\Delta z$ . I just written it with a negative sign outside. And now this term here you can recognize as the derivative  $\frac{dq}{dz}$ . In this case it is a partial derivative because we have variations both in  $z$  as well as some time. So, this is now a partial derivative minus partial  $q$  with respect to  $z$  where  $q$  is the flux in the  $z$  direction plus any source of energy (Refer Time: 24:46). We know that the flux was equal to minus  $k$  times partial  $T$  by partial  $z$ ; that was Fourier law for heat conduction flux was equal to minus  $k$  times partial  $T$  by partial  $z$ .

So, if we substitute that in this equation I will get  $\rho C_p \frac{\partial T}{\partial t}$  is equal to minus  $k$  times  $\frac{\partial^2 T}{\partial z^2}$  plus  $\dot{q}$  energy source per unit volume per unit time. This is of course, assuming that the thermal conductivity  $k$  is independent of temperature, there is the assumption that we are making here. So, if you use that assumption then I will get an equation for the temperature field as  $\rho C_p \frac{\partial T}{\partial t}$  is equal to; I am sorry it should be a plus sign here gladly note this mistake so there is a minus sign over here in  $\frac{dq}{dz}$  and  $q$  itself has a negative sign. So, in the end you will get a positive sign in the equation for the temperature field.

So, we will get  $\alpha \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{\rho C_p}$ . Alternatively, if I divide throughout by  $\rho C_p$  I will get  $\frac{\partial T}{\partial t}$  is equal to  $\frac{k}{\rho C_p} \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{\rho C_p}$ . This of course is the thermal diffusion coefficient this  $\frac{k}{\rho C_p}$  is the thermal diffusion coefficient. So, the final equation that I am getting is of the form  $\frac{\partial T}{\partial t}$  is equal to  $\alpha \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{\rho C_p}$ . Where  $\alpha$  is the thermal diffusion coefficient and you can see that when the thermal diffusion coefficient has dimensions of length square per unit time then you get correct dimensional consistency in this equation between the time derivative and the spatial derivative.

So, this is the thermal diffusion coefficient for unsteady diffusion in one dimension. You see a little bit more about the analogy between this and mass and heat diffusion in the next lecture. And then we look at how to solve these problems. So, we will continue this in the next class, I will see you then.